Frequency as a fifth dimension of reality

Adam Weisser

Corresponding author(s). E-mail(s): weisser@f-m.fm;

Abstract

Over the last century, the quest to formulate physics to account for Reality has attracted a large number of theoreticians to propose various fundamental models that have tended toward a growing level of abstractness. While space and time have been largely recognized as the four fundamental dimensions that make our perceived reality, their completeness has been challenged either by positing hidden dimensions, or by exploring the possibility that space-time itself is an emergent property of measurable physics. One quantity that has not entered this exploration is frequency. Although it is featured in numerous physical contexts, it is normally implied that it is a mere parameter that is determined by the boundary conditions, or that it contains the same information as the time, period, wavelength, or energy—all supporting the notion that frequency is fully dependent on the other dimensions. In contrast, in psychophysics of vision, hearing, and touch, frequency is a quantity that appears independent, so that both its input and output are not directly dependent on perception of space and time. Also, many important engineering applications treat frequency as a variable rather than as a parameter that is constrained alongside time. This paper explores the various conventions with respect to frequency in the physical, mathematical, and engineering literatures. It scrutinizes frequency against the standard dimensions of space-time along nine properties that may be deemed universal. It is argued that in all but the most trivial physical systems frequency should be considered an additional fifth dimension of reality. More generally, it is argued that only one of these claims can be simultaneously true:

1. Time is not a fundamental, obligatory dimension of reality.
2. The universe is fully deterministic with total knowledge of past and future.
3. Frequency is a fundamental dimension of reality.

Keywords: Frequency, waves, oscillations, periodicity, time, Fourier analysis, harmonic analysis, time-frequency analysis, psychophysics, perception, sensation, neuroscience, determinism, quantum mechanics

Contents

1 Introduction

Humans access the external reality through their senses, which function as arrays of detectors of various physical attributes of stimuli from the environment. Once the stimuli are peripherally detected, they are transduced to neural signals that can be perceived by the brain according to the specific stimulus and its modality. The effects and implications of this indirect mediation of the external world on the resultant internal perceived reality have been in ongoing exploration over millennia within philosophy and science. Of late, physics has been informally nominated as the Authority that deals with the external reality most rigorously, whereas perception has been variably dealt with within philosophy, psychology, biology, and neuroscience. The interface between these domains is most directly addressed by psychophysics, itself a multidisciplinary science that does not claim to explain perception. Of these disciplines, physics has been most concerned with getting closer to a true account of reality that is unbiased by the particular layout and wiring of our senses. As a major part of this exploration, a repeating question has been whether the world geometry that is perceived as three dimensions of space plus one dimension of time is, in fact, merely a product of perception. The alternative is that reality is spanned by a different combination of dimensions, whose nature we may not be able to directly perceive or even conceive. In its rush to uncover hidden dimensions of reality, the physics foray has provided exciting ideas that challenge all naive perception of reality, but may have neglected a more mundane contribution from psychophysics that repeatedly highlights the independence of frequency in sensation. This negligence is compounded by a tradition within physics of treating frequency as a de-facto parameter, in contrast with more applied fields where it is de-facto treated as a variable.

1.1 Geometrical and temporal detection in sensation and perception

The starting point for any discussion about reality is our sensory apparatus and the ensuing perception, which mediate all observations of the external world. The different senses tend to provide complementary detection of different body parts, so that every part is covered by at least one or two senses (Fig. 1). Several senses are particularly well equipped to deal with remote stimuli that are external to the body and are carried by radiation (hearing, vision, heat, magnetoreception) or by changes in concentration of chemical compounds (olfaction).

The geometrical arrangement and relevance of the senses are also found inside the brain. Several senses are represented by topographical maps in the brain that reflect the peripheral structure of the senses, such as tonotopy in sound that corresponds to cochlear place [1–3], retinotopy in vision that corresponds to the optical image on the retina [4, 5], somatopy in touch that maps the skin [6, 7], and an odor map that corresponds to primitive dimensions of olfaction [8]. While these maps tend to be distorted at the cortical level [7], they provide a perceptual gateway for mapping the environment, inasmuch as its geometry is causally reflected in the peripheral sensory response. With additional processing of the neural signal, as well as with cross-modal binding of the sensory information that is processed as belonging to the same object [9], the various senses provide information to the perceiver about his/her own position and how he/she is localized with respect to various objects within the environment [e.g. 10]. Bound spatial perception also includes information about the orientation of the perceiver's own body and possible interactions with external objects (Fig. 1, left). The integrated information about the world from the senses is positioned in a three-dimensional (3D) perceptual geometry that includes the sensing individual, who occupies part of that 3D space and serves as an internal reference.

In addition to the spatial information that is provided by the senses, information regarding temporal changes in the sensory stimuli can also be extracted from the detected signals, which can in turn be used to produce the time perception of the individual in the cortex [11]. However, while different senses have different temporal precision associated with their stimulus duration judgments, time perception is not associated with a specific modality and is not anchored to a dedicated sense organ [11, 12]. Hence, time perception may be thought of as a supra-modal sense that is central to the entire perceptual system (Fig. 1, middle).

Despite the difficulty in understanding how perception manipulates or reduces the dimensionality of natural stimuli ([e.g., 13–16]; and somewhat indirectly, [17]), all percepts that originate in the sensory apparatus correspond to certain physical properties of the stimulus and its environment that may not be encapsulated solely in spatial and temporal information[1]. As such, perhaps the most common additional attribute of stimuli in several modalities is their frequency content, or their (power)

[1] For simplicity, we omit from the discussion certain stimuli that are artificially generated and, in some cases, can be designed to "fool" the ecologically evolved correspondence to natural stimuli (e.g., using screens or loudspeakers), so that their perception rarely corresponds to naturally encountered objects in the environment.

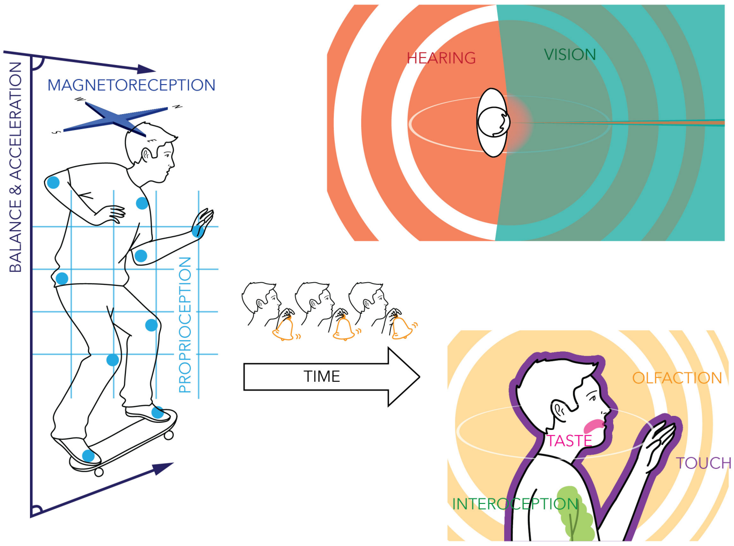

Fig. 1 The sensory mapping of the physical environment is delegated to the different senses that are located in strategic places in the body, which are suitable to detect either direct-contact or remote stimuli. Most body parts are covered by at least one sense (e.g,. touch, chemesthesis, or interoception) with key areas that interface the environment covered by more than one sense. (Original illustration by Jody Ghani.)

spectrum. In humans, vision [18], hearing [19], touch [20–22], and balance [23] all produce perceptions that are causally linked to the stimulus frequency, which is detected by appropriately tuned sensory receptors[2]. In some cases, the receptor tuning has been explicitly analogized to frequency channels that are locally demodulated to obtain baseband (i.e., low-frequency) signals that vary slowly in time and space, at rates that can be directly coded by neuronal spiking [e.g., 29, 30, pp. 122–123]. While different peripheral channels can interact and be segregated or fused in perception, they are understood as providing information that may not be available in the stimulus spatial and temporal attributes alone, as can be gathered, for example, from the effects of color on object recognition [31][3], or speech recognition when some parts of the spectrum are filtered out [32, 33]. The spectral or spectral-equivalent perceptual representation of the stimulus thus provides a "What?" kind of information about the physical object that generates the stimulus[4], which is therefore mandatory in all senses that do not

[2]According to an interesting hypothesis olfaction too may generate differentiated sensations of odorants determined by their vibrational spectra, which in molecular (Raman) spectroscopy are known to be unique in a particular infrared fingerprint region [24–26]. However, under psychophysical and physiological scrutiny, this hypothesis has failed to receive experimental support [27, 28].

[3]Note that even in monochromatic vision as is achieved by the rod ("black and white") photoreceptors as is common in some animals, vision is still tuned, only to less narrow frequency channels than the cone (color) photoreceptors.

[4]The distinction between "Where" and "What" types of processing has been suggested as a fundamental organizing principle of the brain in vision, known as the "dual stream model" [34, 35]. This model was later expanded for hearing as well. See [30, p. 40] for further references.

(exclusively) deal with very slowly varying stimuli (more precisely, at frequency close to 0 Hz)[5].

It therefore appears plausible that the perceptual experience of the external reality has to include, at the very least, five dimensions (5D) with frequency, spectrum, or channel(s) being the fifth dimension on top of the usual 4D space and time.

1.2 Dimensions of reality within physics

While space has always played an obvious role in physics, the inclusion of time in the standard dimensional count is a relatively recent addition that was realized only with the advent of Einstein's special theory of relativity. In fact, both D'Alembert and Lagrange had already proposed that time should count as a fourth dimension, a century and a half before Minkowski came up with space-time [36]. And while the very intangible nature of time is in itself opaque, its inclusion as a an inseparable aspect of space has opened the door for even wilder theoretical proposals of additional dimensions of reality that go beyond the perceptually observable space (beginning from [37–39]; for recent references see [40]). Common to all the various higher-dimensional theories has been the understanding that any such extra dimensions ultimately have to be reducible to the measurable four dimensions that are accessible to our senses [e.g., 41, p. 15]. The extra dimensions in such theories are then either mathematically constructed, or assumed to map to very small geometries that are curled and topologically compact and are not amenable to observation using current measurement methods. Other analyses argue that space-time that is particularly four dimensional has special properties that enable life and physics as we know it [42].

Insights from such physical theories of extra dimensions, along with the realization of how difficult it may be to formulate a consistent physical theory of extreme spatial and temporal scales, have influenced some ideas regarding the validity of perception itself. In its most sensational form, it has been hypothesized that the four-dimensional reality emerges on a macroscopic scale from a higher-dimensional space that exists at scales that are too small to be measured—something that may entail "doom" on the space-time dimensional reality as we naively perceive it [43–45]. In turn, this led some scholars to suggest that perception may deliver an image of reality that is altogether divorced from the "actual" reality and is only geared to satisfy the evolutionary needs of the organism [46].

1.3 The absence of frequency from the physical dimensional count

In the quest to account for both observable and hidden physical dimensions, the ubiquitous frequency variable of key sensory systems (1.1) has been left out of all discussions within the physics and philosophy literature. Why is it that the 4D space-time has become the de-facto standard in physical representation without considering frequency as an extra dimension? While it is only possible to speculate here, at least three explanations are proposed. First, all physicists are trained in Fourier analysis, mainly in

[5]Spectral band-limitation is well-ingrained in modern sensation and perception science, and yet frequency analysis as a general property of sensory channels has not been generalized beyond the specific modalities and no general reviews are available within the sensation and perception literature.

the context of solving partial differential equations. In the modern presentation of this technique, time and frequency appear as reciprocal domains that essentially contain the same information about the system [e.g., 47, p. 21], which suggests that including frequency as an independent dimension would be redundant. Second, unlike space, frequency does not refer to anything that is intuitively or immediately tangible—even less so than time, which has already suffered from this issue [36]. Third, frequency is a much more modern concept than time (introduced only in 1585 by Benedetti, picked up by Galileo, and refined over the subsequent centuries; [48]). Filtering, which constitutes the most fundamental operation in the spectral domain that one can perform on a broadband time signal, is an even more recent concept that was invented for electric circuits only a century ago by Campbell [49]. Nowadays, filters are ubiquitous and any electronic detector can be associated with a filter that effectively limits its applicable spectral range (either deliberately designed or imposed by parasitic elements in the system), but the signal processing theory that facilitates this understanding is also not more than a century old [50][6]. Signal processing theory was anyway developed well after the dominant physical theories of the day became established.

