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Some Third Order Methods for Solving Systems of Nonlinear Equations

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Abstract—Based on Traub's methods for solving nonlinear equation $f(x) = 0$, we develop two families of third-order methods for solving system of nonlinear equations $\mathbf{F}(\mathbf{x}) = 0$. The families include well-known existing methods as special cases. The stability is corroborated by numerical results. Comparison with well-known methods shows that the present methods are robust. These higher order methods may be very useful in the numerical applications requiring high precision in their computations because these methods yield a clear reduction in number of iterations.

Keywords—Nonlinear equations and systems, Newton's method, fixed point iteration, order of convergence.

I. Introduction

Solving systems of nonlinear equations is a common and important problem in science and engineering [1], i.e. for a given nonlinear function $\mathbf{F} : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$, to find a vector $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ such that $\mathbf{F}(\mathbf{r}) = 0$. This solution can be obtained as a fixed point of some function $\boldsymbol{\phi} : \mathbb{R}^n \to \mathbb{R}^n$ by means of fixed point iteration

$$\mathbf{x}^{(k+1)} = \boldsymbol{\phi}(\mathbf{x}^{(k)}), \quad k = 0, 1, \ldots$$

One of the basic procedures for approximating a solution of nonlinear equation $f(x) = 0$, is the quadratically convergent Newton's method defined as

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \qquad k = 0, 1, \ldots \tag{1}$$

For systems of nonlinear equations, Newton's method is given as (see [2, 3])

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \mathbf{J}_{\mathbf{F}}(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}), \tag{2}$$

where $\mathbf{J}_{\mathbf{F}}(\mathbf{x})$ is the jacobian matrix of the function $\mathbf{F}(\mathbf{x})$ and $\mathbf{J}_{\mathbf{F}}(\mathbf{x})^{-1}$ is its inverse. For a system of n equations in n unknowns, the first Fréchet derivative is a matrix with n^2 evaluations while the second Fréchet derivative has n^3 evaluations. The methods like Halley and Chebyshev [4, 5], despite their cubic convergence, are considered less practical from a computational point of view because of costly second derivative.

Recently, many third order iterative methods have been derived and analyzed for systems of nonlinear equations that do not require the computation of second Fréchet derivative. For example, Homeier [6] developed a modified Newton method of order three. Frontini et al. [7], Cordero et al. [8] and Noor et al. [9] developed third order methods from quadrature formulae. Darvishi et al.[10] presented the methods of same order using Adomian decomposition technique. Hueso et al. [11] and Lin et al. [12] also introduced iterative methods of same order for systems of nonlinear equations. Some authors have considered Traub's third order methods [2] for univariate case and generalized to multivariate case. For example, Hernández [13] extended a member of the family given by Traub [2]

$$\begin{aligned} x_{k+1} &= x_k - \left\{ \frac{(2\beta-1)f'(x_k) + f'(x_k + \beta u(x_k))}{2\beta f'(x_k)} \right\} u(x_k), \\ \beta &\neq 0, \quad u(x_k) = \frac{f(x_k)}{f'(x_k)}, \end{aligned} \tag{3}$$

in Banach Space for $\beta = -1/2$. Recently, Babajee et al. [14] extended another member ($\beta = -1$) of this family to systems of equations. Based on the ongoing work in this direction, here, we extend the family (3) to systems of equations for any $\beta \in \mathbb{R}$. We consider another one-parameter family of same order given by Traub [2]

$$x_{k+1} = x_k - \frac{2\beta f(x_k)}{(2\beta+1)f'(x_k) - f'(x_k + \beta u(x_k))}, \tag{4}$$

and generalize this to systems of equations for any $\beta \in \mathbb{R}$. The resulting family includes the methods by Homeier [6] and Noor et al. [9] as particular cases.

The paper is organized in 6 sections. Some basic results relevant to the present work are presented in Section 2. In Section 3, the schemes are developed and behavior is analyzed. In Section 4, particular cases of the families are presented. In Section 5, new methods are compared with closest competitors in a series of numerical examples. Section 6, contains the concluding remarks.

II. Basic results and notations

We consider the following results:

(A) Let $\boldsymbol{\phi}(\mathbf{x})$ be a fixed point function with continuous partial derivatives of order p with respect to all components of $\mathbf{x}$. The iterative method $\mathbf{x}^{(k+1)} = \boldsymbol{\phi}(\mathbf{x}^{(k)})$ is of order p (see [2]) if

$$\boldsymbol{\phi}(\mathbf{r}) = \mathbf{r};$$

$$\begin{aligned} &\frac{\partial^k \phi_j(\mathbf{r})}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_k}} = 0, \text{for all } \ 1 \leq k \leq p-1, \\ &1 \leq j, i_1, i_2, \ldots, i_k \leq n; \end{aligned}$$

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$$\frac{\partial^p \phi_j(\mathbf{r})}{\partial x_{i_1}\partial x_{i_2}\dots\partial x_{i_p}} \neq 0 \text{ for at least one value of } j, i_1, i_2, \dots, i_p, \tag{5}$$

Expanding $\phi_j(\mathbf{x})$ in a Taylor series about $\mathbf{r}$ yields

$$\phi_j(\mathbf{x}) = \phi_j(\mathbf{r}) + \sum_{i_1=1}^{n}\frac{\partial\phi_j(\mathbf{r})}{\partial x_{i_1}}e_{i_1} + \frac{1}{2!}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\frac{\partial^2\phi_j(\mathbf{r})}{\partial x_{i_1}\partial x_{i_2}}e_{i_1}e_{i_2} + \frac{1}{3!}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{i_3=1}^{n}\frac{\partial^3\phi_j(\mathbf{r})}{\partial x_{i_1}\partial x_{i_2}\partial x_{i_3}}e_{i_1}e_{i_2}e_{i_3} + \dots, \tag{6}$$

where

$$e_{i_l} = x_{i_l} - r_{i_l},$$

Thus if $\boldsymbol{\phi}(\mathbf{x})$ is an iteration function of order p, then by equation (5), we have

$$\phi_j(\mathbf{x}) - r_j \sim \frac{1}{p!}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\dots\sum_{i_p=1}^{n}\frac{\partial^p\phi_j(\mathbf{r})}{\partial x_{i_1}\partial x_{i_2}\dots\partial x_{i_p}}e_{i_1}e_{i_2}\dots e_{i_p}. \tag{7}$$

and the quantity

$$\frac{1}{p!}\frac{\partial^p\phi_j(\mathbf{r})}{\partial x_{i_1}\partial x_{i_2}\dots\partial x_{i_p}}.$$

is called the asymptotic error constant.

