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Connectivity Estimation from the Inverse Coherence Matrix in a Complex Chaotic Oscillator Network

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***Abstract*—**We present on the method of inverse coherence matrix for the estimation of network connectivity from multivariate time series of a complex system. In a model system of coupled chaotic oscillators, it is shown that the inverse coherence matrix defined as the inverse of cross coherence matrix is proportional to the network connectivity. Therefore the inverse coherence matrix could be used for the distinction between the directly connected links from indirectly connected links in a complex network. We compare the result of network estimation using the method of the inverse coherence matrix with the results obtained from the coherence matrix and the partial coherence matrix.

***Keywords*—**Chaotic oscillator, complex network, inverse coherence matrix, network estimation.

I. Introduction

RHYTHMIC oscillations are observed ubiquitously in many complex biological, social, and physical systems. As the generators of oscillation are coupled in a large complex network, there have been many efforts to identify the underlying network connectivity from the measured multivariate time series. If two oscillators are connected in a complex network, we expect strong coherence between the two oscillators. Based on the coherence or correlation between the time series, as a reverse engineering, there have been several attempts to estimate the network connectivity [1]. Recently the method of coherence analysis has been applied for the identification of global organization in a brain network using the neuro-physiological data of the EEG, MEG, and fMRI [2][3].

One of the problems in this method is how to distinguish directly connected links from indirectly connected links. If two oscillators are driven by a common node in a network, we expect strong coherence between the two oscillators although there is no direct connection between them. To resolve this problem, the Granger causality problem [4], several attempts to identify directly connected links have been proposed. Recently the partial phase synchronization matrix which is defined as the inverse of the phase synchronization matrix normalized by the diagonal component of the inverse matrix was proposed for the estimation of the network connectivity [5]. In rather small network systems, it was shown that the partial phase synchronization index could distinguish directly connected links from indirectly connected links.

But, in large complex networks with hub nodes and many peripheral nodes as shown in Fig. 1, we observed that the partial phase synchronization index between any two nodes connected directly is strongly attenuated in proportion to the product of degree of two nodes [6]. Therefore identification of directly connected links, especially the links connecting two hub nods, becomes very ambiguous from the measurement of the partial phase synchronization index. We observed that this is due to the scaling of the normalization factor [6], the diagonal component of the inverse phase synchronization index, which grows in proportion to the node degree in a binary network [7]. So we used the inverse phase synchronization index, instead of the partial phase synchronization index, for the estimation of link strength in a weighted complex network. It was shown that the inverse phase synchronization index grows in proportion to the link strength. Therefore it was possible to reconstruct the original network from the phase synchronization index of coupled phase oscillators [6].

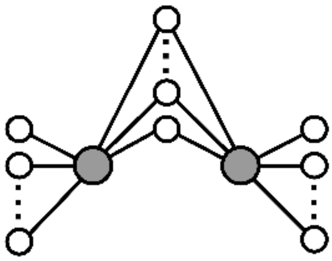

Fig. 1 A schematic diagram shows a complex network with high-degree hub nodes and small-degree peripheral nodes. Two chaotic oscillators on two hub nodes are not connected directly, but could be strongly correlated as they are coupled through many indirect connections

For nonlinear time series, in general, the method of inverse phase synchronization matrix is not applicable as the phase variables are not available [8]. In this paper, we show that network estimation still becomes possible if the inverse phase synchronization matrix is replaced by the inverse coherence matrix of the nonlinear time series. For this, we use the coupled chaotic oscillators on a complex network. The chaotic network of Rössler oscillator is defined as:

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$$\dot{X}_i = -\omega_i Y_i - Z_i + K\sum_j A_{ij}(X_j - X_i) + \xi_k$$
$$\dot{Y}_i = \omega_i X_i + aY_i + K\sum_j A_{ij}(Y_j - Y_i) \quad (1)$$
$$\dot{Z}_i = b + (X_i - c)Z_i + K\sum_j A_{ij}(Y_j - Y_i), \quad i = 1,\cdots,N$$

where the summation is over all neighbors with adjacency matrix elements A_{ij} equal to one. Here we take a=0.15, b=0.2, c=10, and ω_i is taken randomly from the interval [0.95, 1.05]. The white Gaussian noise $\xi_k(t)$ with intensity D satisfies

$$\langle \xi_k(t) \rangle = 0, \quad \langle \xi_k(t)\xi_l(t') \rangle = D\delta_{kl}\delta(t-t'). \quad (2)$$

In this work, we take D=1.0, and the network with N=512 nodes and N_{phys}=1024 links has a scale-free degree distribution $P(k) \propto k^{-\gamma}$, where the degree of a node i is defined as $k_i = \Sigma_j A_{ij}$ and $\gamma \approx 3$ [9].