While these explanations may not fully capture the absence of frequency from the standard dimensional discussion, they highlight the problems involved in attempting to demonstrate that frequency may be, in fact, an independent variable and dimension of reality.

1.4 Logic and outline

The present work is concerned with the dimensions of reality as are phenomenologically perceived by our senses, rather than with hidden dimensions that the senses may or may not be privy to. As such, it focuses on frequency and challenges its current non-dimensional status. The conditions are explored as for whether frequency is:

1. A parameter.
2. A variable that carries the same information as time, may be derived from it, and as such, equivalent to it.
3. An independent variable that may also vary in time.
4. A dimension of reality that is distinct from time and space.

Each option is categorically escalated compared to the previous. The relevance of options 1–3 can be explored using deduction alone based on first principles. Thus, Section 2 deals with 1–3, by referring to the most quintessential appearances of frequency in physics, mathematics, and engineering. Option 4 that entails that frequency, which is neither a parameter nor dependent only on time, can be considered its own dimension is tested against several universal features of the standard space-time dimensions in 3. A theorem that synthesizes the logic of the entire analysis is presented in 4. Challenges and open questions regarding the frequency dimension are discussed in 5.

[6] To the best knowledge of the author, there is no rigorous historical account of early signal processing theory that preceded the digital age [51, pp. 5–8]. Campbell [50] is the first publication that formalized filter theory, following his very own patent of the first electrical filter [49], and as such seems to be the most appropriate milestone to designate the beginning of analog signal processing theory.

2 Evolved approaches to frequency

This section presents a selection of definitional and quintessential occurrences of frequency in physics, engineering, and mathematics. As no new physics or mathematics are presented here, some readers may find certain elements of this review to be overly basic. However, the novelty here is in the uncovering of an otherwise subtextual narrative relating how the concept of frequency evolved much beyond its initial usage and original definition. It is shown how the different definitions or usages of frequency either contain additional supra-dimensional parameters, or they are circular, or they require an ultra-deterministic conception of reality. As such, the first part of this narrative serves as a re-review of many of the familiar topics in basic introductions to oscillatory phenomena, but with emphasis on definitional intricacies, paradoxes, and contradictions that had not been previously put together in writing. The latter part of the review focuses on complementary approaches to frequency within modern time-frequency analysis that may be less familiar to most readers.

Frequency has been treated in two main approaches to modeling of dynamical systems. The historical one is wholly deterministic. It provides analytical (closed-form) solutions and intuition and has been traditionally used to introduce these topics in fundamental physics and engineering courses. Its main results have been studied using different families of differential equations and their solutions have given rise to powerful analytical techniques that carried over to various implementations in both analog and digital signal processing. Similar results keep on appearing in different guises in modern physics, so the relevance of this perspective has not waned. The second, stochastic approach is more recent and is based on the statistical analysis of signals whose particular instantiation is either unknown, unknowable, or unimportant, whereas analysis of the ensemble properties of the signals provide much more salient information. Ultimately, both the deterministic and the statistical approaches to frequency account for the same physics, but they provide different types of intuition about the phenomena at hand.

As the physical and mathematical understanding of dynamical systems has become more sophisticated, so did the concept of frequency has been gradually expanded to include a wider array of conditions that have not originally lent themselves to spectral analysis (Fig. 2). Beginning from oscillatory systems in complete equilibrium, frequency was incorporated in the description of systems with multiple modes of motion, in waves, in the description of lossy systems, in accounting for the effects of external forces. Then, a profound conceptual jump has been to make frequency available for the description of aperiodic oscillations, nonlinear systems, and with arbitrary force driving the system, where periodicity is, at best, local. In dealing with the latter systems, it is impractical to speak about a time-independent frequency, although the classical definition of frequency that is time independent may be applied notwithstanding and may then lead to a description that is mathematically correct, but is of little practical use.

Throughout this analysis we refer somewhat interchangeably to waves, signals, oscillations, vibrations, periodic and cyclic motions, and stimuli. While these terms do not mean the same thing, their mathematical formulation and usages within the sciences is more similar than not. Thus, a “signal” here is taken as a general function of time and frequency. Without the loss of generality, it should be understood that it can represent an arbitrary waveform, time series, variable, measurement, stimulus, etc. This enables us to make use of generic mathematical concepts developed within signal processing theory that may be universally applied.

All signal plots were drawn using MATLAB (MathWorks®).

2.1 Ideal oscillators: frequency as a parameter

2.1.1 Simple harmonic oscillators

Most simply, the frequency f of any periodic motion is defined as the reciprocal of its period T, a fixed duration that repeats and characterizes the motion

$$f = \frac{1}{T} \tag{1}$$

This refers, for example, to the period of an idealized small-amplitude mass-spring system that can be computed from

$$T = 2\pi\sqrt{\frac{m}{s}} \tag{2}$$

with m the mass and s the stiffness of the spring—both of which can be estimated from static mechanical measurements. Similar dynamics accounts for the small-angle periodic movement of the pendulum, where

$$T = 2\pi\sqrt{\frac{g}{l}} \tag{3}$$

with g being the gravity of Earth and l is the length of the pendulum rod. Another fundamental system—an ideal LC (inductor-capacitor) circuit resonator has the period of

$$T = 2\pi\sqrt{LC} \tag{4}$$

with L being the inductance and C the capacitance. The three systems, illustrated in Fig. 3, are examples of simple harmonic oscillators that are described by the same ordinary differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{5}$$

where x is displacement in the mass-spring system, angle of the pendulum, or electric current in the LC circuit, in our examples. The solution to the simple harmonic oscillator is then given by

$$x = x_0 \cos(\omega t - \varphi_0) \tag{6}$$

where the angular frequency $\omega = 2\pi f$ refers to the resonance of the system, the amplitude x_0 and the phase φ_0 are parameters of motion determined by the initial conditions (e.g., the initial displacement and velocity). An equivalent form of the solution as a sum of a sine and a cosine exists, but we shall stick with this more compact one where the amplitude and phase are distinguished, which will enable subsequent generalization below. The simple harmonic oscillation is plotted in Fig. 2 B. Despite its simplicity, the harmonic oscillator model provides the intuition for a large number of vibrational systems. It plays a particularly important role in quantum mechanics, where energy loss and time effects (see 2.2) become relevant only in larger systems and are, therefore, neglected in standard models of subatomic and atomic systems [52, 53].

It should be noted that the standard definition of frequency (Eq. 1) is inherently ambiguous with respect to the time interval that it covers, as well as whether it relates to a constant and time-independent frequency or to an average value (Fig. 2 A–C). These issues will become important further in the analysis.

2.1.2 Coupled simple harmonic oscillators

Simple harmonic oscillators may be combined into more elaborate systems that contain multiple masses and springs, capacitors and inductors, etc. (Fig. 4). Parts of the system are coupled to the others, in a way that can be studied using systems of linear differential equations of the form of Eq. 5 [e.g., 54, 55]. The resultant system can then be characterized by a set of resonances that contains as many frequencies as are degrees of freedom in the system. These frequencies are functions of the individual free-oscillating frequencies of the single simple harmonic oscillators. The total oscillation can be described as a superposition of oscillations at the component frequencies, which also depends on the particular initial conditions and parametric values.

2.1.3 Simple wave motion

In the limit of infinitely many identical coupled oscillators (Figs. 4, right and 5, left), it is possible to arrive at a description of wave motion (e.g., of a string)—an oscillation that is periodic in both space and time and has similar mathematical solutions to the harmonic oscillators [54, pp. 80–91]. The simplest wave equation—the string equation—is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \tag{7}$$

where the motion in one spatial dimension (y), the string amplitude, depends on both (the perpendicular) spatial (x) and time dimensions (Fig. 5, right). The wave frequency is related to its wavelength λ via the wave speed c in the medium

$$f = \frac{c}{\lambda} \tag{8}$$

Hence, for a known c, the frequency contains the same information as the wavelength. Additionally, the spatial frequency k (also called the wavenumber) is defined as

$$k = \frac{2\pi}{\lambda} \tag{9}$$

which hints that the wavelength is analogous to the period ($\omega = 2\pi/T$), only in spatial dimensions. The solutions for Eq. 7 are of the form

$$y(x,t) = y_1 \cos(\omega t - kx + \phi_1) + y_2 \sin(\omega t + kx + \phi_2) \tag{10}$$

with y_1 and y_2 being the amplitudes and ϕ_1 and ϕ_2 are phases that depend on the initial conditions. Another way of expressing the solution uses Euler's identity and taking its real part

$$y(x,t) = \mathrm{Re}\left[C_1 e^{i(\omega t - kx)} + C_2 e^{i(\omega t + kx)}\right] \tag{11}$$

where the complex amplitudes C_1 and C_2 now include the initial phase too. This solution form enables separation of the time-dependent term, so that $y(x,t) = y(x)e^{i\omega t}$, which simplifies the wave equation (7) to

$$\frac{\partial^2 y(x)}{\partial x^2} + k^2 y(x) = 0 \tag{12}$$

In two dimensions, therefore, the string equation can be brought to the same form as the simple harmonic oscillator (Eq. 5)—an ordinary instead of a partial differential equation. Unsurprisingly, the frequency of a string of length l takes the same algebraic form as in the harmonic oscillator with

$$f = \frac{1}{2l}\sqrt{\frac{F_T}{\mu}} \tag{13}$$

with F_T being the tension force in the string and μ its mass per unit length.

In systems of two and three spatial dimensions, the wave equations are more complex, as they admit oscillations that are distributed in all dimensions. For example, the three-dimensional wave equation in Cartesian coordinates is

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} \tag{14}$$

This linear homogenous scalar wave equation is solved for some field function $\psi(x,y,z,t)$, whose exact identity depends on the medium and the type of wave. In the case of sound waves it is pressure, velocity, or density. In the case of electromagnetic radiation, it is electric field or magnetic field. And it is the displacement in elastic waves. The general solution here is of the form $\psi(x,y,z,t) = \psi_1(\alpha x + \beta y + \gamma z - ct) + \psi_2(\alpha x + \beta y + \gamma z + ct)$, as long as $\gamma = \sqrt{1 - \alpha^2 - \beta^2}$. For the basic cases (no loss, no sources, everything is linear), the solutions retain the same form as in Eq. 11,

$$\psi(x,y,z,t) = \psi_1 e^{i(\omega t - \vec{k}\cdot\vec{r})} + \psi_2 e^{i(\omega t + \vec{k}\cdot\vec{r})} \tag{15}$$

for a field defined by the vector $\vec{r} = \{x, y, z\}$. In the most general case, k is the propagation vector of the wave, whose magnitude is the wavenumber $|k| = 2\pi/\lambda$,

$$\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z} = \frac{2\pi}{\lambda}(\alpha\hat{x} + \beta\hat{y} + \gamma\hat{z}) \tag{16}$$

where α, β, and γ are the direction cosines related through the condition on γ, which satisfies the general three-dimensional solution (Fig. 6) [56]. The directional components are then three spatial frequencies, k_x, k_y, and k_z which may be independent from one another.

The oscillations are considered free if the system boundary goes to infinity, but when it is finite—as in the classical case of a string, bar, or membrane—the solutions are often given as a superposition of series of allowed (resonance, natural, normal, or eigen-) frequencies ω_n, each of which is distributed differently in space.

Although the wave description of physical systems tends to be the most accurate one, it will be easier to focus in the following on one-dimensional oscillators, whose solutions share many similarities to wave motion, as was seen through the similarity between the one-dimensional string and the harmonic oscillator equations. Wave propagation and dynamics has been formulated in a large number of partial differential equations at a much higher degree of complexity than is presented here [57]. It is, however, possible to adopt an observer's point of view that is best captured by the signal-processing approach, which often deals with time signals at fixed positions. This means that the contributions of spatial variations through the $\vec{k} \cdot \vec{r}$ term turn into a constant phase that can be incorporated into the initial conditions. The inverse can be done by looking at a spatial array of measurement positions, at a fixed time, which may translate to constant phase differences between the array points. In more complex systems, fewer parameters remain constant, and yet local measurements can still be subjected to signal processing analysis, which in our case entails spectral analysis, as is discussed below.