(B) Let $\mathbf{J}_{ij}(\mathbf{x})$ denotes the (i, j) entry of the matrix $\mathbf{J_F}(\mathbf{x})$ and the elements of $\mathbf{J_F}(\mathbf{x})^{-1}$ are denoted by $H_{ij}(\mathbf{x})$. Then,

$$\sum_{j=1}^{n} H_{ij}(\mathbf{x})J_{jk}(\mathbf{x}) = \delta_{ik}, \tag{8}$$

where δ_{ik} is Kronecker's delta, defined as:

$$\delta_{ik} = \begin{cases} 1 & \text{if } i = k; \\ 0 & \text{if } i \neq k. \end{cases}$$

Let i and q be arbitrary and fixed. Then from (8), it follows that

$$\sum_{j=1}^{n} H_{ij}(\mathbf{x})\frac{\partial J_{jk}(\mathbf{x})}{\partial x_q} + \sum_{j=1}^{n}\frac{\partial H_{ij}(\mathbf{x})}{\partial x_q}J_{jk}(\mathbf{x}) = 0. \tag{9}$$

Thus, we have

$$\sum_{j=1}^{n}\frac{\partial H_{ij}(\mathbf{x})}{\partial x_q}J_{jk}(\mathbf{x}) = -\sum_{j=1}^{n}H_{ij}(\mathbf{x})\frac{\partial J_{jk}(\mathbf{x})}{\partial x_q} = -\sum_{j=1}^{n}H_{ij}(\mathbf{x})\frac{\partial^2 f_j(\mathbf{x})}{\partial x_k\partial x_q}. \tag{10}$$

Differentiating (9) partially with respect to x_r, r being arbitrary and fixed, one can easily obtain

$$\sum_{j=1}^{n}\frac{\partial^2 H_{ij}(\mathbf{x})}{\partial x_r\partial x_q}J_{jk}(\mathbf{x}) = -\left[\sum_{j=1}^{n}\frac{\partial H_{ij}(\mathbf{x})}{\partial x_r}\frac{\partial J_{jk}(\mathbf{x})}{\partial x_q} + \sum_{j=1}^{n}\frac{\partial H_{ij}(\mathbf{x})}{\partial x_q}\frac{\partial J_{jk}(\mathbf{x})}{\partial x_r} + \sum_{j=1}^{n}H_{ij}(\mathbf{x})\frac{\partial^2 J_{jk}(\mathbf{x})}{\partial x_r\partial x_q}\right]. \tag{11}$$

(C) We use the following notations (first two are introduced by Traub [2]):

$$Z_{iqr}(\mathbf{x}) = \sum_{j=1}^{n}H_{ij}(\mathbf{x})\frac{\partial^2 f_j(\mathbf{x})}{\partial x_q\partial x_r}, \tag{12}$$

$$Z_{iqrs}(\mathbf{x}) = \sum_{k=1}^{n}Z_{iqk}(\mathbf{x})Z_{krs}(\mathbf{x}) = \sum_{k=1}^{n}\left[\sum_{l=1}^{n}\left(H_{il}(\mathbf{x})\frac{\partial^2 f_l(\mathbf{x})}{\partial x_q\partial x_k}\right)\sum_{m=1}^{n}\left(H_{km}(\mathbf{x})\frac{\partial^2 f_m(\mathbf{x})}{\partial x_r\partial x_s}\right)\right], \tag{13}$$

and

$$\widetilde{Z}_{iqrs}(\mathbf{x}) = \sum_{j=1}^{n}H_{ij}(\mathbf{x})\frac{\partial^3 f_j(\mathbf{x})}{\partial x_q\partial x_r\partial x_s}, \tag{14}$$

where the last two subscripts of $Z_{iqr}(\mathbf{x})$ and the last three of $\widetilde{Z}_{iqrs}(\mathbf{x})$ may be permuted.
The $Z_{iqrs}(\mathbf{x})$ can also be simplified to (using (8) and (10))

$$Z_{iqrs}(\mathbf{x}) = -\sum_{m=1}^{n}\frac{\partial H_{im}(\mathbf{x})}{\partial x_q}\frac{\partial^2 f_m(\mathbf{x})}{\partial x_r\partial x_s}. \tag{15}$$

III. Development of methods

For system of equations, the families (3) and (4) can be written as

$$\mathbf{y}^{(k)} = \mathbf{x}^{(k)} + \beta\mathbf{J_F}(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}),$$

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \frac{1}{2\beta}\mathbf{J_F}(\mathbf{x}^{(k)})^{-1}[(2\beta-1)\mathbf{J_F}(\mathbf{x}^{(k)}) + \mathbf{J_F}(\mathbf{y}^{(k)})]\mathbf{J_F}(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}) \tag{16}$$

and

$$\mathbf{y}^{(k)} = \mathbf{x}^{(k)} + \beta\mathbf{J_F}(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}),$$

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - 2\beta[(2\beta+1)\mathbf{J_F}(\mathbf{x}^{(k)}) - \mathbf{J_F}(\mathbf{y}^{(k)})]^{-1}\mathbf{F}(\mathbf{x}^{(k)}). \tag{17}$$

Now, we shall prove convergence of (16) and (17). The following lemmas, will be useful in the proof of main theorems:

Lemma 1. *Let*

$$\boldsymbol{\lambda}(\mathbf{x}) = \mathbf{x} + \beta\mathbf{J_F}(\mathbf{x})^{-1}\mathbf{F}(\mathbf{x}), \tag{18}$$

and

$$\lambda_i(\mathbf{x}) = x_i + \beta\sum_{j=1}^{n}H_{ij}(\mathbf{x})f_j(\mathbf{x}), \tag{19}$$

be the coordinate functions of $\boldsymbol{\lambda}(\mathbf{x})$ for $i = 1, 2, \dots, n$. Then,

$$\lambda_i(\mathbf{r}) = r_i, \tag{20}$$

$$\frac{\partial\lambda_i(\mathbf{r})}{\partial x_q} = (1+\beta)\delta_{iq}, \tag{21}$$

$$\frac{\partial^2\lambda_i(\mathbf{r})}{\partial x_r\partial x_q} = -\beta Z_{iqr}(\mathbf{r}). \tag{22}$$

Proof. Since $f_j(\mathbf{r}) = 0$, equation (19) implies that $\boldsymbol{\lambda}(\mathbf{x})$ is fixed point function. i.e.