II. Methods

To estimate the adjacency matrix A_{ij} of the network, we apply the method of inverse coherence matrix. The idea is to remove the contribution of indirect links from the measurement of coherence between two nodes. For the computation of the coherence matrix in this paper, we use the correlation coefficient $R_{i,j}$ for X_i and X_j of two chaotic oscillators. It is defined as:

$$R_{i,j} = \frac{<[X_i(t) - <X_i>_t][X_j(t) - <X_j>_t]>_t}{\sqrt{<[X_i(t) - <X_i>_t]^2>_t}\sqrt{<[X_i(t) - <X_i>_t]^2>_t}} \quad (3)$$

In terms of the correlation coefficient measure, the correlation matrix **R** of N chaotic oscillators is constructed as:

$$\mathbf{R} = \begin{pmatrix} 1 & R_{1,2} & \cdots & R_{1,N} \\ R_{2,1} & 1 & \cdots & R_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{N,1} & R_{N,2} & \cdots & R_{N,N} \end{pmatrix} \quad (4)$$

Note that the correlation matrix **R** is a symmetric matrix.

Recently the partial coherence was proposed to compute the genuine component of the coherence between two nodes X_i and X_j, removing the contributions coming from the indirect connections mediated by the remaining nodes X_z z=1,…,N with z≠i,j. It was shown that the partial coherence computed from the correlation coefficient is given by [10][11]

$$PR_{i,j|z} = -\frac{IR_{i,j}}{\sqrt{IR_{i,i}IR_{j,j}}} \quad (5)$$

We note that the matrix **IR** is defined as the inverse of the correlation matrix:

$$\mathbf{IR} = \mathbf{R}^{-1} \quad (6)$$

Therefore the partial correlation matrix **PR** is defined as the minus of the inverse correlation matrix normalized by the diagonal components of the inverse correlation matrix.

III. Results

In Fig. 2(a), the correlation matrix element for both directly connected and indirectly connected links are plotted versus the degree product of two nodes on the link. When the coupling strength is bigger than the critical coupling strength K_c≈0.1 where global network synchronization occurs, strong enhancement of correlation coefficient between two oscillators is observed. It is noted that the correlation coefficient between two nodes which are not connected directly is also very strong because of the many indirect connections connecting two nodes. As strong overlap between the correlation coefficient of directly connected and indirectly connected links occurs, the distinction between two connections become very difficult from the measurement of the link correlation coefficient [12].

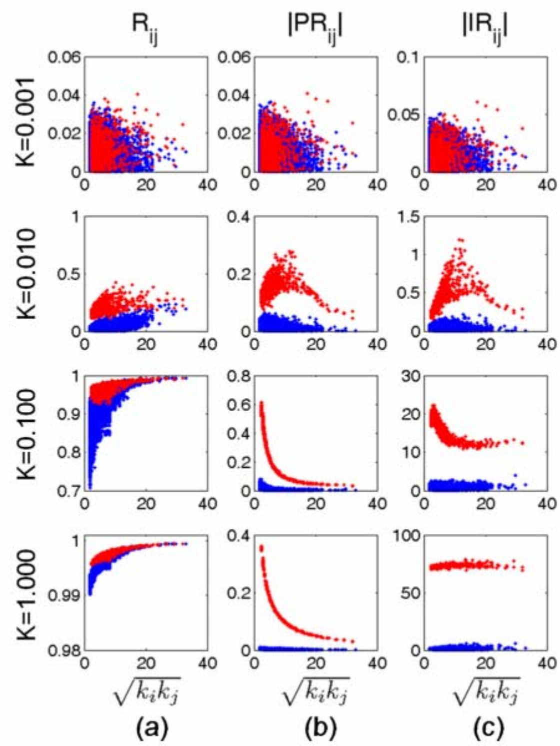

Fig. 2 The correlation (a), partial correlation (b), and inverse correlation (c) matrix element for the directly connected (red) and indirectly connected links (blue) are plotted versus the mean degree of two nodes for K=0.001, 0.01, 0.10, and 1.0. The mean degree of a link connecting two node i and j is defined as $(k_i\, k_j)^{1/2}$

In Fig. 2(c), the plot of the inverse correlation matrix element for strong coupling shows that the inverse correlation matrix elements of the directly connected links are much bigger those of links not connected directly. Therefore the distinction between directly connected links from indirectly connected links is possible by measuring the inverse correlation coefficient matrix elements. For strong coupling strength, the inverse correlation coefficient is almost constant independent of the mean degree. Considering the binary network where the link weight is constant, the constancy of the inverse correlation coefficient implies that the inverse correlation matrix is proportional to the network link strength. This is consistent with our previous result where the inverse phase synchronization matrix is proportional to the link weight matrix in a coupled phase oscillator network [6].

