

Fuzzy restrictions and an application to cooperative games with restricted cooperation

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ABSTRACT

The concept of restriction, which is an extension of that of interior operator, was introduced to model limited cooperation in cooperative game theory. In this paper a fuzzy version of restrictions is presented. We show that these new operators, called fuzzy restrictions, can be characterized by the transitivity of the fuzzy dependence relations that they induce. As an application, we introduce cooperative games with fuzzy restriction, which are used to model cooperative situations in which each player in a coalition has a level of cooperation within the coalition. A value for these games is defined and characterized.

KEYWORDS

restriction; fuzzy set; transitive relation; cooperative game; restricted cooperation; interior operator.

1. Introduction

Cooperative games describe situations in which the cooperation among a set of players gives rise to a profit. Particularly notable are cooperative transferable utility games (or TU-games), in which each coalition (subset of players) can obtain a profit that the players in the coalition can freely share through side payments. When dealing with these games, it is often assumed that when a coalition is formed any player in the coalition will be able to cooperate. However, this does not always correspond to reality. In real life, a cooperation situation may be associated with certain limitations on cooperation, in such a way that when a coalition is formed, some players in the coalition cannot cooperate. The intention to model these situations gave rise to TU-games with restricted cooperation. These games describe not only the profit that can be obtained by any coalition, but also the limitations that there in cooperation. Numerous types of such limitations have been considered in the literature, and different mathematical structures have been used to model them, such as convex geometries (Bilbao 1998), matroids (Bilbao *et al.* 2001), antimatroids (Algaba *et al.* 2004), augmenting systems (Bilbao and Ordóñez 2009) and graphs (Gilles, Owen and van den Brink 1992). Derks and Peters (1993) used an extension of the concept of interior operator in order to model limited cooperation. They introduced the so-called restrictions. Given a finite

set N , they define a restriction on N as a mapping $A : 2^N \rightarrow 2^N$ that satisfies, for every $E \subseteq F \subseteq N$, a) $A(E) \subseteq E$, b) $A(E) \subseteq A(F)$ and c) $A(A(E)) = A(E)$. In the context of restricted cooperation, the elements of N are players and, for any $E \subseteq N$, $A(E)$ is interpreted as the set of players that can cooperate when coalition E is formed. Note that, with this interpretation, properties a) and b) of restrictions are reasonable. However, property c) is not so intuitive. Gallardo, Jiménez and Jiménez-Losada (2016) studied mappings with properties a) and b), which they called authorization operators, and used them to model hierarchies. Our first goal will be to clarify the meaning of property c). It will be shown that the mappings with properties a) and b) induce certain relations on the power set of N which can be interpreted as dependence relations. Indeed, we can interpret that the players in $E \setminus A(E)$ cannot cooperate within E because they have some dependence on the players who are not in E . We will see that property c) of restrictions means that these dependence relations are transitive.

Gallardo *et al.* (2015) studied cooperative situations with fuzzy limitations on cooperation. In order to model these situations, they introduced a fuzzy version of authorization operators. If we take the definition of authorization operator and consider degrees of cooperation within a coalition, the definition of fuzzy authorization operator arises naturally. A fuzzy authorization operator is a mapping $a : 2^N \rightarrow [0, 1]^N$ that satisfies, for every $E \subseteq F \subseteq N$, a) $a(E) \leq \mathbf{1}_E$ ($\mathbf{1}_E$ denotes the characteristic vector of E) and b) $a(E) \leq a(F)$. The goal of this paper is to introduce a fuzzy version of the concept of restriction, which will be called fuzzy restriction. Notice that, to this end, we cannot do an immediate crisp-fuzzy translation of the definition of restriction, as we did to define fuzzy authorization operators. In order to introduce fuzzy restrictions we will have to find an alternative description of restrictions. This alternative description will be crisp-fuzzy translated conveniently. Once we define fuzzy restrictions, it will be shown that, similarly to (crisp) restrictions, they induce certain dependence relations. Moreover, it will be proved that, analogously to the crisp case, fuzzy restrictions are the fuzzy authorization operators which induce transitive dependence relations.

As an application of fuzzy restrictions, we will use them to model cooperative situations with fuzzy-restricted cooperation.

The paper is organized as follows. In Section 2, some preliminaries concerning fuzzy sets, the Choquet integral and TU-games are given. In Section 3, the concepts of restriction and authorization operator are recalled. We introduce the dependence relation induced by an authorization operator, and characterize restrictions as the authorization operators which induce transitive dependence relations. In Section 4, we identify restrictions on a set N with a certain kind of $\{0, 1\}$ functions on the power set of N . This identification is used to introduce fuzzy restrictions. In addition, it is proved that fuzzy restrictions are the fuzzy authorization operators which induce transitive dependence relations. In Section 5, we introduce games with fuzzy restriction. We also define and characterize a value for these games. In Section 6 the conclusions are drawn.

2. Preliminaries

Throughout this paper, N will denote a finite nonempty set with cardinality $n \in \mathbb{N}$.

2.1. Fuzzy sets

Fuzzy sets were introduced by Zadeh (1965). A *fuzzy subset* of N is an element of $[0, 1]^N$. Given e a fuzzy subset of N and $i \in N$, the number e_i is called the *degree of membership* of i in e . The *support* of e is the set $\text{supp}(e) = \{i \in N : e_i > 0\}$. The *image* of e is the set $\text{im}(e) = \{e_i : i \in N\}$. For every $t \in [0, 1]$ the t -*cut* of e is the set

$$[e]_t = \{i \in N : e_i \geq t\}.$$

Given $e, f \in [0, 1]^N$, we say that e is included in f , which is denoted by $e \subseteq f$, if $e_i \leq f_i$ for all $i \in N$. The union and the intersection of e and f are defined, respectively, by $(e \cup f)_i = \max\{e_i, f_i\}$, $(e \cap f)_i = \min\{e_i, f_i\}$ for all $i \in N$.

Every subset E of N can be identified with the fuzzy subset $\mathbf{1}_E \in [0, 1]^N$ defined as

$$(\mathbf{1}_E)_i = \begin{cases} 1 & \text{if } i \in E, \\ 0 & \text{if } i \in N \setminus E. \end{cases}$$

A fuzzy relation on a set X is a fuzzy subset of X^2 . If ρ is a fuzzy relation on X , it is said that ρ is transitive if $\rho(x, z) \geq \min\{\rho(x, y), \rho(y, z)\}$ for every $x, y, z \in X$.

2.2. The Choquet integral

The Choquet integral (Choquet 1953) was introduced for capacities. Later on Schmeidler (1986) studied this integral for other functions. If $v : 2^N \rightarrow \mathbb{R}$ and $e \in [0, 1]^N$, then the *Choquet integral* of e with respect to v is

$$\int e dv = \sum_{l=1}^p (t_l - t_{l-1}) v([e]_{t_l}),$$

where $\{t_l\}_{l=0}^p = \text{im}(e) \cup \{0\}$ and $0 = t_0 < t_1 < \dots < t_p$. If $e = 0$ it is understood that $\int e dv = 0$.