$$\lambda_i(\mathbf{r}) = r_i.$$

Let i and q be arbitrary and fixed. Differentiating (19) partially with respect to x_q, one obtains

$$\begin{aligned}\frac{\partial \lambda_i(\mathbf{x})}{\partial x_q} &= \frac{\partial x_i}{\partial x_q} + \beta \sum_{j=1}^{n} H_{ij}(\mathbf{x}) \frac{\partial f_j(\mathbf{x})}{\partial x_q} \\ &+ \beta \sum_{j=1}^{n} \frac{\partial H_{ij}(\mathbf{x})}{\partial x_q} f_j(\mathbf{x}). \\ &= \delta_{iq} + \beta \sum_{j=1}^{n} H_{ij}(\mathbf{x}) J_{jq}(\mathbf{x}) \\ &+ \beta \sum_{j=1}^{n} \frac{\partial H_{ij}(\mathbf{x})}{\partial x_q} f_j(\mathbf{x}). \qquad (23)\end{aligned}$$

Setting $\mathbf{x} = \mathbf{r}$ and taking into account that $f_j(\mathbf{r}) = 0$. Then using (8), we have

$$\frac{\partial \lambda_i(\mathbf{r})}{\partial x_q} = (1+\beta)\delta_{iq}.$$

Now, we analyze the second derivative of $\lambda_i(\mathbf{x})$. Differentiating (23) partially with respect to x_r, r being arbitrary and fixed, it follows that

$$\begin{aligned}\frac{\partial^2 \lambda_i(\mathbf{x})}{\partial x_r \partial x_q} &= 0 + \beta \sum_{j=1}^{n} H_{ij}(\mathbf{x}) \frac{\partial^2 f_j(\mathbf{x})}{\partial x_r \partial x_q} \\ &+ \beta \sum_{j=1}^{n} \frac{\partial H_{ij}(\mathbf{x})}{\partial x_r} \frac{\partial f_j(\mathbf{x})}{\partial x_q} + \beta \sum_{j=1}^{n} \frac{\partial H_{ij}(\mathbf{x})}{\partial x_q} \frac{\partial f_j(\mathbf{x})}{\partial x_r} \\ &+ \beta \sum_{j=1}^{n} \frac{\partial^2 H_{ij}(\mathbf{x})}{\partial x_r \partial x_q} f_j(\mathbf{x}) \qquad (24)\end{aligned}$$

Evaluating (24) in $\mathbf{x} = \mathbf{r}$ and using (10), we have

$$\frac{\partial^2 \lambda_i(\mathbf{r})}{\partial x_r \partial x_q} = -\beta \sum_{j=1}^{n} H_{ij}(\mathbf{r}) \frac{\partial^2 f_j(\mathbf{r})}{\partial x_r \partial x_q} = -\beta Z_{iqr}(\mathbf{r}).$$

This completes the proof of lemma 1.

Lemma 2. *Let $N_j(\mathbf{x})$ be the jth entry of the vector $\mathbf{N}(\mathbf{x}) = \mathbf{J_F}(\mathbf{x})^{-1}\mathbf{F}(\mathbf{x})$, that is*

$$N_j(\mathbf{x}) = \sum_{k=1}^{n} H_{jk}(\mathbf{x}) f_k(\mathbf{x}). \qquad (25)$$

Then,

$$N_j(\mathbf{r}) = 0, \qquad (26)$$

$$\frac{\partial N_j(\mathbf{r})}{\partial x_q} = \delta_{jq}, \qquad (27)$$

$$\frac{\partial^2 N_j(\mathbf{r})}{\partial x_r \partial x_q} = -Z_{jqr}(\mathbf{r}), \qquad (28)$$

$$\begin{aligned}\frac{\partial^3 N_j(\mathbf{r})}{\partial x_s \partial x_r \partial x_q} &= Z_{jqsr}(\mathbf{r}) + Z_{jrsq}(\mathbf{r}) + Z_{jsrq}(\mathbf{r}) \\ &- 2\widetilde{Z}_{jqrs}(\mathbf{r}). \qquad (29)\end{aligned}$$

Proof. Since $f_j(\mathbf{r}) = 0$, it is clear that

$$N_j(\mathbf{r}) = 0.$$

Let j and q be arbitrary and fixed. Differentiating (25) partially with respect to x_q, one obtains

$$\frac{\partial N_j(\mathbf{x})}{\partial x_q} = \sum_{k=1}^{n} H_{jk}(\mathbf{x}) \frac{\partial f_k(\mathbf{x})}{\partial x_q} + \sum_{k=1}^{n} \frac{\partial H_{jk}(\mathbf{x})}{\partial x_q} f_k(\mathbf{x}). \qquad (30)$$

At $x = \mathbf{r}$, above equation yields

$$\frac{\partial N_j(\mathbf{r})}{\partial x_q} = \sum_{k=1}^{n} H_{jk}(\mathbf{r}) J_{kq}(\mathbf{r}) = \delta_{jq}.$$

Differentiating (30) partially with respect to x_r, we have

$$\begin{aligned}\frac{\partial^2 N_j(\mathbf{x})}{\partial x_r \partial x_q} &= \sum_{k=1}^{n} H_{jk}(\mathbf{x}) \frac{\partial^2 f_k(\mathbf{x})}{\partial x_r \partial x_q} + \sum_{k=1}^{n} \frac{\partial H_{jk}(\mathbf{x})}{\partial x_r} \frac{\partial f_k(\mathbf{x})}{\partial x_q} \\ &+ \sum_{k=1}^{n} \frac{\partial H_{jk}(\mathbf{x})}{\partial x_q} \frac{\partial f_k(\mathbf{x})}{\partial x_r} + \sum_{k=1}^{n} \frac{\partial^2 H_{jk}(\mathbf{x})}{\partial x_r \partial x_q} f_k(\mathbf{x}). \qquad (31)\end{aligned}$$

Setting $x = \mathbf{r}$ and using (10), we obtain

$$\frac{\partial^2 N_j(\mathbf{r})}{\partial x_r \partial x_q} = -\sum_{k=1}^{n} H_{jk}(\mathbf{r}) \frac{\partial^2 f_k(\mathbf{r})}{\partial x_r \partial x_q} = -Z_{jqr}(\mathbf{r}).$$