2.1.4 The simple frequency

All of the above idealized systems admit discrete frequencies, which are theoretically knowable at an arbitrary level of precision. Because these oscillations do not lose any energy, they describe a state of equilibrium, where no internal or external forces disrupt the motion periodicity, and thereby affect its frequency content.

The basic definition of frequency as a reciprocal of the period (Eq. 1), which itself depends on other parameters, goes back to the seminal works on the string by Mersenne [58] and the pendulum by Galileo [59] and is universally found in introductory physics textbooks. According to this definition, the period and frequency are equivalently informative, so measuring only one of the two is sufficient. In theoretical calculations, to avoid circularity— where the period is determined by the frequency and the frequency is determined by the period—it may be necessary to resort to a specific parametric estimation (as in Eqs. 2, 3, and 4).

The temporal regularity of such simple physical systems and others makes them the basis for time measurements. For example, the sundial, pendulum, spring, quartz, and atomic clocks are all based on periodic systems that are highly stable. (In contrast, systems like the hourglass or water clock work on an aperiodic principle, whereby a decay event constitutes a single time unit.) In all cases, to turn the periodic system into a clock, it is necessary to add a counter, which provides the information about the time elapsed according to the number of periods counted. The degree of precision of the clock increases as the oscillation period decreases, but ultimately, evaluating the precision of a finer-unit clock requires other clocks with known precision. Therefore, it is necessary to have a transformation from the temporal or spectral measure to another (say, spatial) measure, which allows for the quantification of precision that is parametric but not circular.

These considerations foreshadow an understanding that both time and frequency can be ultimately used to describe the same physics and may be seen as equivalent.

2.2 Damped oscillations: relaxing the condition of strict periodicity

Relaxing one level of idealization, the simple harmonic oscillator model becomes much more universally applicable when losses are incorporated in the oscillatory motion. For example, the harmonic oscillator (with damping), the pendulum (with friction), and the RLC (resistor inductor capacitor) circuit (Fig. 7) can all be seen as embodiments of this linear ordinary differential equation:

$$\frac{d^2x}{dt^2} + 2a\frac{dx}{dt} + \omega^2 x = 0 \tag{17}$$

where a and ω are determined by the various physical characteristic parameters of the systems (e.g., capacitance, inductance, mass, stiffness, etc.). When $a \neq 0$, there is a damping term that dissipates energy from the system. The general solution here is of the form

$$x = x_0 e^{-at}\cos(\omega_d t - \varphi_0) \quad t > 0 \tag{18}$$

where the additional exponential term represents the decay of the envelope at rate a, as a result of the loss of energy, and ω_d is a lower frequency than that of the simple (lossless) harmonic oscillator, ω, given by

$$\omega_d = \omega\sqrt{1 - \left(\frac{a}{\omega}\right)^2} \tag{19}$$

The motion described by these equations is, strictly speaking, aperiodic, as the amplitude of the motion decreases with every oscillation (Fig. 2 D). Therefore, a narrow definition of periodicity would consider the application of the notion of frequency inadequate [e.g., 54, p. 41]. However, frequency can be readily salvaged, if it is computed only with respect to the phase of the motion, irrespective of the amplitude. Then it would be periodic, as the general solution can be expressed in a similar form to the simple harmonic oscillator, only at frequency ω_d rather than ω. Nevertheless, unlike the free oscillation, the damped oscillation is not defined at $t < 0$, at a time when energy must have been imparted for the motion that can be dissipated later on.

A rearrangement of the solution (18) allows for the definition of so-called complex frequency, which includes both the exponential amplitude as well as the periodic sinusoidal term [60, pp. 18–30]

$$s = \omega + ia \tag{20}$$

The advantage of that is readily seen by using s in the Euler's formula

$$e^{ist} = e^{-at}\left[\cos(\omega t) + i\sin(\omega t)\right] \tag{21}$$

Now each term on the right side of the equation can be used as a solution in the form of Eq. 18. The full expression is also a solution given the superposition property of the linear differential equation 17, although typically only the real part is used for the final result. Using somewhat different reasoning, this solution form has become prevalent in network analysis using the Laplace transform, where cases in which $a < 0$ are of interest too, as regions of circuit instability, as may be the case in some situations implied in the section below. In any event, despite its engineering usefulness, the complex frequency is largely a mathematical convenience that binds together the real frequency parameter and the real decay constant as one complex number that greatly simplifies the analysis of linear networks, such as electronic amplifiers.

2.3 Driven oscillations: the beginning of frequency time-dependence

As is apparent from the damped harmonic oscillator response, any oscillation will eventually cease when enough time has elapsed. Therefore, in order to set such a system in motion in the first place, it is necessary to force[7] it out of its resting state. Thus, the next level of complexity that is added is the effect of an external force, which in one dimension can be described using the inhomogeneous ordinary differential equation [54, pp. 43–60]

$$\frac{d^2x}{dt^2} + 2a\frac{dx}{dt} + \omega^2 x = F(t) \tag{22}$$

The general solution for this equation consists of two terms: the solution to the homogenous equation that is independent of $F(t)$ as in Eq. 18, and a particular solution that depends on the force $F(t)$. Therefore, the oscillator response can be seen as a superposition of the free oscillation—a transient component that eventually dies out—and a forced response that receives energy for its motion from the external force. The particular solution depends on the specific force that drives the oscillator, which is best classified using the force own frequency content. In general, when the force contains discrete frequencies, they also appear in the output of the oscillator, only with modified amplitude and phase compared to how they appear in the force. For example, a force that contains several frequencies that are close in values can give rise to an amplitude modulation type of oscillation (Fig. 2 E). If the force is impulsive, then it starts the oscillator and gives rise to the transient response as in the damped oscillator (Eq. 18). When the force is random (i.e., it does not contain any notable frequencies, but only a certain bandwidth in the spectrum, or "white noise"), then it mirrors the undamped (free) oscillations of the system as in Eq. 6. Forces are often described using the Heaviside step function that models the switching-on moment of the force—it begins at an arbitrary point in time $t = t_0$ before turning into an arbitrary force

[7] Note that the words "forced" and "driven" are used more or less interchangeably in mechanics. Later, in the context of communication and signal processing, it has become more common to refer to "modulation", which often implies that there is an external force at play.

function at $t > t_0$. This classification is particularly useful, taking advantage of the presumed linearity of the system, since many general forces can be represented as a superposition of these basic forces.

Therefore, unlike the free and damped frequency definitions, forced motion may produce frequencies that are observable only from a certain time point as dictated by the external force. This understanding clashes with the requirement for constant (time-independent) frequencies that was entailed by the simple and damped harmonic oscillators. And yet, it is nevertheless possible to retain the definition in which frequencies are time independent and strictly parametric, using a superposition of infinitely long constant frequencies through Fourier analysis, as is explained below.

2.4 Fourier analysis: frequencies that never die out

Fourier analysis is an indispensable set of mathematical tools in the study of all oscillatory phenomena. It originally began from the Fourier series for the study of the heat equation in bounded systems, where it was applied to the string equation as well [61]. In the limit of an unbounded system, the series can be generalized into an integral—the Fourier transform—which has been foundational for the analysis of continuous phenomena. Out of all the integral transforms that are routinely used in harmonic analysis, Fourier analysis rules supreme due to its relative simplicity, comprehensive theory, and due to its common emergence in the solutions of different physical problems. Less common integral transforms all generally rely on the same template, in which the product of an arbitrary function $g(t)$ and a kernel function is integrated over the entire domain to yield the inverse-domain representation of the function $G(\omega)$. Thus, the analysis below holds for other transforms, without loss of generality.

Fourier analysis appears in at least three distinct but complementary contexts in the standard curriculum of undergraduate physics and engineering. First, the Fourier transform organically appears in the derivation of several solutions in wave physics, such as the diffraction integral in optics [56] or in quantum mechanics [62, 63]. Second, along with the closely-related Laplace transform, it is presented as a powerful method for the solution of linear ordinary and partial differential equations [47], which captures the essential dynamics of all oscillatory phenomena. Third, it is a critical tool in signal processing theory, which is used to analyze arbitrary time signals following measurements and synthesis. A related usage is to apply Fourier analysis to get a handle on patterns in the reciprocal domain of various periodic phenomena (e.g., the reciprocal lattice of crystals; [64]).

We will show how the lack of attention in the transition between the Fourier series and the Fourier transform can hint at the mistaken idea that frequency and time are one and the same thing. While the understanding that time and frequency are two separate dimensions is well ingrained in the modern study and applications of harmonic analysis, it is not nearly as obvious from the physics literature, which does not dwell on the definitional intricacies of frequency.

2.4.1 The Fourier series and local or infinite periodicity

Discrete frequency spectrum coincides well with the method of Fourier series expansion, which enables the solution of equations such as Eq. 5 with given boundary conditions. The Fourier series for a piecewise smooth function $x(t)$ over the interval $[t_1, t_2]$ takes the form

$$x(t) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + b_n \sin(n\omega t) \tag{23}$$

where the coefficients a_n and b_n are determined by

$$a_n = \frac{2}{T} \int_{t_1}^{t_2} x(t) \cos\left(\frac{2\pi n t}{T}\right) dt \qquad n = 1, 2, 3, ... \tag{24}$$

$$b_n = \frac{2}{T} \int_{t_1}^{t_2} x(t) \sin\left(\frac{2\pi n t}{T}\right) dt \qquad n = 1, 2, 3, ... \tag{25}$$

and for $n = 0$

$$a_0 = \frac{1}{T} \int_{t_1}^{t_2} x(t) dt \tag{26}$$

The series converges to the original function[8] within the interval $[t_1, t_2]$, with the period defined as $T = t_2 - t_1$. The periodicity gives rise to a series of frequencies $f = \frac{n}{T}$ or angular frequencies $\omega = \frac{2\pi n}{T}$, where for $n = 1$, $f = \frac{1}{T}$ is called the fundamental frequency and the frequencies with $n > 1$ are its harmonics. The parameter a_0 represents the mean of the function, which is a constant by definition, and is often referred to as its DC level, borrowed from electricity[9]. By virtue of the periodicity of all of its components, the Fourier series can be extended to the entire domain $[-\infty, \infty]$ and retain its periodicity in T throughout. An example for the kind of oscillations that can be modeled with Fourier series analysis is given in Fig. 2 F.

Both the original function $x(t)$ and its Fourier series representation are explicit functions of time. By definition, the variable t represents only a segment of the time axis that overlaps with a period corresponding to the time interval. The period is akin to a ruler that is positioned in space to measure the length of an object. Due to the inherent periodicity of the trigonometric functions, the Fourier series is mathematically agnostic as for how much of the time axis is covered by the same ruler shifted, so it can just as well cover the entire time domain with infinitely many periods. For this to work, we can think of the time as being mapped on the unit circle and varying between 0 and T, so that $t = T\phi/2\pi$, where the phase is bound on the interval, $0 \leq \phi \leq 2\pi$. Therefore, unless we count the number of periods within our extended function where $t > t_2$ and $t < t_1$, we are only able to uniquely represent a short temporal duration of

[8]More precisely, the Fourier series converges to the average value between the limits of each value of x, so that it is equal to $(x^+ + x^-)/2$, including at the limits t_1 and t_2, if they are not continuous. In general, the integral over the period must be finite ($\int_{t_1}^{t_2} f(t)dt < \infty$) and the series has to converge in order for the Fourier series to exist. These intricacies are beyond the scope of this discussion, which is concerned with the evolution of the concept of frequency, in part through the wide application of the Fourier analysis.

[9]Direct current (DC) electric power sources such as batteries are typically idealized as a constant voltage with no frequency components.

$t(\mod T)$[10]. The temporal ruler is essentially a clock unit that measures time using a fixed period. Or rather, the entire Fourier-decomposed function is constructed from a set of clocks with diminishing periods, increasing frequency and hence, increasing precision.