The following properties of the Choquet integral are known:

- (C1) $\int \mathbf{1}_E dv = v(E)$, for all $E \subseteq N$.
- (C2) $\int te dv = t \int e dv$, for all $t \in [0, 1]$.
- (C3) $\int e dv \leq \int f dv$, whenever $e \leq f$ and v is monotonic (i.e., $v(E) \leq v(F)$ for all $E \subseteq F \subseteq N$).
- (C4) $\int e d(cv) = c \int e dv$, for all $c \in \mathbb{R}$.
- (C5) $\int e d(v_1 + v_2) = \int e dv_1 + \int e dv_2$.
- (C6) $\int (e + f) dv = \int e dv + \int f dv$, whenever $e + f \leq \mathbf{1}_N$ and e, f are comonotone (i.e., $(e_i - e_j)(f_i - f_j) \geq 0$ for every $i, j \in N$).
- (C7) If $\{t_l\}_{l=0}^q \supseteq \text{im}(e) \cup \{0\}$ and $0 = t_0 < t_1 < \dots < t_q$, then,

$$\int e dv = \sum_{l=1}^q (t_l - t_{l-1}) v([e]_{t_l}).$$

2.3. Cooperative TU-games

A *cooperative transferable utility game* or *TU-game* on N is a function $v : 2^N \rightarrow \mathbb{R}$ that satisfies $v(\emptyset) = 0$. The elements of N are called *players*, and the subsets of N are called *coalitions*. For each coalition E , the number $v(E)$ can be interpreted as the gain that the players in E can achieve when they decide to cooperate. If $v(E) \leq v(F)$ for all $E \subseteq F \subseteq N$ then the game v is said to be *monotonic*.

The set of all TU-games on N is denoted by \mathcal{G}^N . This set is a $(2^n - 1)$ -dimensional real vector space. A well-known basis of this vector space is given by the set $\{u_E : E \in 2^N \setminus \{\emptyset\}\}$, where

$$u_E(F) = \begin{cases} 1 & \text{if } E \subseteq F, \\ 0 & \text{otherwise.} \end{cases}$$

For each nonempty $E \subseteq N$ the game u_E is called the *unanimity game* of E . Every game $v \in \mathcal{G}^N$ can be written as a linear combination of unanimity games. Thus,

$$v = \sum_{E \in 2^N \setminus \{\emptyset\}} \Delta_v(E) u_E$$

where each coordinate $\Delta_v(E)$ of the game v with respect to the basis of the unanimity games is called *dividend of the coalition E* in the game v . The dividends can be obtained recursively:

$$\Delta_v(E) = \begin{cases} v(E) & \text{if } |E| = 1, \\ v(E) - \sum_{\{D \in 2^N \setminus \{\emptyset\} : D \subsetneq E\}} \Delta_v(D) & \text{if } |E| > 1. \end{cases}$$

Given a game, a problem that arises is how to assign a payoff to each player in a fair way. A *value* assigns to each game a payoff vector. Many values have been defined in the literature. The best-known of them is the *Shapley value* (Shapley 1953). For each $v \in \mathcal{G}^N$, the Shapley value of v , denoted by $Sh(v)$, is defined, for all $i \in N$, as

$$Sh_i(v) = \sum_{\{E \subseteq N : i \in E\}} p_E [v(E) - v(E \setminus \{i\})],$$

where $p_E = \frac{(n - |E|)! (|E| - 1)!}{n!}$ and $|E|$ denotes the cardinality of E .

The following properties of the Shapley value are well known:

Efficiency. If $v \in \mathcal{G}^N$, then $\sum_{i \in N} Sh_i(v) = v(N)$.

Linearity. If $v_1, v_2 \in \mathcal{G}^N$ and $t_1, t_2 \in \mathbb{R}$, then $Sh(t_1 v_1 + t_2 v_2) = t_1 Sh(v_1) + t_2 Sh(v_2)$.

Null player property. If $i \in N$ is a null player in $v \in \mathcal{G}^N$ (i.e., $v(E) = v(E \setminus \{i\})$ for all $E \subseteq N$), then $Sh_i(v) = 0$.

Necessary player property. If i is a necessary player in $v \in \mathcal{G}^N$ (i.e., $v(E) = 0$ for every $E \subseteq N \setminus \{i\}$) and v is monotonic, then $Sh_i(v) \geq Sh_j(v)$ for all $j \in N$.

3. Restrictions

Restrictions were introduced by Derks and Peters (1993) in order to provide a model of cooperative games with limited cooperation. We recall the definition.

Definition 3.1. A restriction on N is a mapping $A : 2^N \rightarrow 2^N$ satisfying the following conditions:

- (1) $A(E) \subseteq E$ for every $E \subseteq N$,
- (2) If $E \subseteq F \subseteq N$ then $A(E) \subseteq A(F)$,
- (3) $A(A(E)) = A(E)$ for every $E \subseteq N$.

Restrictions can be seen as special cases of other more general operators, called authorization operators, which were introduced by Gallardo, Jiménez and Jiménez-Losada (2016) to study hierarchical structures. We recall the definition.

Definition 3.2. An authorization operator on N is a mapping $A : 2^N \rightarrow 2^N$ that satisfies the following conditions:

- (1) $A(E) \subseteq E$ for every $E \subseteq N$,
- (2) If $E \subseteq F \subseteq N$ then $A(E) \subseteq A(F)$.

The set of all authorization operators on N is denoted by \mathcal{A}^N .

We interpret authorization operators in terms of permission. From this point of view, the elements in N are agents and the subsets of N are coalitions. Let $A \in \mathcal{A}^N$ and $E, F \in 2^N$. If $F \subseteq A(E)$, it is said that coalition E can authorize coalition F in (N, A) . If $A(E) = E$ it is said that E is *autonomous* in (N, A) .

Notice that if A is a restriction on N then, for every $E \subseteq N$, $A(E)$ is autonomous. The correspondence that assigns, to each restriction A on N , the family of autonomous coalitions in (N, A) , is a bijection from the set of restrictions on N into the set of families of coalitions in N which are union-closed and contain the empty set.

An authorization operator is a restriction if it satisfies idempotence. We aim to provide an interpretation of this condition. We need two previous definitions.

Definition 3.3. Let $A \in \mathcal{A}^N$, $E \subseteq N$ and $i \in N$. It is said that E can veto i in (N, A) if $i \notin A(N \setminus E)$.

Definition 3.4. Let $A \in \mathcal{A}^N$. We define the relation \mathcal{V}_A on 2^N as

$$E \mathcal{V}_A F \text{ if and only if, for every } i \in F, E \text{ can veto } i \text{ in } (N, A)$$

for all $E, F \subseteq N$. Equivalently, $E \mathcal{V}_A F$ if and only if $A(N \setminus E) \subseteq N \setminus F$.

Proposition 3.5. Let $A \in \mathcal{A}^N$. The following statements are equivalent:

- (1) A is a restriction on N .
- (2) \mathcal{V}_A is transitive.

Proof. Let $A \in \mathcal{A}^N$.

(1) \implies (2). Suppose that A is a restriction on N . Let $E, F, H \subseteq N$ such that $E \mathcal{V}_A F$ and $F \mathcal{V}_A H$. By definition, $E \mathcal{V}_A F$ means that $A(N \setminus E) \subseteq N \setminus F$. From the monotonicity of A it follows that $A(A(N \setminus E)) \subseteq A(N \setminus F)$. Now, taking into consideration

that A is a restriction and $F \mathcal{V}_A H$ we have that

$$A(N \setminus E) = A(A(N \setminus E)) \subseteq A(N \setminus F) \subseteq N \setminus H.$$

Thus, $A(N \setminus E) \subseteq N \setminus H$, which is equivalent to $E \mathcal{V}_A H$.