Now, the third derivative of $N_j(\mathbf{x})$ with respect to x_s, s being arbitrary and fixed, is given by

$$\begin{aligned}\frac{\partial^3 N_j(\mathbf{x})}{\partial x_s \partial x_r \partial x_q} &= \sum_{k=1}^{n} H_{jk}(\mathbf{x}) \frac{\partial^3 f_k(\mathbf{x})}{\partial x_s \partial x_r \partial x_q} + \sum_{k=1}^{n} \frac{\partial H_{jk}(\mathbf{x})}{\partial x_s} \frac{\partial^2 f_k(\mathbf{x})}{\partial x_r \partial x_q} \\ &+ \sum_{k=1}^{n} \frac{\partial H_{jk}(\mathbf{x})}{\partial x_r} \frac{\partial^2 f_k(\mathbf{x})}{\partial x_s \partial x_q} + \sum_{k=1}^{n} \frac{\partial^2 H_{jk}(\mathbf{x})}{\partial x_s \partial x_r} \frac{\partial f_k(\mathbf{x})}{\partial x_q} \\ &+ \sum_{k=1}^{n} \frac{\partial^2 H_{jk}(\mathbf{x})}{\partial x_s \partial x_q} \frac{\partial f_k(\mathbf{x})}{\partial x_r} + \sum_{k=1}^{n} \frac{\partial H_{jk}(\mathbf{x})}{\partial x_q} \frac{\partial^2 f_k(\mathbf{x})}{\partial x_s \partial x_r} \\ &+ \sum_{k=1}^{n} \frac{\partial^3 H_{jk}(\mathbf{x})}{\partial x_s \partial x_r \partial x_q} f_k(\mathbf{x}) + \sum_{k=1}^{n} \frac{\partial^2 H_{jk}(\mathbf{x})}{\partial x_r \partial x_q} \frac{\partial f_k(\mathbf{x})}{\partial x_s}. \qquad (32)\end{aligned}$$

Setting $\mathbf{x} = \mathbf{r}$, then substituting (11) in (32), we get

$$\begin{aligned}\frac{\partial^3 N_j(\mathbf{r})}{\partial x_s \partial x_r \partial x_q} &= -\sum_{k=1}^{n} \frac{\partial H_{jk}(\mathbf{r})}{\partial x_q} \frac{\partial^2 f_k(\mathbf{r})}{\partial x_s \partial x_r} - \sum_{k=1}^{n} \frac{\partial H_{jk}(\mathbf{r})}{\partial x_r} \frac{\partial^2 f_k(\mathbf{r})}{\partial x_s \partial x_q} \\ &- \sum_{k=1}^{n} \frac{\partial H_{jk}(\mathbf{r})}{\partial x_s} \frac{\partial^2 f_k(\mathbf{r})}{\partial x_r \partial x_q} - 2\sum_{k=1}^{n} H_{jk}(\mathbf{r}) \frac{\partial^3 f_k(\mathbf{r})}{\partial x_s \partial x_r \partial x_q}. \qquad (33)\end{aligned}$$

Using equations (14) and (15), equation (33) yields

$$\frac{\partial^3 N_j(\mathbf{r})}{\partial x_s \partial x_r \partial x_q} = Z_{jqsr}(\mathbf{r}) + Z_{jrsq}(\mathbf{r}) + Z_{jsrq}(\mathbf{r}) - 2\widetilde{Z}_{jqrs}(\mathbf{r}).$$

This completes the proof of lemma 2.

Theorem 1. *Let $\mathbf{F} : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ be sufficiently differentiable at each point of an open neighborhood of $\mathbf{r} \in \mathbb{R}^n$, that is a solution of the system $\mathbf{F}(\mathbf{x}) = 0$. Let us suppose that $\mathbf{J_F}(\mathbf{x})$ is continuous and nonsingular in $\mathbf{r}$. Then, the sequence $\{\mathbf{x}^{(k)}\}_{k\geq 0}$ ($\mathbf{x}^{(0)} \in D$) obtained by using the iterative expression of method* (16) *converges to $\mathbf{r}$ with convergence order three.*

Proof. Let us consider $\mathbf{r} \in \mathbb{R}^n$ as a fixed point of iteration function $\boldsymbol{\psi} : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined as

$$\boldsymbol{\psi}(\mathbf{x}) = \mathbf{x} - \frac{1}{2\beta} \mathbf{J_F}(\mathbf{x})^{-1} [(2\beta - 1)\mathbf{J_F}(\mathbf{x}) + \mathbf{J_F}(\boldsymbol{\lambda}(\mathbf{x}))] \mathbf{N}(\mathbf{x}). \qquad (34)$$

The above equation can be written as

$$2\beta \mathbf{J_F}(\mathbf{x})(\boldsymbol{\psi}(\mathbf{x}) - \mathbf{x}) + [(2\beta - 1)\mathbf{J_F}(\mathbf{x}) + \mathbf{J_F}(\boldsymbol{\lambda}(\mathbf{x}))] \mathbf{N}(\mathbf{x}) = 0. \qquad (35)$$

The ith component of (35) can be written as

$$2\beta \sum_{j=1}^{n} J_{ij}(\mathbf{x})(\psi_j(\mathbf{x}) - x_j) + \sum_{j=1}^{n} [(2\beta - 1)J_{ij}(\mathbf{x}) + J_{ij}(\boldsymbol{\lambda}(\mathbf{x}))]N_j(\mathbf{x}) = 0. \quad (36)$$

Setting $\mathbf{x} = \mathbf{r}$ in (36) and assuming that $\mathbf{J_F}(\mathbf{r})$ is nonsingular. Then, using (20) and (26), we have

$$\psi_j(\mathbf{r}) = r_j. \quad (37)$$

Differentiating (36) partially with respect to x_q, one obtains

$$2\beta \sum_{j=1}^{n} J_{ij}(\mathbf{x}) \left(\frac{\partial \psi_j(\mathbf{x})}{\partial x_q} - \delta_{jq} \right) + 2\beta \sum_{j=1}^{n} \frac{\partial J_{ij}(\mathbf{x})}{\partial x_q} (\psi_j(\mathbf{x}) - x_j) + \sum_{j=1}^{n} [(2\beta - 1)J_{ij}(\mathbf{x}) + J_{ij}(\boldsymbol{\lambda}(\mathbf{x}))] \frac{\partial N_j(\mathbf{x})}{\partial x_q} + \left[(2\beta - 1) \sum_{j=1}^{n} \frac{\partial J_{ij}(\mathbf{x})}{\partial x_q} + \sum_{j=1}^{n} \left(\sum_{p_1=1}^{n} \frac{\partial J_{ij}(\boldsymbol{\lambda}(\mathbf{x}))}{\partial \lambda_{p_1}(\mathbf{x})} \frac{\partial \lambda_{p_1}(\mathbf{x})}{\partial x_q} \right) \right] N_j(\mathbf{x}) = 0. \quad (38)$$