In Fig. 2(b), in comparison, we plot the partial correlation matrix element is plotted. It is shown that the partial correlation coefficient between any two oscillators connected directly is attenuated with the mean degree. This is to be compared with the inverse correlation matrix, where the strong dependence on the mean degree is completely removed.

As we plot the diagonal component of the inverse correlation matrix in Fig. 3, the diagonal component increases with the node degree for strong coupling strength. Therefore the strong attenuation of the partial correlation coefficient for a link connecting two hub nodes is due to the scaling behavior of the normalization factors, the diagonal component of the inverse correlation coefficient matrix, in the definition of the partial coherence matrix in Eq. (5). As a consequence, the normalization problem does not appear in the inverse correlation matrix. Therefore the inverse correlation matrix, not the partial correlation matrix, is used for the estimation of network connectivity in a complex network.

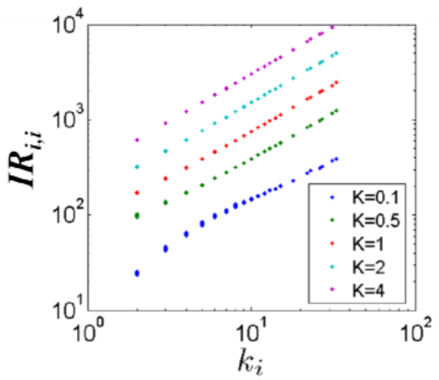

Fig. 3 Plot of the diagonal component of the inverse correlation matrix $IR_{i,i}$ versus the degree k_i of a node i for K=0.1, 0.5, 1.0, 2.0, and 4.0

The result presented in Fig. 2 implies that the network connectivity could be estimates using the inverse correlation matrix. As a test, we apply the method of inverse correlation matrix for the reconstruction of N_{phys} direct connections used in the Eq. (1). For this, we assume that the links with inverse correlation within top N_{phys}-th are connected, and disconnected otherwise. Here the estimation is quantified by the number of physical connections not identified by the method, the false negative connections N_{FN}, and also by the number of indirect connections which was misidentified as connected, the false positive connections N_{FP}. In Fig. 4, the fraction of false positive connection $R_{FP} = N_{FP}/N_{phys}$ and false negative connection $R_{FN} = N_{FN}/N_{phys}$ are plotted versus the coupling strength and mean degree of two nodes on the link.

For the inverse correlation matrix, the estimation is very successful, so the fraction of both false positive and negative connection are very low when the coupling strength is quite strong ($K>K_c$). On the other hand, for the correlation matrix, a significant fraction of false positive connection is produced when the degree product of two nodes is quite large. That is, the links connecting hub nodes are more often misclassified as connected, as the synchronization in a complex network is dominated by hub nodes with higher degree. Instead, the fraction of false negative connection is high when the mean degree of a link connecting peripheral nodes is small. For the partial correlation matrix, the fraction of false negative connection is significant when the mean degree is quite large. This is due to the strong attenuation effect introduced by the normalization of the partial correlation matrix in Eq. (5).

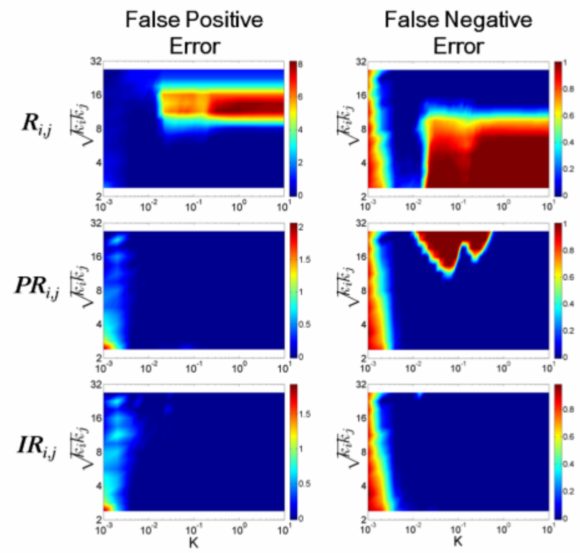

Fig. 4 Plots show the fractions of false positive (left) and false negative (right) estimation versus the coupling strength and the mean degree of two nodes using the correlation (top), partial correlation (middle), and inverse correlation (bottom) matrix

IV. Discussion and Summary

Recently we used the inverse phase synchronization matrix for the estimation of network connectivity in a coupled phase oscillators [6]. It was based on the observation that the inverse phase synchronization matrix element is proportional to the corresponding the link weight [6][7].