(2) \implies (1). Suppose that the relation \mathcal{V}_A is transitive. Let $E \subseteq N$. By definition, it is clear that $(N \setminus E) \mathcal{V}_A (N \setminus A(E))$. Similarly, $(N \setminus A(E)) \mathcal{V}_A (N \setminus A(A(E)))$. From the transitivity of \mathcal{V}_A we obtain that $(N \setminus E) \mathcal{V}_A (N \setminus A(A(E)))$, which means that $A(E) \subseteq A(A(E))$. Thus, it is clear that $A(E) = A(A(E))$. \square

4. Fuzzy restrictions

Gallardo *et al.* (2015) showed that for some purposes it is useful to consider mappings from 2^N into $[0, 1]^N$ with properties analogous to those of authorization operators. They introduced the so called fuzzy authorization operators, which are defined below.

Definition 4.1. A fuzzy authorization operator on N is a mapping $a : 2^N \rightarrow [0, 1]^N$ that satisfies the following conditions:

- (1) $a(E) \subseteq \mathbf{1}_E$ for every $E \subseteq N$,
- (2) If $E \subseteq F \subseteq N$ then $a(E) \subseteq a(F)$.

The goal of this paper is to introduce what we will call fuzzy restrictions, which will be mappings from 2^N into $[0, 1]^N$ with properties analogous to those of restrictions. It is clear that the definition of fuzzy authorization operator comes from the definition of authorization operator by simply considering in the codomain the fuzzy analogue of the set inclusion. Notice that, in order to define fuzzy restrictions, we cannot make a translation like that, since the property of idempotence cannot be translated as such. Therefore, we must proceed differently. To this end, we will reinterpret restrictions through a type of functions. Let

$$\mathcal{I}^N = \{ \Lambda : 2^N \rightarrow \{0, 1\} \mid \Lambda(\emptyset) = 1 \text{ and } \Lambda(E \cup F) \geq \min\{\Lambda(E), \Lambda(F)\} \text{ for all } E, F \subseteq N \}.$$

Notice that if $\Lambda \in \mathcal{I}^N$ and $E_1, \dots, E_m \subseteq N$ then

$$\Lambda(E_1 \cup \dots \cup E_m) \geq \min\{\Lambda(E_1), \dots, \Lambda(E_m)\}.$$

Let $\Lambda \in \mathcal{I}^N$. Consider $R^\Lambda : 2^N \rightarrow 2^N$ defined as

$$R^\Lambda(E) = \bigcup_{\{F \subseteq E : \Lambda(F) = 1\}} F \quad \text{for all } E \subseteq N.$$

The following two propositions show that the expression above provides a bijection between \mathcal{I}^N and the set of restrictions on N . We will use a previous lemma.

Lemma 4.2. If $\Lambda \in \mathcal{I}^N$ and $E \subseteq N$ then $\Lambda(R^\Lambda(E)) = 1$.

Proof. Let $\Lambda \in \mathcal{I}^N$ and let $E \subseteq N$. Then,

$$\Lambda(R^\Lambda(E)) = \Lambda \left(\bigcup_{\{F \subseteq E: \Lambda(F)=1\}} F \right) \geq \min \{ \Lambda(F) : F \subseteq E \text{ and } \Lambda(F) = 1 \} = 1.$$

□

Proposition 4.3. *If $\Lambda \in \mathcal{I}^N$ then R^Λ is a restriction on N .*

Proof. It is clear that $R^\Lambda \in \mathcal{A}^N$. Let us see that R^Λ satisfies the property of idempotence. Take $E \subseteq N$. We must prove that $R^\Lambda(R^\Lambda(E)) = R^\Lambda(E)$. One inclusion is immediate. As for the other inclusion, we have

$$R^\Lambda(R^\Lambda(E)) = \bigcup_{\{F \subseteq R^\Lambda(E): \Lambda(F)=1\}} F \supseteq R^\Lambda(E),$$

where we have used Lemma 4.2. □

Proposition 4.4. *Let A be a restriction on N . Then, there exists a unique $\Lambda \in \mathcal{I}^N$ such that $A = R^\Lambda$.*

Proof. Let A be a restriction on N . Take $\Lambda : 2^N \rightarrow \{0, 1\}$ defined by

$$\Lambda(E) = \begin{cases} 1 & \text{if } A(E) = E, \\ 0 & \text{otherwise.} \end{cases}$$

We want to show that $\Lambda \in \mathcal{I}^N$. It is clear that $\Lambda(\emptyset) = 1$. Let $E, F \subseteq N$. We must see that $\Lambda(E \cup F) \geq \min\{\Lambda(E), \Lambda(F)\}$. We can suppose that $\Lambda(E) = \Lambda(F) = 1$. We have

$$A(E \cup F) \supseteq A(E) \cup A(F) = E \cup F.$$

Clearly, $A(E \cup F) = E \cup F$ and, therefore, $\Lambda(E \cup F) = 1$.

Let us prove that $R^\Lambda = A$. Let $E \subseteq N$. On the one hand,

$$R^\Lambda(E) = \bigcup_{\{F \subseteq E: \Lambda(F)=1\}} F = \bigcup_{\{F \subseteq E: A(F)=F\}} F = \bigcup_{\{F \subseteq E: A(F)=F\}} A(F) \subseteq A(E).$$

On the other hand, since $A(A(E)) = A(E)$ we have $\Lambda(A(E)) = 1$. Hence,

$$A(E) \subseteq \bigcup_{\{F \subseteq E: \Lambda(F)=1\}} F = R^\Lambda(E).$$

We conclude that $R^\Lambda(E) = A(E)$.

It remains to prove the uniqueness of Λ . Let $\Gamma \in \mathcal{I}^N$ be such that $A = R^\Gamma$. Let $E \subseteq N$. Notice that if $\Gamma(E) = 1$ then $R^\Gamma(E) = E$ and, therefore, $A(E) = E$. Conversely, if $A(E) = E$, then

$$\Gamma(E) = \Gamma(A(E)) = \Gamma(R^\Gamma(E)) = 1,$$

where we have used Lemma 4.2. We have proved that $\Gamma(E) = 1$ if and only if $A(E) = E$, for all $E \subseteq N$. Therefore, $\Gamma = \Lambda$. \square

The two previous propositions provide us with an alternative description of restrictions. We will use this to achieve our goal. Recall that we aimed to obtain a family of mappings from 2^N into $[0, 1]^N$ with properties analogous to those of restrictions. It was not clear how to do this from the definition of restriction. Now that we have identified the family of restrictions on N with \mathcal{I}^N , we will tackle our problem in the following way: we will find a family of functions from 2^N into $[0, 1]$ with properties analogous to those of the functions in \mathcal{I}^N and then we will associate to each of these functions a mapping from 2^N into $[0, 1]^N$ in a similar way as we associate R^Λ to each $\Lambda \in \mathcal{I}^N$.

Bearing in mind the definition of \mathcal{I}^N , we consider the following family:

$$\mathcal{FI}^N = \{ \lambda : 2^N \rightarrow [0, 1] \mid \lambda(\emptyset) = 1 \text{ and } \lambda(E \cup F) \geq \min\{\lambda(E), \lambda(F)\} \text{ for all } E, F \subseteq N \}.$$

Taking into consideration the definition of R^Λ , we define, for any $\lambda \in \mathcal{FI}^N$, the mapping $r^\lambda : 2^N \rightarrow [0, 1]^N$ as

$$r^\lambda(E) = \bigcup_{F \subseteq E} \lambda(F) \mathbf{1}_F \quad \text{for every } E \subseteq N.$$

It is clear that r^λ is a fuzzy authorization operator on N .