In the Fourier series there is no ambiguity in the relationship between the time variable t and the frequency. It is clear that t only serves to locally track the phase within the period being analyzed. Using the phase wrapping property of the periodic trigonometric functions allows the extension of the functions over the entire time domain at no extra effort. However, this entails a strong assumption that the period of the extended function remains unchanged over the entire domain of t. This is tantamount to the requirement that no energy will be lost and no external forces will be applied. However, if we subscribe to the belief that the infinite past and future are unknowable, this assumption cannot remain realistic. It means that time and frequency may only be thought of in a very limited and local (both in time and place) sense just as in 2.1. Hence, under the Fourier series analysis, the frequency and period are also parametric—time and frequency convey the same information only in a very restricted sense.

2.4.2 The Fourier transform, aperiodicity, and the convenience of "zero frequency"

Things become more complicated when the Fourier series is generalized to the Fourier transform, whereby the functions involved cover the entire time domain, rather than a limited segment in which the period is well-defined. This enables the expression of aperiodic functions as the superposition of a continuum of periodic functions with known frequencies (for example, see Fig. 2 G). To obtain the transform, it is instructive to rewrite the Fourier series in its exponential form that is equivalent to Eq. 23,

$$x(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{-\frac{2i\pi n}{T}t} \tag{27}$$

where the coefficients c_n are given by

$$c_n = \frac{1}{T}\int_{t_1}^{t_2} x(t) e^{\frac{2i\pi n}{T}t} dt \tag{28}$$

The original function can then be reconstructed by combining the two expressions

$$x(t) \sim \frac{1}{T}\sum_{n=-\infty}^{\infty}\int_{t_1}^{t_2} x(\hat{t}) e^{\frac{2i\pi n}{T}\hat{t}} e^{-\frac{2i\pi n}{T}t} d\hat{t} \tag{29}$$

In the limit of a very large period $T \to \infty$, equivalent to $\omega \to 0$, a substitution in enabled of the sum with infinitely many terms with a continuous integral, $1/T \to$

[10]By virtue of the phase wrapping property of the trigonometric functions: $\sin(\phi + 2\pi) = \sin(\phi)$ and $\cos(\phi + 2\pi) = \cos(\phi)$.

$\delta\omega/2\pi$, yielding the Fourier integral identity

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\hat{t}) e^{i\omega\hat{t}} e^{-i\omega t} d\hat{t} d\omega \tag{30}$$

where the small frequency interval $\delta\omega$ was replaced with the differential $d\omega$. The Fourier transform itself is then defined as

$$X(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt \tag{31}$$

And similarly the inverse Fourier transform is

$$x(t) = \int_{-\infty}^{\infty} X(\omega) e^{i\omega t} d\omega \tag{32}$$

In the limit of $T \to \infty$, the Fourier transform yields a continuous function of frequency, so frequency is no longer a discrete set of parameters as in all previous cases. An example of the application of the Fourier transform on an aperiodic signal and the effect of diminishing $\delta\omega$ (and an increase of T toward infinity) is illustrated in Fig. 8.

The limit in which $T \to \infty$ also results in mapping of the time axis onto the complex unit circle. In particular, the transform includes the special value of $\omega = 0$ —zero frequency (DC)—whose infinite period $T = \infty$ spans the entire time domain. Just like the a_0 (or c_0) coefficient in the Fourier series, this is essentially the mean of the time function, but here, for mathematical convenience, it stitches the integration around zero to make frequency continuous between the negative and the positive portions of the frequency axis $[-\infty, 0^-]$ and $[0^+, \infty]$ (or, it connects the period axis at $T = -\infty$ and $T = +\infty$). However, zero frequency is an oxymoron—it refers to a mean value of a constant, which has nothing cyclical about it and, hence, neither periodicity nor frequency in the physical sense[11]. But this is not a small detail, as is argued below[12].

2.4.3 Are frequency and time the same thing?

One of the strengths of the Fourier transform is that it models the complete signal or wave and it is not an idealization of periodicity over a fixed interval as is the Fourier series. Therefore, it provides a complete and correct reciprocal representation of the modeled phenomenon. The nuanced zero frequency limit allows for a conceptual switch between the information afforded by the time dimension and that which is given by a continuous frequency variable. It produces frequency-domain representations of the

[11] Mathematically, every constant function $f(t) = C$ satisfies the periodicity condition, by definition, where $f(t+T) = f(t)$ for all t. However, for this case only of a constant function, there is no associated oscillation that corresponds to this condition.

[12] In some applications, it is common to invoke the analytic signal—a complex function whose real part is identical to the measured signal, but whose spectrum does not contain negative frequencies, which are taken to be redundant [65]. Even in this alternative formulation, the zero frequency is always included. The complex signal can be obtained directly from the real signal through the Hilbert transform, which itself has a singularity at $t = 0$ that requires using Cauchy's principal value in order to obtain the limit. Our zero frequency is not a singularity in the mathematical sense, but rather in the physical sense—we argue that it is a categorically different quantity that is referred to by $T = 0$ and $t = 0$ and it their merging that complicates the analysis.

form of $X(\omega)$ that do not depend on time explicitly. Now, with the transformation between $x(t)$ and $X(\omega)$ that has become habitual in analysis (for example, for the solution of dynamical systems in the form of differential equations), the two representations are put on equal footing—each with its own merits—and it is not unheard of to arrive at an implicit understanding that time and frequency domains convey the same information at all times (even if time is taken as the more physical variable of the two; [e.g., 66, pp. 2 and 27]).

There are two important theorems that bolster the view that time and frequency representations are equivalent, both in terms of their total energy content and the information they carry. The first one is Plancharel theorem, which states that the time signal and its spectrum contain equal energy

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega \tag{33}$$

This equality also holds for the more limited case of Fourier series, where the sum of the harmonic component amplitudes in the series contain the same energy as the original signal (Parseval's theorem) [67].

The second relevant theorem states that a continuous bandlimited time signal can be accurately reconstructed from a discrete time series of its amplitude values as long as it is regularly sampled at a frequency that is double or more than the signal bandwidth. Shannon's sampling theorem—a fundamental result for all digital signal processing applications—invokes both the Fourier transform and the Fourier series in its original proof [68, 69]. While real signals have an infinite bandwidth, for all intents and purposes the sampling theorem provides a perfect prescription on how to discretely (digitally) capture continuous (analog) signals and release them back as analog signals with little to no measurable distortion in comparison to the original.

Even with these powerful theorems at hand, their usefulness is limited to the present degree of knowledge of the signal. With a deterministic knowledge of the spectrum, we also get access to perfect predictability of the time signal, and hence completely determined future, whose dynamics is expressed as the superposition of a continuum of sinusoids with constant periods. The time signal and its spectrum are fully accounted for as long as all inputs to the system (external forces) or losses (dissipation of energy) are taken into account in the input to the Fourier analysis. If the energy is not conserved, then the problem formulation must be corrected so to include all the changes, so that Plancharel theorem (Eq. 33) still holds. Otherwise, there is nothing in the unmodified spectrum or time signal that can predict the future with unknown external effects on the system.

This reasoning entails that the idea that time and frequency are the same thing may only be entertained in the case of perfect knowledge of the time signal and its evolution, or alternatively, of the spectrum at infinite bandwidth. No such conflation between time and frequency—really between time and periodicity—would have been possible in the first place, if it were not for the inclusion of zero frequency and the complete mapping of the time and frequency axes in the Fourier integral.

2.4.4 The compact support paradox and the cost of complete determinism

Full determinism of the signal and its Fourier transform encapsulates a deeper discord with the observed reality. A well-known property of the Fourier transform is that a signal can be finite only in one domain (technically referred to as having a bounded or compact support). In other words, the Fourier transform of a signal with finite duration has an infinite bandwidth, whereas a signal with a finite bandwidth has an infinite duration. Slepian [70, 71] commented on this deeply unsatisfactory discrepancy between the mathematics and the reality in which the signals of interest in engineering are finite both in time and in frequency. Slepian attempted to resolve this paradox by making a distinction between the abstract nature of the mathematical constructs that are used in signal analysis and reality itself. He suggested that it is easy to conflate the observed reality and the mathematics, but they are not the same thing. The very notion of frequency, to him, is a construct of convenience and utility that need not have any meaning for the real signal[13]. He finally goes on to identify signals whose energy is effectively concentrated in finite intervals both in the time and in the energy domains and has negligible residual energy outside. Adhering to these signals is what enables sampling, as was prescribed by Shannon's sampling theorem, which works well in real-time and effectively turns this uncomfortable paradox moot.

It should be underlined that there is nothing at fault with Fourier transform itself that ushers the compact support paradox. It may describe reality perfectly well, only at a level we have no access to: inability to garner perfect knowledge about signals in the remote past and future, and inability to measure infinitesimally small amplitudes at arbitrary frequencies, as is predicted to exist by the Fourier calculus.

2.4.5 The uncertainty principle

Signal determinism in Fourier transform analysis is served a final blow in the form of the uncertainty principle, which becomes a thorny issue exactly for the signals that are not as well contained as those highlighted by Slepian [70, 71]. The uncertainty principle first appeared in quantum mechanics, where it was shown that it is impossible to simultaneously measure the position and momentum of a particle, or alternatively, to simultaneously measure its energy and time [63][14]. However, the uncertainty principle is a more general property of any pair of functions that are the Fourier transform of each other (as are the quantum position and momentum and the energy and time

[13]From Slepian [70, p. 293]: "...the words 'bandlimited,' 'start,' 'stop,' and even 'frequency' describe secondary constructs from Facet B of our field. They are abstractions we have introduced into our paper and pencil game for our convenience in working with the model. They require precise specification of the signals in the model at times in the infinitely remote past and in the infinitely distant future. These notions have no meaningful counterpart in Facet A. We are no more able to determine by measurements whether a 'real signal' was always 'zero' before noon today than we are able to determine its continuity with time." He referred to the observed reality as Facet A and to the various analytical tools that are employed to describe and manipulate it as Facet B.

[14]It is arguable whether Heisenberg [63] actually proved the uncertainty principle in his seminal paper, where it appeared in a limited form as $\Delta p \Delta q \sim \hbar$ [72, 73]. In his lecture series later, Heisenberg referred to a rigorous proof by Kennard [74] that came shortly after, which does not generalize to arbitrary wave functions and is therefore flawed [72]. It was followed by a rigorous proof by Weyl [75, pp. 77 and 393–394], credited to Wolfgang Pauli, based on the Schwartz inequality. Other proofs for the uncertainty principle exist, beginning with Robertson [76].

operators), as was proved by Gabor [65] in an analogous way for any time signal to quantum mechanics[15]. In this version, the product of the variances of the time and frequency of any signal—based on its duration and its Fourier spectral bandwidth—has a minimum, true for any arbitrary signal

$$\Delta t \Delta f \geq \frac{1}{4\pi} \tag{34}$$

with equality achieved only in the case of Gaussian-shaped signals, whose spectra are Gaussian as well. In practice, the uncertainty principle constrains the precision in which the frequency content of very short time signals can be determined. Hence, it is also referred to as the time-bandwidth product theorem in signal analysis. Other transforms in harmonic analysis are all constrained by similar uncertainty bounds [79, 80]. While the uncertainty principle in quantum mechanics has been discussed, interpreted, and contested in innumerable texts of physics and philosophy, its signal-analysis counterpart has been accepted rather matter-of-factly as an inevitable constraint to be reckoned with in applied time-frequency analysis [e.g., 81, 82]. According to Cohen [81, p. 45], the signal-analytical uncertainty principle is a misnomer that merely formulates the reciprocal relations between the time signal duration and its bandwidth, similarly to that mentioned in 2.4.4.

2.4.6 Perceptual discrepancy between time-invariant spectra and time-varying signals

As was implied above, equivalence between the time-domain and frequency-domain representations of the system dynamics is retained if all inputs and outputs of the system are accounted for. Therefore, it has been a common practice, especially in physics, to a priori specify the types of forces that drive the system in order to have no ambiguity as for how the system behaves (the simplest cases were discussed in 2.2 and 2.3; see 2.5.1). For linear systems, it also entails the existence of a deterministic, time-independent Fourier spectrum.