Let us substitute $\mathbf{x} = \mathbf{r}$ and apply (20), (21), (26), and (27). Then, above equation yields

$$\frac{\partial \psi_j(\mathbf{r})}{\partial x_q} = 0. \quad (39)$$

Differentiating (38) partially with respect to x_r, we have

$$2\beta \sum_{j=1}^{n} J_{ij}(\mathbf{x}) \frac{\partial^2 \psi_j(\mathbf{x})}{\partial x_r \partial x_q} + 2\beta \sum_{j=1}^{n} \frac{\partial J_{ij}(\mathbf{x})}{\partial x_r} \left(\frac{\partial \psi_j(\mathbf{x})}{\partial x_q} - \delta_{jq} \right) + 2\beta \sum_{j=1}^{n} \frac{\partial J_{ij}(\mathbf{x})}{\partial x_q} \left(\frac{\partial \psi_j(\mathbf{x})}{\partial x_r} - \delta_{jr} \right) + 2\beta \sum_{j=1}^{n} \frac{\partial^2 J_{ij}(\mathbf{x})}{\partial x_r \partial x_q} (\psi_j(\mathbf{x}) - x_j) + \sum_{j=1}^{n} [(2\beta - 1)J_{ij}(\mathbf{x}) + J_{ij}(\boldsymbol{\lambda}(\mathbf{x}))] \frac{\partial^2 N_j(\mathbf{x})}{\partial x_r \partial x_q} + \left[(2\beta - 1) \sum_{j=1}^{n} \frac{\partial J_{ij}(\mathbf{x})}{\partial x_r} + \sum_{j=1}^{n} \left(\sum_{p_1=1}^{n} \frac{\partial J_{ij}(\boldsymbol{\lambda}(\mathbf{x}))}{\partial \lambda_{p_1}(\mathbf{x})} \frac{\partial \lambda_{p_1}(\mathbf{x})}{\partial x_r} \right) \right] \frac{\partial N_j(\mathbf{x})}{\partial x_q} + \left[(2\beta - 1) \sum_{j=1}^{n} \frac{\partial J_{ij}(\mathbf{x})}{\partial x_q} + \sum_{j=1}^{n} \left(\sum_{p_1=1}^{n} \frac{\partial J_{ij}(\boldsymbol{\lambda}(\mathbf{x}))}{\partial \lambda_{p_1}(\mathbf{x})} \frac{\partial \lambda_{p_1}(\mathbf{x})}{\partial x_q} \right) \right] \frac{\partial N_j(\mathbf{x})}{\partial x_r} + \left[(2\beta - 1) \sum_{j=1}^{n} \frac{\partial^2 J_{ij}(\mathbf{x})}{\partial x_r \partial x_q} + \sum_{j=1}^{n} \left(\sum_{p_1=1}^{n} \frac{\partial J_{ij}(\boldsymbol{\lambda}(\mathbf{x}))}{\partial \lambda_{p_1}(\mathbf{x})} \frac{\partial^2 \lambda_{p_1}(\mathbf{x})}{\partial x_r \partial x_q} \right) \right] N_j(\mathbf{x}) + \sum_{j=1}^{n} \left[\sum_{p_1=1}^{n} \left(\sum_{p_2=1}^{n} \left(\frac{\partial^2 J_{ij}(\boldsymbol{\lambda}(\mathbf{x}))}{\partial \lambda_{p_2}(\mathbf{x}) \partial \lambda_{p_1}(\mathbf{x})} \frac{\partial \lambda_{p_2}(\mathbf{x})}{\partial x_r} \right) \times \frac{\partial \lambda_{p_1}(\mathbf{x})}{\partial x_q} \right) \right] N_j(\mathbf{x}) = 0. \quad (40)$$

Setting $\mathbf{x} = \mathbf{r}$. Using (26), (27), (28) and results of lemma 1 in above equation, we obtain

$$\frac{\partial^2 \psi_j(\mathbf{r})}{\partial x_r \partial x_q} = 0. \quad (41)$$

Now, we analyze the third derivative of (36). Therefore, differentiating (40) partially with respect to x_s, and evaluating the resulting expression in $\mathbf{x} = \mathbf{r}$ using results of lemma 1, lemma 2 and (14), it can be easily proved that

$$2\beta \frac{\partial^3 \psi_j(\mathbf{r})}{\partial x_s \partial x_r \partial x_q} + (3\beta^2 + 2\beta) \widetilde{Z}_{jqrs}(\mathbf{r}) - 2\beta \big[Z_{jqsr}(\mathbf{r}) + Z_{jrsq}(\mathbf{r}) + Z_{jsrq}(\mathbf{r}) \big] = 0. \quad (42)$$

This yields

$$\frac{\partial^3 \psi_j(\mathbf{r})}{\partial x_s \partial x_r \partial x_q} = Z_{jqsr}(\mathbf{r}) + Z_{jrsq}(\mathbf{r}) + Z_{jsrq}(\mathbf{r}) - \left(1 + \frac{3\beta}{2} \right) \widetilde{Z}_{jqrs}(\mathbf{r}) \neq 0 \quad (43)$$

So, by (5), we conclude that (16) presents a one-parameter family of third order methods for systems of equations. We denote this family by GTM1, as this family is generalization of Traub method (3). Substituting (43) into (6) and noting that the summation is performed over all q, r and s from 1 to n, yields

$$e_j^{(k+1)} = \sum_{q=1}^{n} \sum_{r=1}^{n} \sum_{s=1}^{n} \left[\frac{1}{2} Z_{jqrs}(\mathbf{r}) - \frac{1}{6} \left(1 + \frac{3\beta}{2} \right) \widetilde{Z}_{jqrs}(\mathbf{r}) \right] e_q^{(k)} e_r^{(k)} e_s^{(k)}. \quad (44)$$

This completes the proof of theorem 1. □

Remark 1. The error equation of (3) is given as (see [2])

$$e^{(k+1)} = \left[\frac{1}{2} \left(\frac{f''(x^\star)}{f'(x^\star)} \right)^2 - \frac{1}{6} \left(1 + \frac{3\beta}{2} \right) \left(\frac{f'''(x^\star)}{f'(x^\star)} \right) \right] [e^{(k)}]^3, \quad (45)$$

where $x^\star$ is the root of nonlinear equation $f(x) = 0$. Noting that the Jacobian matrix is a generalization of f' and that the inverse of jacobian matrix, whose elements are denoted by H_{ij}, is generalization of $1/f'$, shows that

$$\sum_{i_1=1}^{n} \sum_{i_2=1}^{n} Z_{j i_1 i_2}(\mathbf{r}) e_{i_1}^{(k)} e_{i_2}^{(k)},$$

is generalization of $f''(x^\star)/f'(x^\star)[e^{(k)}]^2$ and

$$\sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \sum_{i_3=1}^{n} Z_{j i_1 i_2 i_3}(\mathbf{r}) e_{i_1}^{(k)} e_{i_2}^{(k)} e_{i_3}^{(k)},$$

is generalization of $(f''(x^\star)/f'(x^\star))^2[e^{(k)}]^3$. Also

$$\sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \sum_{i_3=1}^{n} \widetilde{Z}_{j i_1 i_2 i_3}(\mathbf{r}) e_{i_1}^{(k)} e_{i_2}^{(k)} e_{i_3}^{(k)}$$

is generalization of $f'''(x^\star)/f'(x^\star)[e^{(k)}]^3$. Thus, equation (44) is the generalization of (45).