Now we define the concept of fuzzy restriction.

Definition 4.5. A mapping $a : 2^N \rightarrow [0, 1]^N$ is said to be a fuzzy restriction on N if there exists $\lambda \in \mathcal{FI}^N$ such that $a = r^\lambda$.

In the following proposition we show that the set of fuzzy restrictions on N is identified with \mathcal{FI}^N .

Proposition 4.6. Let a be a fuzzy restriction on N . Then there exists a unique $\lambda \in \mathcal{FI}^N$ such that $a = r^\lambda$.

Proof. Let a be a fuzzy restriction on N . Let $\lambda \in \mathcal{FI}^N$ be such that $a = r^\lambda$. In order to prove the uniqueness of λ , it is enough to prove that

$$\lambda(E) = \min \{a_i(E) : i \in E\} \quad \text{for all } E \in 2^N \setminus \{\emptyset\}.$$

Take $E \in 2^N \setminus \{\emptyset\}$. On the one hand, we have

$$a(E) = r^\lambda(E) = \bigcup_{F \subseteq E} \lambda(F) \mathbf{1}_F \supseteq \lambda(E) \mathbf{1}_E.$$

Hence,

$$\min \{a_i(E) : i \in E\} \geq \lambda(E).$$

On the other hand, from

$$a(E) = \bigcup_{F \subseteq E} \lambda(F) \mathbf{1}_F$$

it follows that, for each $i \in E$, there exists $F_i \subseteq E$ such that $i \in F_i$ and $a_i(E) = \lambda(F_i)$. We have

$$\lambda(E) = \lambda\left(\bigcup_{i \in E} F_i\right) \geq \min\{\lambda(F_i) : i \in E\} = \min\{a_i(E) : i \in E\}.$$

□

Recall that we characterized the restrictions on N as the authorization operators on N for which the induced veto relation is transitive. We aim to do something similar in the case of fuzzy restrictions. To this end, we need some previous results and definitions.

Let $\lambda : 2^N \rightarrow [0, 1]$. For any $t \in (0, 1]$ consider $\lambda_t : 2^N \rightarrow \{0, 1\}$ defined as

$$\lambda_t(E) = \begin{cases} 1 & \text{if } \lambda(E) \geq t, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.7. *Let $\lambda : 2^N \rightarrow [0, 1]$. Then the following statements are equivalent:*

- (1) $\lambda \in \mathcal{FI}^N$.
- (2) $\lambda_t \in \mathcal{I}^N$ for every $t \in (0, 1]$.

Proof. Let $\lambda : 2^N \rightarrow [0, 1]$.

(1) \implies (2). This is obvious from the definitions.

(2) \implies (1). Suppose that $\lambda_t \in \mathcal{I}^N$ for every $t \in (0, 1]$. From $\lambda_1(\emptyset) = 1$ we obtain that $\lambda(\emptyset) = 1$. Let $E, F \subseteq N$. Let $s = \min\{\lambda(E), \lambda(F)\}$. We must see that $\lambda(E \cup F) \geq s$. We can suppose that $s > 0$. From $\lambda_s(E) = \lambda_s(F) = 1$ and $\lambda_s \in \mathcal{I}^N$ it follows that $\lambda_s(E \cup F) = 1$. Hence, $\lambda(E \cup F) \geq s$. □

Definition 4.8. Let a be a fuzzy authorization operator on N and let $t \in (0, 1]$. The t -cut of a is the mapping $[a]_t : 2^N \rightarrow 2^N$ defined as $[a]_t(E) = [a(E)]_t$ for every $E \subseteq N$.

It is clear that if a is a fuzzy authorization operator on N and $t \in (0, 1]$ then $[a]_t$ is an authorization operator on N .

In the following proposition we show that the cuts of a fuzzy restriction on N are restrictions on N .

Proposition 4.9. *Let $\lambda \in \mathcal{FI}^N$ and $t \in (0, 1]$. Then*

$$[r^\lambda]_t = R^{\lambda_t}$$

Proof. Let $\lambda \in \mathcal{FI}^N$, $t \in (0, 1]$ and $E \subseteq N$. We must prove that $[r^\lambda(E)]_t = R^{\lambda_t}(E)$. On the one hand, $i \in [r^\lambda(E)]_t$ means, by definition of r^λ , that there exists $F \subseteq E$ such that $i \in F$ and $\lambda(F) \geq t$. On the other hand, $i \in R^{\lambda_t}(E)$ means, by definition of R^λ , that there exists $F \subseteq E$ such that $i \in F$ and $\lambda_t(F) = 1$. □

Proposition 4.10. *Let a be a fuzzy authorization operator on N . The following statements are equivalent:*

- (1) a is a fuzzy restriction on N .
- (2) $[a]_t$ is a restriction on N for every $t \in (0, 1]$.

Proof. Let a be a fuzzy authorization operator on N .

(1) \implies (2). Direct consequence of the definition of fuzzy restriction and Propositions 4.3, 4.7, 4.9.

(2) \implies (1). Suppose that $[a]_t$ is a restriction for every $t \in (0, 1]$. Consider $\lambda : 2^N \rightarrow [0, 1]$ defined as

$$\lambda(E) = \begin{cases} \min\{a_i(E) : i \in E\} & \text{if } E \neq \emptyset, \\ 1 & \text{if } E = \emptyset. \end{cases}$$

It can be easily verified, by using that a is a fuzzy authorization operator, that $\lambda \in \mathcal{FI}^N$. We will prove that $a = r^\lambda$. Let $E \in 2^N \setminus \{\emptyset\}$. Let us show that

$$a(E) = r^\lambda(E).$$

Firstly, we will show that $a(E) \subseteq r^\lambda(E)$. Let $i \in E$. If $a_i(E) = 0$, then $a_i(E) \leq r_i^\lambda(E)$. Suppose that $a_i(E) > 0$. Let $t = a_i(E)$. Since $[a]_t$ is a restriction, we know that $[a]_t([a]_t(E)) = [a]_t(E)$. This implies that $\lambda([a(E)]_t) \geq t$. We have

$$r^\lambda(E) = \bigcup_{F \subseteq E} \lambda(F) \mathbf{1}_F \supseteq \lambda([a(E)]_t) \mathbf{1}_{[a(E)]_t} \supseteq t \mathbf{1}_{[a(E)]_t}.$$

From this inclusion and $i \in [a(E)]_t$ it follows that $r_i^\lambda(E) \geq t = a_i(E)$. Now, we will prove that $r^\lambda(E) \subseteq a(E)$. Let $i \in E$. From the definition of r^λ , it follows that there exists $F \subseteq E$ with $i \in F$ such that $r_i^\lambda(E) = \lambda(F)$. We have

$$r_i^\lambda(E) = \lambda(F) = \min\{a_j(F) : j \in F\} \leq a_i(F) \leq a_i(E).$$

□

We defined the veto relation induced by an authorization operator. Now we proceed to define the analogous concept in the case of fuzzy authorization operators.

Definition 4.11. Let a be a fuzzy authorization operator on N . For every $E, F \subseteq N$, the level of veto of E over F is defined as

$$\nu_a(E, F) = \begin{cases} 1 - \max\{a_i(N \setminus E) : i \in F\} & \text{if } F \neq \emptyset, \\ 1 & \text{if } F = \emptyset. \end{cases}$$

Notice that ν_a is a fuzzy relation on 2^N .