In many engineering applications, however, it is necessary to deal with time variations that are not well captured by the time-independent spectrum. The most instructive example of a dynamic signal are different forms of frequency modulation (FM). Although it was originally introduced as a technique for radio communication, its relevance has been shown for naturally occurring signals such as human voice and animal vocalizations [e.g., 83], or as Doppler shift as a result of a moving radiating source. For example, sinusoidal frequency modulation has a time-dependent phase term

$$x(t) = \cos\left[\omega_c t + m \sin(\omega_m t)\right] \tag{35}$$

where ω_c is referred to as the carrier frequency, ω_m is the modulation frequency, and m is the modulation index [84]. The time-dependent phase term implies that it has

[15] The uncertainty property was known in rougher forms before Heisenberg, as was noted by Gabor himself. The most well known one in signal analysis may be Küpfmüller [77], but earlier accounts in physics were noted. In his autobiography, Norbert Wiener recounted a talk he gave in Göttingen in 1925, in which he discussed the uncertainty principle in harmonic analysis [78, pp. 105-108]. He hinted that both Max Born and his student Werner Heisenberg may have attended the talk and could have been influenced by his ideas.

frequency that varies in time. When it is implemented as a stimulus in the right sensory modality, it may be also perceived as such. For example, at audible frequencies with a very slow modulation frequency and large modulation index, this signal sounds like a siren. However, as with all Fourier series representations, the spectrum of this periodic signal is time invariant and contains an infinite series of sinusoids whose amplitudes, in this case, scale as Bessel function of the first kind $J_n(m)$ [85][16]

$$x(t) = \sum_{n=-\infty}^{\infty} J_n(m) \cos(\omega_c t + n\omega_m t) \tag{36}$$

While mathematically exact and analytically important, this is an unsatisfying result conceptually, as it does not capture any spectral changes that are perceived in real-time [e.g., 66, pp. 383–395]. The same goes for linear FM (up- or down-chirp), which has a prohibitively complex Fourier transform [86]. Both signals and their respective spectra are displayed in Fig. 9.

2.5 Beyond the classical Fourier transform

At this point we have pointed at several aspects of the ubiquitous Fourier transform that are not readily reconciled with reality[17]:

- The requirement for complete determinism in order to precisely calculate the spectrum.
- Mathematical signals have infinite support in time or frequency or both, unlike real signals that appear to be finite both in time and in frequency.
- The duration and bandwidth of signals are not freely manipulable or measurable and they have a lower bound imposed by the uncertainty principle.
- The spectra of signals that have a distinct nonstationary character (perceived or measured) do not intuitively reflect their time-varying nature[18].

Many methods have been devised to overcome these limitations, primarily with the intent to be able to analyze nonstationary signals, often in real time. Some methods salvage the classical notion of frequency as a parameter (or rather ignore its intricacies), whereas others only use it as a special case when the signals are stationary. It is not the intention to survey these methods in any detail, but rather pick some of the

[16] In the transition between Fourier series and Fourier transform, every term in the series is transformed into a delta-function pair, representing an infinitely narrow frequency line (e.g., $x(t) = \cos(\omega t) \rightarrow X(\omega) = \delta(-\omega) + \delta(\omega)$). While not strictly valid in the classical usage of the transform that requires finite signals with finite energy content, it nevertheless retains the physical intuition and is routinely used in practical applications.

[17] To this list we can add challenges that are more technical in nature and do not threaten the very definition of frequency as the ones listed in the text: the appearance of negative frequencies, the limitation of applying Fourier analysis to nonlinear systems, and the existence of stochastic signals that do not have a Fourier representation and can only be analyzed statistically.

[18] The distinction between stationary and nonstationary signals and spectra comes from the statistical approach to time series analysis [e.g., 87, pp. 33–36]. Roughly defined, a stationary process is characterized by statistical parameters (e.g., average frequency, covariance, higher-order moments) that are independent of the absolute time point at which they are sampled or calculated and depend only on the relative durations. In the statistical approach to time-frequency analysis, processes (signals) are taken to be instances from ensembles that follow certain probability distributions (see 2.5.2). Nonstationary signals are then those that depend on the absolute time point at which they are sampled.

underlying assumptions and contrast them with the definition and status of frequency we are contending with.

2.5.1 Retaining determinism by force inclusion

The most common method that is employed in classical science and engineering analyses is to include the applied force with the complete time signal model, so that no external forces are ever truly external to the analysis (Fig. 10). The most typical examples are the inclusion of loss (see 2.2), discontinuity in applied force or medium, an impulse, a periodic force, and amplitude-modulated forces. Each one of these types of forces can be readily represented in closed form. They lend themselves to Fourier analysis as well, so they inject their own spectral content into the system. Superposition allows for the generation of arbitrary forces based on these simple building blocks (see 2.3), which is often complemented by multiplication of the force by functions that further shape it and that transform to convolution in the reciprocal domain.

Therefore, this is a "meta-method" of a sort, which presupposes knowledge of all forces impacting the system, so over the evolution of its dynamics, energy is conserved and no new information is introduced into it. Here, frequency and time representations carry exactly the same information by the very statement of the problem. However, as the external forces get more complex and variable, the usefulness of this method quickly degrades, as the resultant spectrum becomes less and less intuitive, as was seen in 2.4.6. And critically, this type of analysis is only as good as the predictability of external forces is guaranteed, as well as the complete knowledge of the system history. This somewhat confusing problem statement is rarely admitted in physics textbooks, whereas openly acknowledging it has been the basis for all modern time-frequency analytical methods.

There is a sense of circularity in this important way of dealing with system dynamics, as it implies that time and frequency representations are equivalent by definition. Because, if we added another force, then it would have been an altogether different problem with its own time and frequency representations. As these problems are defined over the entire time axis, the system under analysis is closed, including losses (Fig. 10 C). Once again, this way of thinking is only possible because of the switch between periodicity and time in the Fourier integral, facilitated by the inclusion of the zero frequency.

2.5.2 Time windowing

By far, the most important analytical step in curtailing the infinitude associated with the fully deterministic Fourier transform is the introduction of time windowing, first introduced by Lord Rayleigh [88]. A time window is a real function that limits the duration of the time signal, so that it is forced to be zero outside of a well-defined time interval. It weights the contributions of the signal at different times in the non-zero portion and completely suppresses any remote past and future contributions. Typical examples are a rectangular window, a triangular window, and a sinusoidal window (i.e., half a period of sine function). The windowed signal itself has a modified spectrum that is the convolution of the full spectrum with the Fourier transform of the window function.

The most typical application of time windowing is to analyze the signal in time frames, so that each frame has its own local spectrum. This procedure produces a two-dimensional time-frequency analysis grid of the signal (see Fig. 11). Most familiarly, this is the underlying procedure in the spectrogram [89] and in the short-time Fourier transform [e.g., 90]. Concatenating the different time frames captures the spectral changes, even though each time frame has constant time-invariant frequencies that are computed at worse precision due to the limited window duration—each time frame is constrained by its own uncertainty principle that is valid for its modified duration and bandwidth [81, pp. 44–52]. However, the spectrum in each time frame comprises of time-independent frequency components. Any apparent change in the frequency content of the system can only be gathered from the changes between the discrete spectra over consecutive time frames. In contrast, analyzing the entire time signal using a single Fourier transform would have contained the same amount of data, but at a much higher spectral resolution that would have not enabled us to intuitively appreciate the time-varying nature of the underlying dynamics, because of the unvarying nature of the Fourier frequencies (for example, compare the three spectrograms in Fig. 11 to their respective long term spectra in the bottom right corner of the figure).

Time windowing is the basis for all time-frequency analyses, where signals are taken as joint probability distributions, or energy density functions, in both time and frequency [65, 91]. In this approach, we define the signal energy density in time and frequency $p(t, \omega)$, so that the fractional energy in each time-frequency grid point of duration $\Delta t = t_2 - t_1$ and bandwidth $\Delta\omega = \omega_2 - \omega_1$ is [81, p. 82–92]

$$p(t, \omega)\Delta t \Delta\omega \tag{37}$$

Typically, the distribution p is normalized, so that the total energy is set to unity.

This approach is as far as it is possible to apply the traditional, parametric frequency definition to nonstationary signal analysis without explicitly making the frequency time dependent.

2.5.3 Instantaneous frequency

The final stop in the process of decisively differentiating between frequency and time was the explicit introduction of time-dependent frequency. Although it was originally introduced as an ad-hoc engineering quantity in radio communication, and despite several associated paradoxes and issues, it is indispensable in communication and electronic engineering and has become a central concept in time-frequency analysis over the century of its existence.

Instantaneous frequency is defined as the time derivative of the phase function $\theta(t)$ [92]

$$\omega(t) = \frac{d\theta(t)}{dt} \qquad\qquad f(t) = \frac{1}{2\pi}\frac{d\theta(t)}{dt} \tag{38}$$

with $\omega(t)$ being the instantaneous angular frequency and $f(t)$ the instantaneous frequency. Therefore, in the case of sinusoidal FM (Eq. 35), the instantaneous frequency is $\omega_c + m\omega_m \cos(\omega_m t)$, whereas for a standard (unmodulated) sinusoidal signal it is simply $\omega = \omega_c$. The general FM signal therefore takes the form [85]

$$x(t) = a \exp\left[i\left(\omega_c t + m\int_{-\infty}^{t}\omega(\tau)d\tau\right)\right] \tag{39}$$

in which the argument of the complex exponential is its instantaneous phase. For example, see Fig. 12 (left) for the instantaneous frequency of a linear FM (up chirp).

In the probabilistic framework of the signal as a two-dimensional probability distribution, the instantaneous frequency is the first frequency moment of $p(t,\omega)$

$$\langle\omega\rangle_t = \frac{1}{p(t)}\int \omega p(t,\omega)d\omega \tag{40}$$

where $p(t)$ is the marginal distribution of the signal with respect to time. This definition entails that the instantaneous frequency is the local frequency average of the signal at time frame Δt (Fig. 12, left and middle). When the signal is stationary, then the local average is equal to the global average, which is then identical to the classical definition of frequency (see Fig. 2 C).

Another definition of the instantaneous frequency that is often invoked is directly derivable from the analytic signal—a complex representation of the time signal, whose real part is equal to (half) the measured signal and its spectrum does not contain any negative frequencies [65]. Due to the symmetry properties of the Fourier transform, it turns out that the real and imaginary parts of the analytic signal are related through the Hilbert transform, so that

$$z(t) = x(t) + i\mathcal{H}[x(t)] \tag{41}$$

where the operator $\mathcal{H}$ designates the Hilbert transform, which is defined as

$$\mathcal{H}[x(t)] \equiv \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{x(t')}{t-t'}dt' \tag{42}$$

The integral is evaluated using the Cauchy principle value (denoted by $\mathcal{P}$) at $t' = t$. For a rigorous derivation of the analytic signal, see, for example, [93, pp. 92–97].

There are many advantages for using the analytic signal in time-frequency analysis, where it has become a key tool, along with the Hilbert transform [81, 94]. One of the conveniences in employing the analytic signal is the ability to represent signals in polar form

$$z(t) = a(t)e^{i\varphi(t)} \tag{43}$$

where $a(t)$ represents instantaneous amplitude and $\varphi(t)$ is the instantaneous phase of the signal. Depending on the context, both may count as a form of modulation whenever it is possible to define a stationary carrier around which the instantaneous variations occur (see Fig. 2 H). The instantaneous frequency can be therefore obtained directly from this expression, by differentiating the argument (the unwrapped phase) according to Eq. 38. For example, see Fig. 12 (middle).

By its very definition, the instantaneous frequency is time dependent and thus any suggestion that it is completely equivalent to time itself would be incoherent. Nevertheless, there are several issues that arise with the definition of the instantaneous frequency and its interpretation, which have contributed to its less than universal adoption.