Theorem 2. *Under the hypothesis of theorem* 1*, scheme* (17) *converges to* $\mathbf{r}$ *with convergence order three.*

Proof. Let us consider a solution $\mathbf{r} \in \mathbb{R}^n$ of $\mathbf{F}(\mathbf{x}) = 0$ as a fixed point of the iteration function $\boldsymbol{\phi} : \mathbb{R}^n \to \mathbb{R}^n$ defined as

$$\boldsymbol{\phi}(\mathbf{x}) = \mathbf{x} - 2\beta[(2\beta + 1)\mathbf{J_F}(\mathbf{x}) - \mathbf{J_F}(\boldsymbol{\lambda}(\mathbf{x}))]^{-1}\mathbf{F}(\mathbf{x}). \quad (46)$$

Equation (46) is equivalent to

$$[(2\beta + 1)\mathbf{J_F}(\mathbf{x}) - \mathbf{J_F}(\boldsymbol{\lambda}(\mathbf{x}))](\boldsymbol{\phi}(\mathbf{x}) - \mathbf{x}) + 2\beta\mathbf{F}(\mathbf{x}) = 0. \quad (47)$$

Let $\phi_i(\mathbf{x}), i = 1, 2, \ldots, n$ be the coordinate functions of $\boldsymbol{\phi}(\mathbf{x})$. Then, *i*th component of equation (47) is

$$\sum_{j=1}^{n}[(2\beta+1)J_{ij}(\mathbf{x}) - J_{ij}(\boldsymbol{\lambda}(\mathbf{x}))](\phi_j(\mathbf{x})-x_j)+2\beta f_i(\mathbf{x}) = 0. \quad (48)$$

From above equation, it is clear that if $\beta \neq 0$, $J_{\mathbf{F}}(\mathbf{r})$ is assumed to be nonsingular and $f_j(\mathbf{r}) = 0$, then $\boldsymbol{\phi}(\mathbf{x})$ is fixed point function, i.e.

$$\phi_j(\mathbf{r}) = r_j. \quad (49)$$

Now, differentiating equation (48) partially with respect to x_q, we have

$$\begin{aligned}&\Bigg[(2\beta+1)\sum_{j=1}^{n}\frac{\partial J_{ij}(\mathbf{x})}{\partial x_q}\\ &-\sum_{j=1}^{n}\left(\sum_{p_1=1}^{n}\frac{\partial J_{ij}(\boldsymbol{\lambda}(\mathbf{x}))}{\partial \lambda_{p_1}(\mathbf{x})}\frac{\partial \lambda_{p_1}(\mathbf{x})}{\partial x_q}\right)\Bigg](\phi_j(\mathbf{x})-x_j)\\ &+\left((2\beta+1)\sum_{j=1}^{n}J_{ij}(\mathbf{x})-\sum_{j=1}^{n}J_{ij}(\boldsymbol{\lambda}(\mathbf{x}))\right)\left(\frac{\partial \phi_j(\mathbf{x})}{\partial x_q}-\delta_{jq}\right)\\ &+2\beta\frac{\partial f_i(\mathbf{x})}{\partial x_q} = 0. \quad (50)\end{aligned}$$

Evaluating (50) in $\mathbf{x} = \mathbf{r}$ and using (49), it can be easily seen that

$$\frac{\partial \phi_j(\mathbf{r})}{\partial x_q} = 0. \quad (51)$$

Now by partial differentiation of (50) with respect to x_r, we have

$$\begin{aligned}&\Bigg[(2\beta+1)\sum_{j=1}^{n}\frac{\partial J_{ij}(\mathbf{x})}{\partial x_q}\\ &-\sum_{j=1}^{n}\left(\sum_{p_1=1}^{n}\frac{\partial J_{ij}(\boldsymbol{\lambda}(\mathbf{x}))}{\partial \lambda_{p_1}(\mathbf{x})}\frac{\partial \lambda_{p_1}(\mathbf{x})}{\partial x_q}\right)\Bigg]\left(\frac{\partial \phi_j(\mathbf{x})}{\partial x_r}-\delta_{jr}\right)\\ &+\Bigg[(2\beta+1)\sum_{j=1}^{n}\frac{\partial^2 J_{ij}(\mathbf{x})}{\partial x_r\partial x_q}\\ &-\sum_{j=1}^{n}\left(\sum_{p_1=1}^{n}\frac{\partial J_{ij}(\boldsymbol{\lambda}(\mathbf{x}))}{\partial \lambda_{p_1}(\mathbf{x})}\frac{\partial^2 \lambda_{p_1}(\mathbf{x})}{\partial x_r\partial x_q}\right)\Bigg](\phi_j(\mathbf{x})-x_j)\\ &-\sum_{j=1}^{n}\left[\sum_{p_1=1}^{n}\left(\sum_{p_2=1}^{n}\frac{\partial^2 J_{ij}(\boldsymbol{\lambda}(\mathbf{x}))}{\partial \lambda_{p_2}(\mathbf{x})\partial \lambda_{p_1}(\mathbf{x})}\frac{\partial \lambda_{p_2}(\mathbf{x})}{\partial x_r}\right)\frac{\partial \lambda_{p_1}(\mathbf{x})}{\partial x_q}\right]\\ &\times(\phi_j(\mathbf{x})-x_j)+\left((2\beta+1)\sum_{j=1}^{n}J_{ij}(\mathbf{x})-\sum_{j=1}^{n}J_{ij}(\boldsymbol{\lambda}(\mathbf{x}))\right)\\ &\times\frac{\partial^2 \phi_j(\mathbf{x})}{\partial x_r\partial x_q}+2\beta\frac{\partial^2 f_i(\mathbf{x})}{\partial x_r\partial x_q}\\ &+\left((2\beta+1)\sum_{j=1}^{n}\frac{\partial J_{ij}(\mathbf{x})}{\partial x_r}-\sum_{j=1}^{n}\left(\sum_{p_1=1}^{n}\frac{\partial J_{ij}(\boldsymbol{\lambda}(\mathbf{x}))}{\partial \lambda_{p_1}(\mathbf{x})}\frac{\partial \lambda_{p_1}(\mathbf{x})}{\partial x_r}\right)\right)\\ &\times\left(\frac{\partial \phi_j(\mathbf{x})}{\partial x_q}-\delta_{jq}\right) = 0. \quad (52)\end{aligned}$$