Proposition 4.12. *Let a be a fuzzy authorization operator on N . The following statements are equivalent:*

- (1) ν_a is transitive.
- (2) $\mathcal{V}_{[a]_t}$ is transitive for every $t \in (0, 1]$.

Proof. Let a be a fuzzy authorization operator on N .

(1) \implies (2). Suppose that ν_a is transitive. Let $t \in (0, 1]$. Let $E, F, G \subseteq N$ be such that $E \mathcal{V}_{[a]_t} F$ and $F \mathcal{V}_{[a]_t} G$. If $F = \emptyset$ or $G = \emptyset$ it is trivial to check that $E \mathcal{V}_{[a]_t} G$. Suppose that $F, G \neq \emptyset$. We have that $[a(N \setminus E)]_t \subseteq N \setminus F$ and $[a(N \setminus F)]_t \subseteq N \setminus G$. This means that

$$\begin{aligned} a_i(N \setminus E) &< t && \text{for every } i \in F, \\ a_j(N \setminus F) &< t && \text{for every } j \in G, \end{aligned}$$

whence $\nu_a(E, F) > 1 - t$ and $\nu_a(F, G) > 1 - t$. Since ν_a is transitive we conclude that $\nu_a(E, G) > 1 - t$, or, equivalently, $E \mathcal{V}_{[a]_t} G$.

(2) \implies (1). Suppose that $\mathcal{V}_{[a]_t}$ is transitive for every $t \in (0, 1]$. Let $E, F, G \subseteq N$. We must show that $\nu_a(E, G) \geq \min\{\nu_a(E, F), \nu_a(F, G)\}$. We suppose that $F, G \neq \emptyset$ and $\min\{\nu_a(E, F), \nu_a(F, G)\} > 0$, since otherwise the proof is trivial. It suffices to prove that $\nu_a(E, G) > t$ for every $t \in (0, \min\{\nu_a(E, F), \nu_a(F, G)\})$. If $t \in (0, \min\{\nu_a(E, F), \nu_a(F, G)\})$ we have

$$\begin{aligned} a_i(N \setminus E) &< 1 - t && \text{for every } i \in F, \\ a_j(N \setminus F) &< 1 - t && \text{for every } j \in G. \end{aligned}$$

Therefore, $E \mathcal{V}_{[a]_{1-t}} F$ and $F \mathcal{V}_{[a]_{1-t}} G$. Since $\mathcal{V}_{[a]_{1-t}}$ is transitive, $E \mathcal{V}_{[a]_{1-t}} G$, whence $\nu_a(E, G) > t$. \square

Finally, we state the result analogous to Proposition 3.5 for fuzzy restrictions.

Proposition 4.13. *Let a be a fuzzy authorization operator on N . The following statements are equivalent:*

- (1) a is a fuzzy restriction on N .
- (2) ν_a is transitive.

Proof. The result follows from Propositions 3.5, 4.10 and 4.12. \square

5. TU-games with fuzzy restriction

When studying games with restricted cooperation, it is often assumed that, when a coalition is formed, a player in the coalition either can fully cooperate within the coalition or cannot cooperate at all. However, in some cases a player in a coalition has a degree of freedom to cooperate within the coalition. For instance, there are game situations in which each player owns some kind of resources such as goods, raw materials or labor force, and where the cooperation within a coalition consists of putting the resources at the disposal of the coalition. In these cases, when a coalition is formed, a player in the coalition might be able to offer only a part of the resources that she/he owns, due to dependence relations with players that are not in the coalition. These situations can be modeled by means of the set of players N , a TU-game $v \in \mathcal{G}^N$ and a mapping $a : 2^N \rightarrow [0, 1]^N$ which associates, to each $E \subseteq N$ and each $i \in E$, the proportion $a_i(E)$ of the resources of player i that she/he can use within coalition E . Notice that, bearing in mind our interpretation, a must satisfy $a(E) \subseteq \mathbf{1}_E$ and $a(E) \subseteq a(F)$ for every $E \subseteq F \subseteq N$, that is, a will be a fuzzy authorization operator.

Moreover, if the dependence relations associated to the situation are transitive, then, by Proposition 4.13, a will be a fuzzy restriction, and, by Proposition 4.6, there exists a unique $\lambda \in \mathcal{FI}^N$ such that $a = r^\lambda$. We have established the model that we aim to study. Firstly, we assign a name to these games with fuzzy-restricted cooperation.

Definition 5.1. A game with fuzzy restriction on N is a pair (v, λ) where $v \in \mathcal{G}^N$ and $\lambda \in \mathcal{FI}^N$.

Our goal is to define a value for games with fuzzy restriction on N , that is, a mapping $\psi : \mathcal{G}^N \times \mathcal{FI}^N \rightarrow \mathbb{R}^N$ that, to each $v \in \mathcal{G}^N$, $\lambda \in \mathcal{FI}^N$ and $i \in N$, assigns a real number $\psi_i(v, \lambda)$, which will be the profit allocated to i . We aim to find a value with nice properties, that is, a value that establishes a reasonable way of sharing the profits derived from each game with fuzzy restriction.

Firstly, we will assign to each game with fuzzy restriction an auxiliary game that combines the information from both the original game and the fuzzy restriction.

Definition 5.2. Let $v \in \mathcal{G}^N$ and $\lambda \in \mathcal{FI}^N$. We define $v^\lambda \in \mathcal{G}^N$ as

$$v^\lambda(E) = \int r^\lambda(E) dv \quad \text{for all } E \subseteq N,$$

where the integral symbol denotes the Choquet integral.

Now we are prepared to introduce the value that we will study.

Definition 5.3. The fuzzy-restricted Shapley value is the mapping $\varphi : \mathcal{G}^N \times \mathcal{FI}^N \rightarrow \mathbb{R}^N$ defined as

$$\varphi(v, \lambda) = Sh(v^\lambda)$$

for every $v \in \mathcal{G}^N$ and every $\lambda \in \mathcal{FI}^N$.

We aim to characterize the fuzzy-restricted Shapley value. It will be proved that this is the unique value that satisfies certain reasonable properties. Firstly, we state these properties.

- **Efficiency.** A value $\psi : \mathcal{G}^N \times \mathcal{FI}^N \rightarrow \mathbb{R}^N$ satisfies efficiency if for every $v \in \mathcal{G}^N$ and every $\lambda \in \mathcal{FI}^N$ such that $im(r^\lambda(N)) \subseteq \{0, 1\}$,

$$\sum_{i \in N} \psi_i(v, \lambda) = v(supp(r^\lambda(N))).$$

- **Additivity.** A value $\psi : \mathcal{G}^N \times \mathcal{FI}^N \rightarrow \mathbb{R}^N$ satisfies additivity if for every $v, w \in \mathcal{G}^N$ and every $\lambda \in \mathcal{FI}^N$,

$$\psi(v + w, \lambda) = \psi(v, \lambda) + \psi(w, \lambda).$$

- **Necessary player property.** A value $\psi : \mathcal{G}^N \times \mathcal{FI}^N \rightarrow \mathbb{R}^N$ satisfies the necessary player property if for every monotonic $v \in \mathcal{G}^N$, every $\lambda \in \mathcal{FI}^N$, every necessary player $i \in N$ in v and every $j \in N$,

$$\psi_i(v, \lambda) \geq \psi_j(v, \lambda).$$

Definition 5.4. Let $\lambda \in \mathcal{FI}^N$, $E \subseteq N$ and $i, j \in E$. We say that j depends partially on i within E according to λ if $r_j^\lambda(E) > r_j^\lambda(E \setminus \{i\})$.