The first class of issues with instantaneous frequency relates to the clash with the traditional concept of frequency in physics and the lack of intuition that it garners to more complex signals. Already in its introduction, Carson [92] noted that the notion of variable or instantaneous frequency is difficult to reconcile with our physical intuition of what frequency means. Van der Pol [96] also underlined the unintuitive nature of instantaneous frequency compared to the classical frequency concept. As a solution, he analogized it to nonuniform angular motion from classical mechanics, where the angular velocity $\omega(t)$ is determined in an identical way to the instantaneous frequency, by differentiating the phase function.

The second class of issues with the instantaneous frequency are those of mathematical inconsistency and uniqueness applying the particular definition of Eq. 38 in arbitrary cases. Shekel [97] strongly argued against the usage of instantaneous frequency, at least in its standard definition, since it is not unique for a given signal and its usage is both paradoxical and inconsistent. Mandel [98] distinguished between the classical definition of frequency as infinitely periodic and that of the mean frequency of a narrowband signal. He emphasized that the instantaneous frequency as the derivative of the phase may produce values that do not actually appear in the measured (Fourier) spectrum. He went as far as to suggest that the two quantities should not be both thought of as frequency, since it produces an unfortunate ambiguity in our analytical understanding. Difficulties arise also when dealing with broadband signals (for example, see Fig. 2 I), which are best expressed as a sum of narrowband signals, but may not be amenable to a unique decomposition at that [99, 100]. Even then, the instantaneous frequency may give rise to out-of-bandwidth frequencies, or to negative frequencies even after they were eliminated from the spectrum, and be dependent on the signal remote past and future [81, p. 40–41]. Some of these issues may be a result of the mathematical formalism related to the analytic signal itself [94], as there is usually a persistent ambiguity regarding a unique representation of the signal, with respect to the allocation of signal variations to the instantaneous amplitude or to the instantaneous phase and frequency [100]. Some of these challenges make the estimation and interpretation of the instantaneous frequency of real-world, arbitrary signals (of the kind of Fig. 2 I) nontrivial. Different methods in applied harmonic analysis and signal processing contend with this problem, as is illustrated in the the example of Fig. 12 (middle, right).

All in all, the concept of instantaneous frequency has not made it into any mainstream signal processing or physics curricula. While applied researchers still grapple with the intricacies of the term, it is typically omitted from the introduction to the topic of periodic phenomena and harmonic analysis. Its appearances in physics may have been limited to specific problems that tend to be either nonlinear [e.g., 57, 101][19]

[19] For a short review of select appearances of instantaneous frequency in the form of chirps in physics, biology, and engineering, see Flandrin [102, pp. 9–20].

or manifestly modulatory [e.g., 98][20]. It is therefore not featured in any standard introduction to physics, as the Fourier analysis and classical periodicity are—i.e., in the solution of standard differential equations, in the derivation of solution to various physical problems, or in standard signal analysis and processing (2.4). This means that the very idea of a time-dependent frequency remains relatively esoteric in mainstream science.

2.6 Interim discussion

The review of the concept of frequency above loosely followed the historical relaxation of the assumptions that have classically constrained the applicability of the original definition of frequency to strictly periodic oscillations. This eventually led to the analysis of arbitrary waveforms and signals, including aperiodic ones, using tools and concepts that were developed with periodicity in mind. As frequency is calculated from the time signal periodicity, it is inherently intertwined with time, to the point that the two can seem one and the same—two reciprocal quantities that encompass their own domains. According to traditional thinking, the time and frequency domains are complementary and effectively contain the same information, only in different forms.

What this review has attempted to prove, though, is that frequency cannot be considered only dependent on the period and it is also not equivalent to time. In its simplest parametric definition, the period and frequency are always dependent on additional non-temporal parameters. In more advanced formulations, a time-independent frequency entails a completely deterministic worldview, which is analytically impractical and epistemically fantastic. With the addition of Fourier transform to the harmonic analytic toolbox, it has become possible to dispense with strict periodicity, using a one-to-one map between the periodicity axis to that of the time dimension. It gave us access to frequency as a continuous variable, but also to a potential conflation between time and frequency, or rather, time and periodicity. In modern time-frequency analysis, however, a clear, upfront distinction is made between stationary and nonstationary processes, which makes the time-frequency modeling explicitly two dimensional. This is also where the various paradoxes, constraints, and lacunae in the transition between the time and frequency domains are not swept under the carpet. For reasons that we can only speculate over, a similar, explicit recognition that frequency can be independent of time has not made it into physics or philosophy.

The implications of frequency being both time and space independent and the possibility that it is a dimension of reality in its own right are analyzed in the next sections. It should be clarified, however, what we mean by frequency, given the barrage of definitions, nuances, and analytical methods that were presented above, which have not yet converged to a universally agreed upon definition across the sciences. While ideally the instantaneous frequency reduces to the parametric frequency and Fourier spectra in stationary cases, this is not the case in general. Furthermore, there is ambiguity with respect to the choice of signal representation, which means that the instantaneous frequency is relative to the method used to extract it and assumptions behind it. While this is certainly not encouraging when one attempts to study the

[20] An instantaneous-frequency operator in quantum mechanics was derived in Tsang et al. [103].

problem of frequency, we can live with this ambiguity in the context of the proposition put forth in this work: frequency should be counted as a mandatory dimension of reality, regardless of how it is estimated. In simple systems, the frequency would be constant and all its definitions neatly converge. However, rejecting a wholly deterministic view of reality, we have to accept that this applies only to a small subset of systems that are encountered in the real world.

3 Frequency and general properties of the dimensions of perceived reality

The analysis in the previous section has attempted to establish that frequency is a variable that is different from time, despite being tightly interwoven with it. Physically, this implies that frequency corresponds to at least one more degree of freedom in the system dynamic equations it appears in, which can be readily translated to another mathematical dimension. Perceptually, frequency is detected through different dedicated receptors in several sensory modalities, including vision, hearing, and touch, where it gives rise to percepts that are distinct from time and space.

We would like to go further by asking if this mathematical and perceptual degree of freedom may be also cast as an additional dimension of reality that is on equal footing with space and time. In order to do that, it will be instructive to elucidate what properties the four dimensions of reality that are in consensus have that may be generalizable. The following may not be an exhaustive list of properties, but it aims to capture the most key ones that can be applied mathematically, physically, perceptually, and conceptually. Each property listed is explained in terms of space and time and then analyzed also with respect to frequency.

3.1 Nine properties of the known dimensions of reality

The analysis below primarily pertains to a Newtonian conception of space-time, which is largely in line with our phenomenological, sensory, and perceptual version of reality. While taken as a starting point for each property explored, in the extreme cases of very small and very large or fast physics, we have to consult with quantum mechanics and relativity theory, respectively, which complicates the generalization of these phenomenological properties. Nevertheless, the properties listed below are usually general enough to hold at all scales, in spite of occasional strange effects associated with them at the extremities.

When scrutinizing frequency against these general properties, we shall use two primary types of arguments. First, straightforward application of frequency to the logic of the known dimensions. Second, a fortiori arguments about the nature of frequency, given the peculiarity of time and the fact that it has already been widely accepted to be a fundamental dimension of reality.

The property list is summarized in Table 1.

3.1.1 Mandatory coordinates

The most fundamental property of space and time, as we phenomenologically perceive them, is that nothing physical (we know of) exists outside of them. In other words, all elements of matter and energy fields are associated with a specific region in space and interval in time. This is the same for actions and events that take place in specific locations and moments, respectively. Also information, which is less tangible, is still physical and must be stored somewhere [104]. The location and duration of all these things can be therefore translated to particular coordinates—either points in space and moments in time—or to zones defined by coordinates—regions in space and durations in time that are occupied by continuous objects and events.

Both the quantum and relativistic points of view significantly complicate the generalization of this otherwise straightforward property. In quantum physics, nonlocality is the hallmark of quantum entanglement, whereby a particle that is spatially separated from its entangled particle somehow "knows" when the wave function of the other particle collapses [105–109]. However, nonlocality too is defined by spatial coordinates, as limited information makes it across space, between coordinates.

On the other extreme, relativity theory advises us that there may be no meaning for absolute coordinates, given the impossibility to have an agreed upon "now" moment between moving objects at velocities approaching the speed of light [e.g., 110], the space-time expansion of the universe itself and the nonexistence of a universe center [e.g., 111]. According to general relativity, space does not exist independently of matter and the gravitational field that is formed as a result [112]. However, if we entertain that space-time has been given rise to with a particular metric, then we can use a set of coordinates—relative, ad-hoc, local or others. Here, the best that we can do is to confine our relative coordinate system to the light cone on which our inertial system travels, in order to retain its meaningfulness, which would otherwise be lost between non-intersecting light cones. Nevertheless, this limitation is more than acceptable for any phenomenological and perceptual perspective that is pursued in the present context.

It is possible to abstract certain processes and models from this binding space-time framework (for example, in pure mathematics and statistical analysis), or from time only (by setting it as a parameter). An example for dispensing with the time coordinate is the Wheeler-DeWitt equation for quantum gravity [113]. Despite these mathematical representations, they do necessarily entail that anything can exist outside of space-time, or at least outside of the 3D space only. Once an arbitrary reference point in space and time is chosen, every object and event may be associated with coordinates, or a region within space-time, to within some degree of uncertainty. In practice, the mandatory coordinate property of the dimensions has been used to express the dynamical equations of all mechanical, electromagnetic, and quantum systems, which are formulated using the observed functions and their derivatives in space and time.

Can frequency be considered a mandatory coordinate for any element of matter or energy field? A different way to ask this question is whether there is anything in the universe that does not vibrate or oscillate, or can be associated with a frequency or a spectrum of frequencies. In the quantum universe, the answer is trivial, given the two basic relations that are universally applicable—Planck formula $E = \hbar\omega$ and de

Broglie wavelength (matter wave) $k = \frac{\hbar}{p}$, where p is the momentum of the particle and k is its spatial frequency, related to temporal frequency through the particle's velocity. Frequencies may even be applied to electrostatic fields, which can be modeled in quantum field theory as absorption between an electron and itself [e.g., 114]. In macroscopic systems, matter (as small as diatomic molecules) vibrations can manifest as rotations, or complex vibrations in three dimensions, which are often modeled in mechanics as occupying their own three degrees of freedom, apart from the standard three dimensions—all together may be considered the generalized coordinates of the system. In relativistic formulation, frequency may occupy a latent degree of freedom through the speed of light, which is a constant (frequency-independent) only in vacuum where $c = \frac{\omega}{k}$, but is generally dispersive as $k = k(\omega)$.

The generalized notion of frequency afforded by the Fourier transform entails that even aperiodic entities can be expressed using periodic functions. It also associates constant, unvarying functions with zero frequency. Problematic as it may be (2.4.2), we can invoke this classical framework to attach frequency coordinates (values or regions from the Fourier spectrum) to all matter and energy distributions. Uncertainty and limits to deterministic knowledge would then ensure that it is at least partially independent of the time coordinate. The more elaborate and realistic instantaneous-frequency definition can be applied to any time-varying function with no ambiguity with regards to its independence from the time coordinate (2.5.3).

Perceptually, all of our modalities that interface with the external environment are frequency dependent—either directly through its sensory organ filters (vision, hearing, touch, balance) or indirectly in all other senses, including those that may be non-spectral (olfaction, gustation, pain, etc.). An indirect frequency sensation can be attributed to any sense once we apply time-frequency analytical tools to the functions that describes the stimuli or their sensed response (e.g., the sensation of sweetness as a function of the spatial-temporal concentration of sugar on the tongue; [115, 116]). This frequency-dependence—likely an aperiodic one—may better express the modulation domain of the signal rather than a carrier frequency per se.

In summary, there is no difficulty to assign a frequency coordinate to any physical variable that is characterized by space and time. In the simplest cases, the frequency is either reduced to a constant, or is assigned the 0 Hz value, without loss of generality.

3.1.2 Movement

The basic dynamic property of space is that material objects and radiation of any kind can move about the geometry spanned by the spatial three dimensions, as long as the path is contiguous (i.e., without jumps[21]). As for time, movement appears to be both contiguous and restricted to one direction—only from past to future—despite numerous works of fiction that dispensed with this limitation [118], beginning with H. G. Wells [119]. This produces the fundamental relationship between cause and effect, where the former must precede the latter in order to comply with our understanding of reality [120]. However, at high velocities, the relative speed in which there is movement to the future does appear to vary between observers moving at different velocities than

[21] Note that even the speculative wormholes of general theory of relativity only allow for *apparent* jumps in Euclidean space due to extreme local features in curved space-time topology [e.g., 117].

the moving object they observe. The relativistic "proper time" of the moving object captures this difference.