Setting $\mathbf{x} = \mathbf{r}$ and then substituting (20), (21), (49) and (51) in above equation, it can be proved that

$$\frac{\partial^2 \phi_j(\mathbf{r})}{\partial x_r\partial x_q} = 0. \quad (53)$$

Differentiating equation (52) partially with respect to x_s, and evaluating the resulting expression in $\mathbf{x} = \mathbf{r}$. Applying (13), (14), (49), (51), (53) and results of lemma 1 it can be shown that

$$\begin{aligned}2\beta\frac{\partial^3 \phi_j(\mathbf{r})}{\partial x_s\partial x_r\partial x_q} &+ (3\beta^2+2\beta)\widetilde{Z}_{jqrs}(\mathbf{r})\\ &-\beta\big[Z_{jqsr}(\mathbf{r})+Z_{jrsq}(\mathbf{r})+Z_{jsrq}(\mathbf{r})\big] = 0, \quad (54)\end{aligned}$$

which implies

$$\frac{\partial^3 \phi_j(\mathbf{r})}{\partial x_s\partial x_r\partial x_q} \neq 0 \quad (55)$$

Therefore, we conclude that (17) presents a one-parameter family of third order methods for systems of equations. We denote this family by GTM2. Substituting (55) into (6), the error equation for family (17) is given as

$$\begin{aligned}e_j^{(k+1)} = \sum_{q=1}^{n}\sum_{r=1}^{n}\sum_{s=1}^{n}\Bigg[&\frac{1}{4}Z_{jqrs}(\mathbf{r})\\ &-\frac{1}{6}\left(1+\frac{3\beta}{2}\right)\widetilde{Z}_{jqrs}(\mathbf{r})\Bigg]e_q^{(k)}e_r^{(k)}e_s^{(k)}. \quad (56)\end{aligned}$$

This completes the proof of theorem 2.

Remark 2. The error equation in case of (4) is given as [2]

$$\begin{aligned}e^{(k+1)} = \Bigg[&\frac{1}{4}\left(\frac{f''(x^\star)}{f'(x^\star)}\right)^2\\ &-\frac{1}{6}\left(1+\frac{3\beta}{2}\right)\left(\frac{f'''(x^\star)}{f'(x^\star)}\right)\Bigg][e^{(k)}]^3, \quad (57)\end{aligned}$$

Thus, by the similar argument as in remark 1 the equation (56) is generalization of (57).

IV. Particular Cases

(A) *Particular Cases of GTM1*

For $\beta = -1/2$, we get third order method developed by Hernández [13]

$$\mathbf{y}^{(k)} = \mathbf{x}^{(k)} - 1/2\mathbf{J_F}(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}),$$

$$\begin{aligned}\mathbf{x}^{(k+1)} =& \mathbf{x}^{(k)} + (\mathbf{J_F}(\mathbf{x}^{(k)}))^{-1}(\mathbf{J_F}(\mathbf{y}^{(k)}) - 2\mathbf{J_F}(\mathbf{x}^{(k)}))\\ &\times(\mathbf{J_F}(\mathbf{x}^{(k)}))^{-1}\mathbf{F}(\mathbf{x}^{(k)}). \quad (58)\end{aligned}$$

If $\beta = -1$, we obtain third order method developed by Babajee et al. [14]

$$\mathbf{y}^{(k)} = \mathbf{x}^{(k)} - \mathbf{J_F}(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}),$$

$$\begin{aligned}\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} &+ \frac{1}{2}(\mathbf{J_F}(\mathbf{x}^{(k)}))^{-1}(\mathbf{J_F}(\mathbf{y}^{(k)}) - 3\mathbf{J_F}(\mathbf{x}^{(k)}))\\ &\times(\mathbf{J_F}(\mathbf{x}^{(k)}))^{-1}\mathbf{F}(\mathbf{x}^{(k)}). \quad (59)\end{aligned}$$

(B) *Particular Cases of GTM2*

The value $\beta = -1/2$, gives third order method developed by Homeier [6] which is given as

$$\mathbf{y}^{(k)} = \mathbf{x}^{(k)} - 1/2\mathbf{J_F}(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}),$$

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - (\mathbf{J_F}(\mathbf{y}^{(k)}))^{-1}\mathbf{F}(\mathbf{x}^{(k)}). \quad (60)$$

If we let $\beta = -2/3$, equation (17) generates third order method developed by Noor et al. [9].

$$\mathbf{y}^{(k)} = \mathbf{x}^{(k)} - 2/3\mathbf{J_F}(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}),$$

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - 4(\mathbf{J_F}(\mathbf{x}^{(k)}) + 3\mathbf{J_F}(\mathbf{y}^{(k)}))^{-1}\mathbf{F}(\mathbf{x}^{(k)}). \quad (61)$$

V. Numerical examples

Here we present some examples to test the performance of present families GTM1 and GTM2 by taking $\beta = 0.1$ and 1. The comparison is carried out with Newton's method (NM) and with the special members of the families; Hernández method (HM), formula (58) and Homeier method (HMM), formula (60). To check the theoretical order of convergence, we obtain the computational order of convergence (ρ) using the formula [15]

$$\rho \approx \frac{\ln(\|\mathbf{x}^{(k+1)} - \mathbf{r}\|/\|\mathbf{x}^{(k)} - \mathbf{r}\|)}{\ln(\|\mathbf{x}^{(k)} - \mathbf{r}\|/\|\mathbf{x}^{(k-1)} - \mathbf{r}\|)}. \quad (62)$$

We consider the following systems of nonlinear equations:

(a)
$$\begin{cases} x_1 + e^{x_2} - \cos(x_2) = 0, \\ 3x_1 - x_2 - \sin(x_2) = 0. \end{cases}$$

(b)
$$\begin{cases} x_1^2 - 2x_1 - x_2 + 0.5 = 0, \\ x_1^2 + 4x_2^2 - 4 = 0. \end{cases}$$

(c)
$$\begin{cases} x_1^2 + x_2^2 - 1 = 0, \\ x_1^2 - x_2^2 + 0.5 = 0. \end{cases}$$

(d)
$$\begin{cases} x_1^2 - x_2^2 + 3\log(x_1) = 0, \\ 2x_1^2 - x_1x_2 - 5x_1 + 1 = 0. \end{cases}$$

(e)
$$\begin{cases} e^{x_1} + x_1x_2 - 1 = 0, \\ \sin(x_1x_2) + x_1 + x_2 - 1 = 0. \end{cases}$$