Definition 5.5. Let $v \in \mathcal{G}^N$, $E \subseteq N$ and $j \in E$. We say that j is a null player for v within E if $v(F) = v(F \setminus \{j\})$ for every $F \subseteq E$.

Definition 5.6. Let $v \in \mathcal{G}^N$, $\lambda \in \mathcal{FI}^N$ and $i \in N$. We say that i is irrelevant in (v, λ) if for every $E \subseteq N$ and every $j \in E$ such that j depends partially on i within E according to λ , j is a null player for v within E .

- **Irrelevant player property.** A value $\psi : \mathcal{G}^N \times \mathcal{FI}^N \rightarrow \mathbb{R}^N$ satisfies the irrelevant player property if for every $v \in \mathcal{G}^N$, every $\lambda \in \mathcal{FI}^N$ and every irrelevant player $i \in N$ in (v, λ) ,

$$\psi_i(v, \lambda) = 0.$$

Definition 5.7. Let $\lambda \in \mathcal{FI}^N$ and $E \subseteq N$. We say that E is inessential in λ if $\lambda(E) = 0$.

- **Inessential coalition property.** A value $\psi : \mathcal{G}^N \times \mathcal{FI}^N \rightarrow \mathbb{R}^N$ satisfies the inessential coalition property if for every $v, w \in \mathcal{G}^N$, every $\lambda \in \mathcal{FI}^N$ and every inessential coalition $E \subseteq N$ in λ such that $v(F) = w(F)$ for all $F \in 2^N \setminus \{E\}$,

$$\psi(v, \lambda) = \psi(w, \lambda).$$

If $\lambda \in \mathcal{FI}^N$ and $t \in (0, 1)$ we define

$$\lambda_{[0,t]} = \min \left(1, \frac{\lambda}{t} \right), \quad \lambda_{[t,1]} = \max \left(0, \frac{\lambda - t}{1 - t} \right).$$

It can be easily verified that $\lambda_{[0,t]}, \lambda_{[t,1]} \in \mathcal{FI}^N$.

- **Reduction property.** A value $\psi : \mathcal{G}^N \times \mathcal{FI}^N \rightarrow \mathbb{R}^N$ satisfies the reduction property if for every $v \in \mathcal{G}^N$, every $\lambda \in \mathcal{FI}^N$ and every $t \in (0, 1)$,

$$\psi(v, \lambda) = t\psi(v, \lambda_{[0,t]}) + (1 - t)\psi(v, \lambda_{[t,1]}).$$

In the following theorem we characterize the fuzzy-restricted Shapley value.

Theorem 5.8. *A value for games with fuzzy restriction is equal to the fuzzy-restricted Shapley value if and only if it satisfies the properties of efficiency, additivity, necessary player, irrelevant player, inessential coalition and reduction.*

Proof. Firstly we will prove that the fuzzy-restricted Shapley value satisfies the properties mentioned in the theorem.

EFFICIENCY. Let $v \in \mathcal{G}^N$ and let $\lambda \in \mathcal{FI}^N$ be such that $im(r^\lambda(N)) \subseteq \{0, 1\}$. Taking into consideration the efficiency property of the Shapley value, we obtain

$$\sum_{i \in N} \varphi_i(v, \lambda) = \sum_{i \in N} Sh_i(v^\lambda) = v^\lambda(N) = \int r^\lambda(N) dv = v(supp(r^\lambda(N))).$$

ADDITIVITY. Let $v, w \in \mathcal{G}^N$ and let $\lambda \in \mathcal{FI}^N$. Notice that, for every $E \subseteq N$,

$$(v + w)^\lambda(E) = \int r^\lambda(E) d(v + w) = \int r^\lambda(E) dv + \int r^\lambda(E) dw = v^\lambda(E) + w^\lambda(E).$$

Therefore, $(v + w)^\lambda = v^\lambda + w^\lambda$. We have

$$\varphi(v + w, \lambda) = Sh((v + w)^\lambda) = Sh(v^\lambda + w^\lambda),$$

which, by the additivity property of the Shapley value, is equal to

$$Sh(v^\lambda) + Sh(w^\lambda) = \varphi(v, \lambda) + \varphi(w, \lambda).$$

NECESSARY PLAYER PROPERTY. Suppose that $v \in \mathcal{G}^N$ is monotonic and $i \in N$ is a necessary player in v . Let $\lambda \in \mathcal{FI}^N$ and let $j \in N$. We must prove that $\varphi_i(v, \lambda) \geq \varphi_j(v, \lambda)$. Taking into consideration that $\varphi(v, \lambda) = Sh(v^\lambda)$ and the necessary player property of the Shapley value, it suffices to prove that v^λ is a monotonic game and that i is necessary in v^λ . The monotonicity of v^λ follows from the monotonicity of r^λ , the monotonicity of v and property (C3) of the Choquet integral. The fact that i is necessary in v^λ can be easily verified.

IRRELEVANT PLAYER PROPERTY. Let $v \in \mathcal{G}^N$, $\lambda \in \mathcal{FI}^N$ and $i \in N$ be such that i is an irrelevant player in (v, λ) . We must prove that $\varphi_i(v, \lambda) = 0$. Taking into consideration that $\varphi(v, \lambda) = Sh(v^\lambda)$ and the null player property of the Shapley value, it suffices to prove that i is a null player in v^λ . Let $E \subseteq N$. Let $\{t_l\}_{l=0}^q = im(r^\lambda(E)) \cup im(r^\lambda(E \setminus \{i\})) \cup \{0, 1\}$ with $0 = t_0 < t_1 < \dots < t_q = 1$. By property (C7) of the Choquet integral, we have

$$v^\lambda(E) = \int r^\lambda(E) dv = \sum_{l=1}^q (t_l - t_{l-1}) v([r^\lambda(E)]_{t_l})$$

and

$$v^\lambda(E \setminus \{i\}) = \int r^\lambda(E \setminus \{i\}) dv = \sum_{l=1}^q (t_l - t_{l-1}) v([r^\lambda(E \setminus \{i\})]_{t_l}).$$

Let $l \in \{1, \dots, q\}$. Firstly, notice that $[r^\lambda(E \setminus \{i\})]_{t_l} \subseteq [r^\lambda(E)]_{t_l}$. Notice also that if $j \in [r^\lambda(E)]_{t_l} \setminus [r^\lambda(E \setminus \{i\})]_{t_l}$ then $r_j^\lambda(E) > r_j^\lambda(E \setminus \{i\})$. This means that j depends partially on i within E according to λ . Since i is irrelevant in (v, λ) , it follows that j is a null player for v within E . This implies that $v([r^\lambda(E)]_{t_l}) = v([r^\lambda(E)]_{t_l} \setminus \{j\})$. Following the same reasoning, if $k \in ([r^\lambda(E)]_{t_l} \setminus \{j\}) \setminus [r^\lambda(E \setminus \{i\})]_{t_l}$, then $v([r^\lambda(E)]_{t_l} \setminus \{j\}) = v([r^\lambda(E)]_{t_l} \setminus \{j, k\})$. Repeatedly applying this reasoning with all players in $[r^\lambda(E)]_{t_l} \setminus [r^\lambda(E \setminus \{i\})]_{t_l}$ we obtain that $v([r^\lambda(E)]_{t_l}) = v([r^\lambda(E \setminus \{i\})]_{t_l})$. We conclude that $v^\lambda(E) = v^\lambda(E \setminus \{i\})$. As this is true for any $E \subseteq N$, we obtain that i is a null player in v^λ .