What can the meaning be of "movement in frequency"? "Moving about" in frequency is qualitatively different from moving in the spatial dimensions and is unlike movement in time (which is ordinarily thought to be restricted to the future direction). The answer to this question depends on the frequency definition that is being looked at. Fourier spectrum is by definition stationary and each frequency component is infinitely long and can be thought of as inertial if taken in isolation. The totality of (infinitely) many such inertial components gives rise to dense spectra that can appear as frequency varying in the time domain (Fig. 9). Despite its mathematical correctness, this solution seems to be missing the point. A more accessible vantage point may be to look at the general decomposition of signals to carrier and envelope terms (Eq. 43; see example in Fig. 2 H). In the simplest of cases, there is a high-frequency carrier, or a mean frequency, which remains fixed and all spectral changes in time can be associated with the slow-varying complex envelope around the carrier. But this decomposition is not unique and it may be difficult to pinpoint as for where the change lies—is the normal mode (associated with the carrier) being changed, or only the force that impacts it (associated with the envelope)? More complex systems contain multiple normal modes (i.e., frequency components or carriers), which tend to have an even more ambiguous decomposition (e.g., Fig. 2 I). These systems are generally not continuous in spectrum, which is concentrated around the carriers and exhibit "spectral holes" between them. Movement in frequency in these cases may be complex and not uniform across all modes, so multiple trajectories may be required to describe it in the frequency dimension. Despite this marked ambiguity and high degree of complexity, there is no conceptual difficulty in associating spectral changes with particular frequency components, which may then appear to be moving like objects in space, at least locally.

It is perhaps instructive to make a distinction between measurable spectral changes that are inertial versus those that require energy transfer into or out of the system. This is because time-frequency analysis alone may not be able to distinguish between the two without additional information about the system and its boundary conditions. For example, the classical Doppler shift effect can be measured as frequency modulation by an observer relative to a moving source, even if both observer and source are inertial in their own systems. For the static observer, the moving object may well count as entering its otherwise static system and injecting energy into it. If the system is taken as both observer and source, then the net energy is constant and the entire movement can show in the stationary Fourier spectrum as numerous spectral lines, as in linear frequency modulation, for example (Fig. 9; see 2.5.1 regarding including all forces in the problem).

In another instructive example, if we return to Van Der Pol's analogy between the instantaneous frequency definition and angular velocity—both being the derivative of the phase function with respect to time ([96]; 2.5.3)—then the spectral interpretation of the time-dependent motion of a planet in an elliptical orbit around a star (i.e., with variable angular velocity) may be a puzzling case, since there is no net energy transfer there between the star and the planet and conservation of energy is maintained by instantaneously varying radial and angular velocities [55, pp. 70–127][22]. Therefore, in this case, the inertial decomposition offered by the Fourier analysis may be much more intuitive and correct, as any apparent modulation in the observation is fully accounted for by all the observable forces, all of which are conservative. Therefore, we may choose to not register any movement in frequency in this system.

Unlike movement in space and time, frequency jumps are possible (i.e., between two frequencies $f_1 \neq f_2$) without having to sweep across all the values in between the two. It is the standard observation in the spectrum of quantum transitions between energy states. Macroscopically, frequency jumps are possible in every modality when a generator is swapped or modulated quickly (e.g., a loudspeaker may produce two well-separated tones with no detectable sweep or non-tonal noise between them).

Therefore, while spectral movement is certainly possible and common, frequency jumps appear to be just as common, unlike jumps that are prohibited in the spatial and temporal dimensions.

3.1.3 Collisions and interactions

For objects and fields that overlap in their coordinates or are positioned within reach of a certain far field, it is expected to observe some kind of interaction (object-object, object-field, or field-field). Depending on the specifics, these include collisions, deformations, attraction and repulsion, phase transformations, chemical and nuclear reactions, and others.

Interference between two waves is observed when their carrier frequencies are either identical or very close (it is a given that the waves overlap in space and time). There is some associative resemblance here to interaction between rigid objects in three dimensions: interference may be thought of as an extension of the concept of collision into the spectral dimension, where the impact depends on the frequency (as well as phase and amplitude) difference between the waves and the final product may not resemble the input waves, suggesting a significant interaction effect. This aspect of the 5D representation that includes frequency was alluded to by Wiener and Struik [121], who suggested that coherence could be explained more readily through the addition of another dimension to the quantum wave functions, in line with the 5D theories of Kaluza and Klein.

More complex interactions between frequencies in different modalities may be possible in the context of special phenomena, such as the acousto-optic effect, or the piezoelectric effect. Unlike interference, these interactions generally lead to modulation and energy transformation between waves, so they are observable at a much broader range of frequencies than interference, as long as the waveforms overlap over the same spatial and temporal coordinates.

3.1.4 Mathematical independence

Mathematically, quantities that take up their own dimensions cannot be expressed using other dimensional quantities alone. Thus, each dimension contains some information that is not found in the other dimensions. Realistically, however, quantities that

[22] Note that it is not customary to look at the rotation spectrum of planets or talk about their instantaneous frequency, but rather about their average periods and time-dependent angular velocities.

manifest within the spatial and temporal dimensions are often interdependent, so there may be fewer degrees of freedom than dimensions, due to various constraints that tie the dimensional dependencies together. For example, in certain mechanical problems (e.g., in the central force problem) this enables the parametrization of the trajectory using time—effectively eliminating at least one variable / coordinate / dimension from the solution.

When it comes to frequency, its interdependence with time is very high in conservative systems, but even there it not total, as was argued throughout 2. It is not possible to arbitrarily specify a signal both in time and frequency without some constraints applying. The uncertainty principle is one such constraint that is most evident for very narrow distributions in time (duration) or frequency (bandwidth) (2.4.5). Cohen [81, pp. 127–128] discussed the concept of signal representability (or realizability), where arbitrary two-dimensional time-frequency (Wigner-Ville) distributions can be specified, in spite of the fact that they may not be correspond to any realizable signals in actuality.

Another fundamental physical constraint ties the relation between the temporal and spatial frequencies for a particular medium. This goes back to the frequency dependence of the wave velocity in the medium, which is constant in vacuum for electromagnetic radiation, nearly constant for light frequencies in air, but is variable in most other conditions. Audio-frequency sound waves in standard atmospheric conditions are also nearly dispersionless, at least for short distances and relatively low frequencies [122, pp. 122–124]. In all other media, some dispersion should be assumed [123], so that

$$k = k(\omega) \qquad \omega = \omega(k) \tag{44}$$

both are alternative expressions that satisfy the condition that the propagation speed in the medium is $c = \omega/k$. The dispersion relations are defined by the medium, which is defined by spatial parameters such as material type and density. Some regions may have those properties time dependent as well. This makes the dimensions interdependent in a complex way, possibly leading to the number of degrees of freedom to be smaller than the number of dimensions. This is visually summarized in Fig. 13, which is titled somewhat facetiously "the frequency accessibility paradox", which illustrates that frequency (be it a dimension or other) cannot be completely disentangled from the temporal and spatial dimensions.

3.1.5 Scalability

Objects both in space and in time are scalable. There is neither a mathematical nor a conceptual difficulty to stretch and compress them either spatially or temporally, although in practice it is not always physically feasible. The fabric of space-time itself seems to be continuous, so that scaling objects within it does not lead to odd discretization effects, at least not on a macroscopic level of observation. Also, for every intent and purpose, both space and time are of the same size as the universe itself, so in practice one does not run into a ceiling effect as a result of overstretching objects in space-time. On the quantum level, discretization effects (and subsequent limitations on observations) of the order of the Planck constant—the Planck length $l_p \approx 1.616 \cdot 10^{-35}$

m and Planck time $t_p \approx 5.390 \cdot 10^{-44}$ s—have been theorized [124–127]. If correct, these would lead to respective discretization effects on scaling.

Macroscopic frequency is scalable, depending on how it is generated. Mathematically, the frequency range may be infinite and real, so there are no minimum or maximum values that it can take. Physically, though, the frequencies are sometimes defined to be strictly non-negative (see Footnote 12). Additionally, there may be an upper bound to how high the frequency can be. In vibrational systems, the spectrum usually becomes compressed if the spatial and temporal dimensions of the system are stretched (and vice versa—a compressed or stretched spectrum suggests a respective change in the spatial and temporal dimensions). Here, frequency is constrained by the other dimensions, yet in order to be fully determined it depends on additional extra-dimensional parameters of the system (e.g., medium density, elastic properties, etc.). Note that to the extent that the modulation and carrier frequencies can be distinguished, they are usually associated with distinct functions of space and time, and hence with a distinct spectrum that may be not independently scalable, depending on the spatial and temporal sources of the carrier and modulation domain functions. On the quantum level, the energy states of bounded quantum systems (unlike free particles) are generally determined by combinations of constants, which are not as malleable as macroscopic parameters may be. Thus, free scaling is generally unavailable here due to discretization. A continuous frequency scale is then obtained only through the effects of decoherence in larger (classical / macroscopic) scales.

In summary, frequency is scalable, but in a way that is generally interdependent on the scaling of spatial and temporal properties of the system, and on discretization effects on the quantum scale.

3.1.6 Modulability

Every force or parameter that is associated with the dynamics of the wave (or signal) may be spatially or temporally varied, in what is referred to as modulation. Typically, modulation refers to external forcing of a parameter or variable that would be otherwise stationary. In general, since the wave is defined over space and time, modulating over either a spatial or the temporal dimension would necessarily have an effect on the other.

As was already implied above, frequency may be directly modulated, as is commonly done in radio communication, acoustics, and music, among many other domains. In the sound modality, different musical instruments employ various methods of modulating their pitch either continuously or discretely, to produce certain timbral, melodic, and harmonic aspects of the music. In hearing research, it is common to measure the perceptual response to spectrotemporal modulation that is defined both on the temporal and the spectral dimensions of the signal [128, 129]. In vision and optics, changes of the carrier frequency[23] of the optical objects generally lead to changes in perceived color, as is encoded by the visual system [e.g., 130, 131]. Changes in the spatial frequency content of the optical object relate to how its coarse or fine details appear, as

[23] Within the visual range, it is typical to refer to the wavelength of the light waves, rather than to their frequencies. Most references to frequencies relate to spatial frequencies instead of the temporal frequency of the carrier, which are used in describing the geometry of the optical object.

can be predicted from the modulation transfer function of the imaging system, that is the eye [56]. All in all, disentangling the dimensional contribution of the modulation may be somewhat artificial, since few (if any) changes to the signal can truly manifest one-dimensionally.

3.1.7 Invariance typicality

As the substrate of physical existence, space tends to be remarkably tolerant to any change in the absolute coordinates (homogeneity) and to the direction of movement (isotropy). A similar property can be related to change in absolute time coordinates (stationarity). Put differently, numerous phenomena appear to be both time-invariant and space-invariant (translation invariant). Effects of memory and nonlinearity locally disrupt these invariances, but most events and systems seem to be indifferent to where they are positioned in the universe, as long as all the relative relationships to the various surrounding media, fields, and forces are equal between the two positions.

There are many situations in which frequency does not have a direct effect on the wave dynamics, which are captured by the geometrical approximation that is regularly used in both optics and acoustics [132, 133]. In this case, the only frequency effects would be those dictated by the medium properties, such as the wave speed, and ultimately the wave phase may be neglected as it is the intensity that is being detected. This describes well problems in which the wavelength is much shorter than the relevant spatial boundaries of the system (e.g., in concert hall acoustical design of sound incidence and reflections between the orchestra to the audience). Spectral invariance may also apply to musical melodies, which may be transposed to a different scale or register—a relative change in frequencies. In this case, it would still retain its melodic identity, which is determined by the relative intervals between the notes in the melody, and their respective durations. However, perceptually, transposition only applies as long as it is within the melodic audio range of hearing [134, 135].