(f)
$$\begin{cases} x_1 + 2x_2 - 3 = 0, \\ 2x_1^2 + x_2^2 - 5 = 0. \end{cases}$$

(g)
$$\begin{cases} x_1^2 + x_2^2 + x_3^3 = 9, \\ x_1x_2x_3 = 1. \\ x_1 + x_2 - x_3^2 = 0. \end{cases}$$

(h)
$$\begin{cases} x_2x_3 + x_4(x_2 + x_3) = 0, \\ x_1x_3 + x_4(x_1 + x_3) = 0, \\ x_1x_2 + x_4(x_1 + x_2) = 0, \\ x_1x_2 + x_1x_3 + x_2x_3 - 1 = 0. \end{cases}$$

(i)
$$\begin{cases} x_ix_{i+1} - 1 = 0, \quad i = 1, 2, \ldots, 98 \\ x_{99}x_1 - 1 = 0. \end{cases}$$

whose solutions, respectively are:

(a) $\mathbf{r} = (0, 0)^t$.
(b) $\mathbf{r} = (1.9006767263670658, 0.31121856541929427)^t$.
(c) $\mathbf{r} = (0.50000000000000000, 0.86602540378443865)^t$ and $\mathbf{s} = (-0.50000000000000000, -0.86602540378443865)^t$.
(d) $\mathbf{r} = (1.3192058033298924, -1.6035565551874148)^t$.
(e) $\mathbf{r} = (0, 1)^t$.
(f) $\mathbf{r} = (1.4880338717125849, 0.75598306414370757)^t$.

(g) $\mathbf{r} = (2.2242448288477843, 0.28388497407293814, 1.5837076128252723)^t$,

and

$$\mathbf{s} = (0.28388497407293814, 2.2242448288477843, 1.5837076128252723)^t$$

(h) $\mathbf{r} = (0.57735026918962576, 0.57735026918962576, 0.57735026918962576, -0.28867513459481288)^t$,

and

$$\mathbf{s} = (-0.57735026918962576, -0.57735026918962576, -0.57735026918962576, 0.28867513459481288)^t.$$

(i) $\mathbf{r} = (1, 1, \ldots, 1)^t$ and $\mathbf{s} = (-1, -1, \ldots -1)^t$.

All computations are done using MATHEMATICA [16] software, where the solutions $\mathbf{r}$ and $\mathbf{s}$ are computed with a precision of 200 decimal digits. The stopping criterion used is $\|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\| + \|\mathbf{F}(\mathbf{x}^{(k)})\| < 10^{-100}$. For every method, we analyze the number of iterations needed to converge to the solution and computational order of convergence. The results are displayed in Table 1. It can be observed that the numerical results are in accordance with the theory developed in paper. We can also observe that the present methods show good stability and require less number of iterations (k) than NM to reach the required solution.

VI. Conclusions

We have presented two one-parameter families of third order methods for solving systems of equations. These families are the generalization of known univariate families by Traub. Analysis of convergence is supplied in Theorem 1 and 2. Numerical results are presented and performance is compared with well known methods. These numerical results overwhelmingly support the theoretical results we have derived. Similar experimentations have been performed on number of other problems and results are found at par with those presented here. Finally, we conclude the paper with the remark that there is no clear winner among the presented methods in the sense that in some situation one method may be the winner while in some different the other method may be the winner.

TABLE I
COMPARISON OF THE PERFORMANCE OF METHODS

$\mathbf{F}(\mathbf{x})$	$\mathbf{x}^{(0)}$	k							ρ							Sol.
		NM	HM	HMM	GTM1	GTM2	GTM1	GTM2	NM	HM	HMM	GTM1	GTM2	GTM1	GTM2	
					$\beta=0.1$		$\beta=1$					$\beta=0.1$		$\beta=1$		
(a)	$(1.5,.1)^t$	7	5	4	5	4	5	4	2.0	3.0	3.0	3.0	3.0	3.0	3.0	**r**
	$(.3,.5)^t$	8	5	5	5	5	5	5	2.0	3.0	3.0	3.0	3.0	3.0	3.0	**r**
(b)	$(3,2)^t$	9	6	6	5	6	6	6	2.0	2.99	3.0	3.0	3.0	2.99	3.0	**r**
	$(1.6,0)^t$	8	6	5	6	5	6	5	2.0	3.01	3.0	3.0	3.0	3.0	3.0	**r**
(c)	$(.7,1.2)^t$	7	5	5	5	5	5	5	2.0	2.99	2.99	3.0	3.0	3.0	3.0	**r**
	$(-1,-2)^t$	8	6	6	6	6	6	5	2.0	3.0	3.0	3.0	3.0	3.0	3.0	**s**
(d)	$(.91,-2)^t$	8	6	5	6	5	5	5	2.0	3.0	2.96	3.0	3.0	3.0	3.0	**r**
	$(1.5,-1.5)^t$	7	5	5	5	5	5	5	2.0	3.0	2.98	3.0	3.0	3.0	3.0	**r**
(e)	$(.9,.9)^t$	8	5	5	5	5	6	6	1.99	3.01	2.97	3.0	2.95	3.0	3.0	**r**
	$(-0.1,0.2)^t$	7	5	5	5	5	5	5	2.0	3.0	3.0	3.0	2.98	2.93	2.95	**r**
(f)	$(.9,.5)^t$	8	6	5	6	5	6	5	2.0	3.0	3.0	3.0	3.0	3.0	3.0	**r**
	$(1.5,1)^t$	7	5	4	5	4	5	4	2.0	3.0	3.0	3.0	3.0	3.0	3.0	**r**
(g)	$(2,1.5,1.9)^t$	9	6	6	6	6	7	6	2.0	3.0	3.0	3.0	3.0	3.0	3.0	$(\mathbf{r},\mathbf{s})^*$
	$(3,.5,2)^t$	8	5	5	5	5	5	5	2.0	3.0	3.0	3.0	3.0	3.0	3.0	**r**
(h)	$(-1,-1,-1,-1)^t$	8	5	5	5	5	5	5	2.0	3.0	3.01	3.0	3.09	3.0	3.0	**s**
	$(2,2,2,0)^t$	9	6	6	6	6	6	6	2.0	3.0	3.01	3.0	3.0	3.0	3.0	**r**
(i)	$(2,2,\ldots,2)^t$	9	6	6	6	6	6	6	2.0	3.0	3.0	3.0	3.0	3.0	3.0	**r**
	$(-4,-4,\ldots-4)^t$	10	7	7	7	7	7	7	2.0	3.0	3.0	3.0	3.0	3.0	3.0	**s**

*NM, HMM and GTM2 converge to **r**, while HM and GTM1 converge to **s**.

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