INESSENTIAL COALITION PROPERTY. Let $\lambda \in \mathcal{FI}^N$ and $E \subseteq N$ be such that $\lambda(E) = 0$. Let $v, w \in \mathcal{G}^N$ be such that $v(F) = w(F)$ for all $F \in 2^N \setminus \{E\}$. We must prove that $\varphi(v, \lambda) = \varphi(w, \lambda)$. Taking into account the definition of the fuzzy-restricted Shapley value, it is enough to show that $v^\lambda = w^\lambda$. To this end, it is clear that it suffices to prove

that $v([r^\lambda(D)]_t) = w([r^\lambda(D)]_t)$ for every $D \subseteq N$ and every $t \in (0, 1]$. Let $D \subseteq N$ and let $t \in (0, 1]$. By Proposition 4.9 and Lemma 4.2,

$$\lambda_t([r^\lambda(D)]_t) = \lambda_t(R^{\lambda_t}(D)) = 1.$$

Therefore, $\lambda([r^\lambda(D)]_t) \geq t > 0$. Hence, $[r^\lambda(D)]_t \neq E$. Thus, $v([r^\lambda(D)]_t) = w([r^\lambda(D)]_t)$.

REDUCTION. Let $v \in \mathcal{G}^N$, $\lambda \in \mathcal{FI}^N$ and $t \in (0, 1)$. Firstly we will prove that

$$v^\lambda = t v^{\lambda_{[0,t]}} + (1-t) v^{\lambda_{[t,1]}}. \quad (1)$$

Let $E \subseteq N$. We have

$$t v^{\lambda_{[0,t]}}(E) + (1-t) v^{\lambda_{[t,1]}}(E) = t \int r^{\lambda_{[0,t]}}(E) dv + (1-t) \int r^{\lambda_{[t,1]}}(E) dv,$$

which, by property (C2) of the Choquet integral, is equal to

$$\int t r^{\lambda_{[0,t]}}(E) dv + \int (1-t) r^{\lambda_{[t,1]}}(E) dv. \quad (2)$$

We want to show that $r^{\lambda_{[0,t]}}(E)$ and $r^{\lambda_{[t,1]}}(E)$ are comonotone. To this end, it suffices to prove that if $i, j \in E$ and $r_i^{\lambda_{[0,t]}}(E) > r_j^{\lambda_{[0,t]}}(E)$, then $r_i^{\lambda_{[t,1]}}(E) \geq r_j^{\lambda_{[t,1]}}(E)$. Let $i, j \in E$ be such that $r_i^{\lambda_{[0,t]}}(E) > r_j^{\lambda_{[0,t]}}(E)$. This implies that $r_j^{\lambda_{[0,t]}}(E) < 1$. This means that for every $F \subseteq E$ with $j \in F$, $\lambda_{[0,t]}(F) < 1$. Notice that $\lambda_{[0,t]}(F) < 1$ is equivalent to $\lambda(F) < t$, which implies that $\lambda_{[t,1]}(F) = 0$. Therefore, we have that for every $F \subseteq E$ with $j \in F$, $\lambda_{[t,1]}(F) = 0$. We conclude that $r_j^{\lambda_{[t,1]}}(E) = 0$, and hence $r_i^{\lambda_{[t,1]}}(E) \geq r_j^{\lambda_{[t,1]}}(E)$. Since $r^{\lambda_{[0,t]}}(E)$ and $r^{\lambda_{[t,1]}}(E)$ are comonotone, it is trivial that $t r^{\lambda_{[0,t]}}(E)$ and $(1-t) r^{\lambda_{[t,1]}}(E)$ are comonotone. By property (C6) of the Choquet integral, (2) is equal to

$$\int \left(t r^{\lambda_{[0,t]}}(E) + (1-t) r^{\lambda_{[t,1]}}(E) \right) dv. \quad (3)$$

Let us see that

$$t r^{\lambda_{[0,t]}}(E) + (1-t) r^{\lambda_{[t,1]}}(E) = r^\lambda(E). \quad (4)$$

If $r_i^\lambda(E) \leq t$ then it can be easily verified that $r_i^{\lambda_{[0,t]}}(E) = \frac{r_i^\lambda(E)}{t}$ and $r_i^{\lambda_{[t,1]}}(E) = 0$. Therefore, $t r_i^{\lambda_{[0,t]}}(E) + (1-t) r_i^{\lambda_{[t,1]}}(E) = r_i^\lambda(E)$. If $r_i^\lambda(E) > t$ then $r_i^{\lambda_{[0,t]}}(E) = 1$ and $r_i^{\lambda_{[t,1]}}(E) = \frac{r_i^\lambda(E) - t}{1-t}$. Therefore, $t r_i^{\lambda_{[0,t]}}(E) + (1-t) r_i^{\lambda_{[t,1]}}(E) = r_i^\lambda(E)$. By (4), (3) is equal to

$$\int r^\lambda(E) dv = v^\lambda(E).$$

We have proved (1). Now we can write

$$\varphi(v, \lambda) = Sh(v^\lambda) = Sh\left(t v^{\lambda_{[0,t]}} + (1-t) v^{\lambda_{[t,1]}}\right),$$

which, by the linearity of the Shapley value, is equal to

$$t Sh(v^{\lambda_{[0,t]}}) + (1-t) Sh(v^{\lambda_{[t,1]}}) = t \varphi(v, \lambda_{[0,t]}) + (1-t) \varphi(v, \lambda_{[t,1]}).$$

We have seen that the fuzzy-restricted Shapley value satisfies the properties mentioned in the theorem. It remains to prove the uniqueness. Suppose that $\psi : \mathcal{G}^N \times \mathcal{FI}^N \rightarrow \mathbb{R}^N$ satisfies the properties of efficiency, additivity, necessary player, irrelevant player, inessential coalition and reduction. We must show that $\psi = \varphi$.

For every $\lambda \in \mathcal{FI}^N$ we define $m(\lambda) = |\{\lambda(E) : E \subseteq N\} \setminus \{0, 1\}|$. We are going to prove that $\psi(v, \lambda) = \varphi(v, \lambda)$ for every $v \in \mathcal{G}^N$ and every $\lambda \in \mathcal{FI}^N$ by strong induction on $m(\lambda)$.

BASE CASE. $m(\lambda) = 0$. Let $\lambda \in \mathcal{FI}^N$ be such that $m(\lambda) = 0$, that is, $\lambda(E) \in \{0, 1\}$ for every $E \subseteq N$. We must prove that $\psi(v, \lambda) = \varphi(v, \lambda)$ for every $v \in \mathcal{G}^N$. If $\lambda(E) = 0$ for every $E \in 2^N \setminus \{\emptyset\}$ then, by the irrelevant player property, it is clear that $\psi(v, \lambda) = 0 = \varphi(v, \lambda)$ for every $v \in \mathcal{G}^N$.

Let $\lambda \in \mathcal{FI}^N$ be such that $\{E \in 2^N \setminus \{\emptyset\} : \lambda(E) = 1\} \neq \emptyset$. Firstly we will study the case of the games with the form cu_E with $\lambda(E) = 1$ and $c \in \mathbb{R}$.