Here we have shown that the method of the inverse phase synchronization matrix could be extended to the case of couple chaotic oscillators where the phase variable is not defined properly. We have shown that the inverse correlation matrix in a complex chaotic oscillator network is proportional to the network connectivity, so it could distinguish the directly connected links from the indirectly connected links. On the other hand, the partial correlation matrix suffers a serious normalization problem: the partial correlation matrix element connecting two hub nodes is strongly attenuated. This is due to the scaling behavior of the diagonal component of the inverse correlation matrix which was used as the normalization factors in Eq. (5). Because of the normalization problem, the inverse correlation matrix, not the partial correlation matrix, should be used for the estimation of network connectivity.

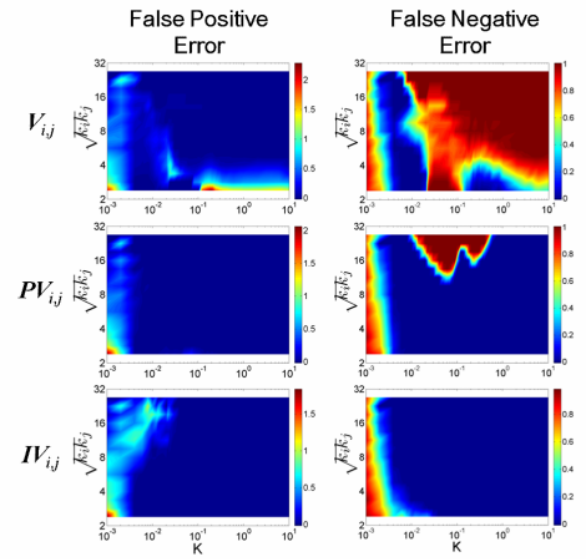

Fig. 5 Plots show the fractions of false positive (left) and false negative (right) estimation versus the coupling strength and the mean degree of two nodes using the covariance (top), partial covariance (middle), and inverse covariance (bottom) matrix

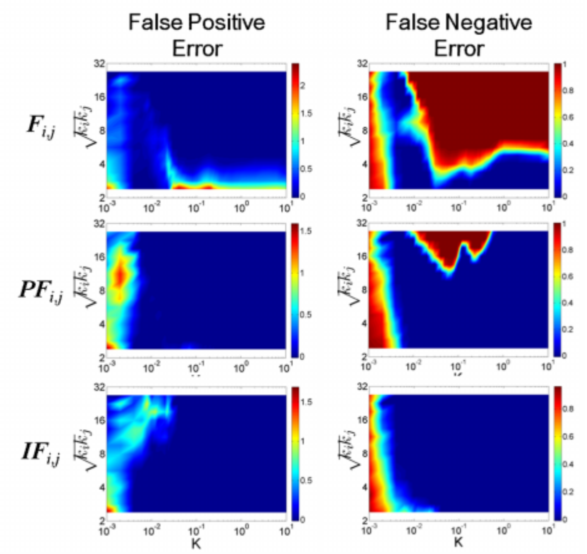

Fig. 6 Plots show the fractions of false positive (left) and false negative (right) estimation versus the coupling strength and the mean degree of two nodes using the co-fluctuation (top), partial co-fluctuation (middle), and inverse co-fluctuation (bottom) matrix

Here the method of inverse coherence matrix was developed using the correlation coefficient. But the proposed method could be also extended to other measures of coherence, such as, the covariance **V**, and co-fluctuation **F** [7] defined as follows:

$$\begin{aligned} V_{i,j} &= <[X_i(t) - <X_i>_t][X_i(t) - <X_i>_t]>_t , \\ F_{i,j} &= <[X'_i(t) - <X'_i>_t][X'_j(t) - <X'_j>_t]>_t , \\ &X'_i(t) = X_i(t) - \overline{X}(t) , \\ &\overline{X}(t) = \frac{1}{N}\sum_{i=1}^{N} X_i(t) \end{aligned} \tag{7}$$

The plots of the fraction of false positive and false negative connection obtained from the inverse covariance and inverse co-fluctuation in Fig. 5 and 6, respectively, show that the method of inverse coherence matrix is also successful if the inverse covariance or the inverse co-fluctuation matrix is used.

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