Let $c > 0$ and let $E \in 2^N \setminus \{\emptyset\}$ be such that $\lambda(E) = 1$. Notice that cu_E is monotonic and any player in E is necessary in cu_E . By the necessary player property, it is clear that there exists $b \in \mathbb{R}$ such that

$$\psi_i(cu_E, \lambda) = b \quad \text{for every } i \in E. \quad (5)$$

Taking into account that $r^\lambda(E) = \mathbf{1}_E$, it can be easily verified that any player in $N \setminus E$ is irrelevant in (cu_E, λ) . By the irrelevant player property,

$$\psi_i(cu_E, \lambda) = 0 \quad \text{for every } i \in N \setminus E. \quad (6)$$

From (5), (6) and the efficiency property it follows that

$$\psi_i(cu_E, \lambda) = \begin{cases} \frac{c}{|E|} & \text{if } i \in E, \\ 0 & \text{if } i \in N \setminus E. \end{cases}$$

Hence, $\psi(cu_E, \lambda) = \varphi(cu_E, \lambda)$. Notice that, by the irrelevant player property, $\psi(0, \lambda) = 0$. Now, by additivity,

$$\psi(-cu_E, \lambda) = \psi(0, \lambda) - \psi(cu_E, \lambda) = -\psi(cu_E, \lambda) = -\varphi(cu_E, \lambda) = \varphi(-cu_E, \lambda).$$

Therefore, we have proved that

$$\psi(ku_E, \lambda) = \varphi(ku_E, \lambda) \quad (7)$$

for every $k \in \mathbb{R}$ and every $E \in 2^N \setminus \{\emptyset\}$ with $\lambda(E) = 1$.

Let $v \in \mathcal{G}^N$. Notice that, for every $E \subseteq N$, if $v(E) \neq v^\lambda(E)$ then $\lambda(E) = 0$. By repeatedly applying the inessential coalition property, it can be easily proved that

$$\psi(v, \lambda) = \psi(v^\lambda, \lambda) \quad \text{and} \quad \varphi(v, \lambda) = \varphi(v^\lambda, \lambda). \quad (8)$$

Let us show that

$$\Delta_{v^\lambda}(E) = 0 \quad \text{for every } E \in 2^N \setminus \{\emptyset\} \text{ with } \lambda(E) = 0. \quad (9)$$

Let us prove it by strong induction on $|E|$.

BASE CASE. $|E| = 1$. Suppose that $i \in N$ and $\lambda(\{i\}) = 0$. We have

$$\Delta_{v^\lambda}(\{i\}) = v^\lambda(\{i\}) = \int r^\lambda(\{i\}) dv = \int 0 dv = 0.$$

INDUCTIVE STEP. Suppose that $E \subseteq N$, $|E| > 1$ and $\lambda(E) = 0$. Let $D = \text{supp}(r^\lambda(E))$. Notice that $\lambda(D) = \lambda(\text{supp}(r^\lambda(E))) = \lambda_1(\text{supp}(r^\lambda(E))) = \lambda_1([r^\lambda(E)]_1) = \lambda_1(R^{\lambda_1}(E))$, which, by Lemma 4.2, is equal to 1. It easily follows that $D \subsetneq E$ and $v^\lambda(D) = v(D)$. Notice also that if $F \subseteq E$ and $F \not\subseteq D$ then $\lambda(F) = 0$ (otherwise, by definition of r^λ , we would have $\mathbf{1}_F \subseteq r^\lambda(E)$, and thus $F \subseteq \text{supp}(r^\lambda(E))$). Therefore, by induction hypothesis, we conclude that $\Delta_{v^\lambda}(F) = 0$ for every $F \subsetneq E$ with $F \not\subseteq D$. We have

$$\begin{aligned} \Delta_{v^\lambda}(E) &= v^\lambda(E) - \sum_{\{F \in 2^N \setminus \{\emptyset\} : F \subsetneq E\}} \Delta_{v^\lambda}(F) \\ &= v(D) - \sum_{\{F \in 2^N \setminus \{\emptyset\} : F \subseteq D\}} \Delta_{v^\lambda}(F) \\ &= v^\lambda(D) - \sum_{\{F \in 2^N \setminus \{\emptyset\} : F \subseteq D\}} \Delta_{v^\lambda}(F) \\ &= v^\lambda(D) - \sum_{\{F \in 2^N \setminus \{\emptyset\} : F \subsetneq D\}} \Delta_{v^\lambda}(F) - \Delta_{v^\lambda}(D) \\ &= \Delta_{v^\lambda}(D) - \Delta_{v^\lambda}(D) = 0. \end{aligned}$$

Therefore, we have proved (9). Taking into account the definition of the dividends, we conclude that

$$v^\lambda = \sum_{\{E \in 2^N \setminus \{\emptyset\} : \lambda(E)=1\}} \Delta_{v^\lambda}(E) u_E. \quad (10)$$

By (8) and (10),

$$\psi(v, \lambda) = \psi(v^\lambda, \lambda) = \psi \left(\sum_{\{E \in 2^N \setminus \{\emptyset\} : \lambda(E)=1\}} \Delta_{v^\lambda}(E) u_E, \lambda \right),$$

which, by additivity, is equal to

$$\sum_{\{E \in 2^N \setminus \{\emptyset\} : \lambda(E)=1\}} \psi(\Delta_{v^\lambda}(E) u_E, \lambda),$$

which, by (7), is equal to

$$\begin{aligned} & \sum_{\{E \in 2^N \setminus \{\emptyset\} : \lambda(E)=1\}} \varphi(\Delta_{v^\lambda}(E) u_E, \lambda) \\ = & \varphi \left(\sum_{\{E \in 2^N \setminus \{\emptyset\} : \lambda(E)=1\}} \Delta_{v^\lambda}(E) u_E, \lambda \right) \\ = & \varphi(v^\lambda, \lambda) = \varphi(v, \lambda). \end{aligned}$$

INDUCTIVE STEP. Let $\lambda \in \mathcal{FI}^N$ be such that $m(\lambda) > 0$. Let $t \in \{\lambda(E) : E \subseteq N\} \setminus \{0, 1\}$. It is clear that $m(\lambda_{[0,t]}) < m(\lambda)$ and $m(\lambda_{[t,1]}) < m(\lambda)$. Let $v \in \mathcal{G}^N$. By induction hypothesis, $\psi(v, \lambda_{[0,t]}) = \varphi(v, \lambda_{[0,t]})$ and $\psi(v, \lambda_{[t,1]}) = \varphi(v, \lambda_{[t,1]})$. From this and the reduction property, we have

$$\begin{aligned} \psi(v, \lambda) &= t \psi(v, \lambda_{[0,t]}) + (1-t) \psi(v, \lambda_{[t,1]}) \\ &= t \varphi(v, \lambda_{[0,t]}) + (1-t) \varphi(v, \lambda_{[t,1]}) = \varphi(v, \lambda). \end{aligned}$$

□

6. Conclusions

We have introduced a fuzzy version of restrictions. Since a direct crisp-fuzzy translation of the definition of restriction was not possible, we have used the identification between restrictions and functions in \mathcal{I}^N . Dependence relations induced by restrictions and fuzzy restrictions have been described. It has been proved that, both restrictions and fuzzy restrictions are characterized by the transitivity of such dependence relations. This result supports the suitability of the fuzzy extension of restrictions obtained. As an application of the concepts introduced, games with fuzzy restriction have been presented. A value for these games has been defined and characterized.

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