Spectral invariance appears to break down more often in common situations than do spatial and temporal invariances. A strong spectral dependence is found with various interaction effects that are exclusive for certain frequencies or wavelengths, which in analogy could be compared to very crowded regions in space, or a large density of events (in time). For example, molecular spectroscopy is concentrated on mid-infrared frequency with the Raman "fingerprint region" loosely defined to be at 1300-900 cm^{-1} [e.g., 136]. Such a molecular spectrum cannot be thought of as relative, since its absolute frequencies determine the very identity of the molecule.

All in all, while spectral invariance can characterize many classical systems, in reality parts of the electromagnetic spectrum have unique interactions that make many systems spectral variant. Also, as all sensory systems are bandlimited, the stimulus spectrum invariance is of limited extent.

To the above seven properties, we shall add two more that are more narrowly related to our perceptual experience as humans and possibly to our non-human relatives.

3.1.8 Tangibility

Tangibility refers primarily to the property of objects that can be perceived by touch—material objects that evoke tactile sensations when touched. Somewhat more ambiguously, tangibility also refers to the property of perceived by the senses: "*real and not imaginary; able to be shown, touched, or experienced*", or "*a real thing that exists in a physical way*"(Cambridge Dictionary[24]). Or, "*capable of being perceived especially by the sense of touch: palpable*", or "*substantially real: material*", "*capable of being precisely identified or realized by the mind*" (Merriam-Webster Dictionary[25]).

A dimensional perspective on the concept of tangibility would attribute it to the space that the objects occupy. Arguably, solids are more tangible than liquids, whereas gases may be altogether intangible, especially if they are colorless. Also microscopic objects the size of microbes or smaller are not amenable to touch, and macroscopic objects that are too large, can be touched but their full size cannot be truly appreciated (like a wall, or soil, or a planet). In all other cases, the information combined from the touch and visual modalities is often consistent and complementary, so what looks tangible is indeed tangible, and what feels tangible is generally visible.

In contrast to the spatial attribute of the objects, the time dimension is not directly tangible—only indirectly, through the understanding of dynamics and cause and effect and how objects change as a result. Objects that continually change in time may be perceived as lacking in tangibility if their properties cannot be confidently pinned down. Auditory objects in themselves are also not tangible if they are not accompanied by additional inputs from other modalities [e.g. 137].

Is frequency tangible? Yes and no. If tangibility relates exclusively to touch, then the effects of frequency are certainly felt across space and time. For example, the spatial frequency spectrum of objects relates to their contour and texture. Their temporal frequency content relates to felt vibrations upon touching. Touching an object dynamically (stroking, rubbing, hitting, etc.) excites a combined response of its spatial and temporal frequencies. Sound is not tangible per se, but has no meaning without frequency or pitch (even if perceived as pitch-less, as white noise is, for example). And in vision, frequency gives us color, which is not a property that can be felt by touch either. All of these are no less tangible than the time dimension, but they are less tangible than the spatial dimensions, which are inseparable from our senses of positioning and movement of objects.

3.1.9 Sensory association

Spatial coordinates are most immediately associated with vision and touch (see Fig. 1), which elicit the effect of tangibility. Time is much more abstract than space and we become conscious of it as a supra-modal percept that is not peripherally detected with any one sense. The passage of time has been most strongly linked with hearing [30, pp. 5–8], yet stimuli to all senses have indispensable temporal as well as spatial cues, which become mandatory in eliciting the actual perceptions and the resultant information that maps the objects in the environment (Fig. 1).

[24] https://dictionary.cambridge.org/dictionary/english/tangible, accessed 30.11.2023.
[25] https://www.merriam-webster.com/dictionary/tangible, accessed 30.11.2023.

As was noted in the introduction, both vision and hearing are strongly associated with frequency. Hearing is strongly associated with temporal frequencies—the frequencies that determine pitch, timbre, harmony, and melody. Temporal frequencies in hearing can double up both in the carrier and in the modulation domain, which can sometimes have complex interrelationships. Slow modulatory frequencies determine rhythm, beating, level changes, etc. Vision, in contradistinction, is split between temporal and spatial frequencies. Colors—the percepts stemming from the broad tuning of photoreceptors to three different ranges of the electromagnetic spectrum—are associated with the temporal frequencies of the light spectrum from the objects (usually discussed in terms of wavelengths), factored as carrier frequencies. The objects themselves are often defined using spatial frequencies, which are the go-to spectrum when visual images are discussed [e.g., 56]. As was discussed in 3.1.8, the object surface can be thought of as spatial frequencies, which when dynamically moved, transform to temporal frequencies. If the object internally vibrates, then the vibration frequency is temporal and associated with the carrier domain, whereas the textural frequencies are modulation domain. Although olfactory detection does not seem to be based on spectral principles, as long as different substances can be uniquely identified using their vibrational spectra, then their spectrum becomes a relevant parameter in objectively characterizing olfactory stimuli. Other senses are more narrowly designed to target very specific objects, where frequency may not be the most important parameter.

3.2 Frequency as a dimensional property of reality

Ultimately, this analysis only supplements the one in the previous section (2), as we only have two kinds of dimensions to infer from the properties of a general dimension, which may be insufficient. Space and time do not behave identically, and frequency too does not exactly follow these general properties in an identical manner to either space or time. The alternative possibility—that frequency is an important variable but not a dimension in its own right—may be considered where the properties do not apply as well to frequency, unlike space and time.

Three properties listed above may be considered odd when applied to frequency: movement (3.1.2), mathematical independence (3.1.4), and invariance typicality (3.1.7). Movement in frequency stands out, because unlike space and time, it appears that jumping between frequencies in a discontinuous way is possible. This may be due to quantum effects, but can also apply to classical systems, depending on the definition of the frequency source. However, time too is subjected to a unique rule of movement in a single direction only, at least according to the current knowledge. Therefore, this may not be a significant oddity. As for mathematical independence, this was argued throughout 2, but is an elusive thing to demonstrate, mainly because of the interdependence frequency has with time. In many physical models, frequency is absent and can only be made explicit through the inclusion of dispersion. This is different than space and time, which tend to be mathematically explicit. Once again, it may not be a significant difference in its own right to disqualify it from being a dimension, but rather a unique feature it has, which may have historically led to it being elusive.

The last property that stands out—that of systems often being not frequency-invariant—may be the most interesting one, because it reflects many of the properties

Property	Space	Time	Frequency
Mandatory coordinates	Yes. Nonlocality may be taken to violate it, in case of entanglement. Relativistic effects confine the meaningful coordinates to the light cone of the inertial system on which we travel.		Yes, depending on the level of analysis
Movement	In any direction, continuous	Only forward, continuous	Backward and forward, jumps are allowed, depending on the definition
Collisions and interactions	With intersecting coordinates, or overlapping fields	With co-occuring events	When the (carrier) frequencies are nearly the same; interference
Mathematical independence	Yes	Yes	Partially time-dependent. It is down to the uncertainty principle in conservative systems. In nonconservative systems it has much less dependence on time.
Scalability	Yes	Yes	Yes, but usually interdependent on space and time.
Modulability	Yes	Yes	Yes
Invariance typicality	Yes	Yes	Limited
Tangibility	Yes	Indirectly	More than time, but less than space
Sensory association	Vision, touch	Hearing	Hearing and color vision (temporal frequencies), vision (spatial frequencies), touch (temporal and spatial frequencies)

Table 1 A summary of the nine properties of 3.1 and how they apply to the space, time, and arguably, frequency dimensions.

that make our reality the way it is. These spectral "islands" correspond to particles, atoms, molecules, object sizes and shapes, duration and progression of events, etc. In many cases, our senses are tuned to receive information at these frequencies and not in others, in a way that ends up being perceived uniquely (as color, sound, touch, etc.). A shift in these frequencies cannot be done without affecting the entire cascade of physical, chemical, and biological filters that depend on the absolute values of these frequencies. Whether this interaction between frequency and reality is a cause for disqualifying frequency from the dimensional count may ultimately be a philosophical choice. We argue that this is what makes frequency special, as it ultimately leads for answers to the "What?" question, just as space answers the "Where?" and time the "When?" questions.

4 Synthesis

The case for frequency as a mandatory dimension of reality has been argued in the above. While the idea of including a complex concept such as frequency in the standard count of the dimensions may come across as an abstract imposition, it was shown that logically, perceptually, physically, and mathematically, excluding frequency would necessarily be inconsistent with modern science and engineering. According to the analysis in 2 and 3, the exclusion of frequency as a dimension may only be justified if either

1. Time is rejected from the standard count of the obligatory dimensions of reality, or
2. The universe is fully deterministic with total knowledge of past and future, so that frequency can be retained as either a parameter or a stationary variable

Both alternatives carry substantial metaphysical weight that may not stand to reason with the normal intuitive, phenomenological perception of reality, nor with standard (Newtonian) physics. This does not mean that they are impossible, but rather that they either constitute a step backwards from the perceptual understanding that is a pillar of this work (i.e., that frequency is not constant and the universe is not completely deterministic), or that it is a step backwards in our physical knowledge (i.e., that time is not a fundamental dimension—rather, it is emergent from space or from yet more primitive, irreducible physical entities). Thus, while we may reject these two alternative explanations as not corresponding well to our standard perceptual and experiential reality, we logically acknowledge that they represent possibilities that exist notwithstanding and may better apply to some physical systems. This reasoning is synthesized into the following theorem, which is a corollary of all of the above.

Theorem 1. *Only one of these three propositions can be simultaneously true:*

P1. Time is not a fundamental, obligatory dimension of reality.

P2. The universe is fully deterministic with total knowledge of past and future.

P3. Frequency is a fundamental dimension of reality.

The three propositions may be immediately interpreted as referring to 3D, 4D, and 5D conceptualizations of reality.

The choice of wording "*only one is simultaneously correct*" in the theorem should be clarified. The three propositions are clearly mutually exclusive, in a logical sense. But we would like to underscore that the particular proposition that is in effect may not be permanent, as might be implied by the logical relation alone. This would normally be worded by stating it as "only one of the three propositions can occur at one time". But since the propositions themselves directly frame the existence of time, this seems to be somewhat circular, as though the choice takes place in time but is outside of time. The concept of simultaneity usually refers to things that happen together in time as well. But the word "simultaneous" is also defined as "*satisfied by the same values of the variables*[26]." Therefore, "simultaneous" seems somewhat more appropriate and less committing in this context, although it results in this odd wording.

The theorem is directly deduced from the analysis of 2 and 3. A shorter informal proof of this theorem is as following. In this proof we take physics and physical systems

[26] Merriam Webster Dictionary, accessed 11.1.2024, https://www.merriam-webster.com/dictionary/simultaneous.

as a sample representation of reality, and in turn, of the universe (but see 4.1). Starting from P1, we assume that time is not a dimension. This can mean either one of two things: time is a parameter, or the system can be described as a stationary process (a statistical distribution), for which there is no difference in the choice of a reference time point, as there is no meaning to past and future—all is present. This directly entails that there are neither causes nor effects—the system is conservative and its dynamics has no beginning and will have no end. Thus, the notion of determinism is meaningless. Frequency, if it has any meaning, describes infinite (unvarying) periodicity that is observable along arbitrary time intervals—durations that are abstracted from time as a dimension that has past and future. Thus, frequency describes stationary physics and is not a free variable either, and hence not a dimension. In P2, we assume determinism, so strict cause and effect exist, as do past and future. Hence, time is a dimension. Given determinism, we are allowed to apply the Fourier integral and obtain a frequency representation from any time measurement (that can represent any physical quantity). The frequency representation is completely determined by the time function, which means that frequency on its own is not independent, and thus not a dimension. Finally, in P3, we start from frequency being an independent dimension, which means that it can vary more or less independently in time. A fortiori, therefore, time is a dimension too. Inasmuch as the spectrum represents the physics of a given system, it is not constant in time, so it is only as informative as the information gathered in the present moment that defines it (i.e., over a finite time window). Therefore, it contains only limited or no information about the remote past and future, and therefore does not correspond to a reality whereby the remote past inevitably causes the present, which will cause the future. In other words, different pasts may have led to the present, which can in turn lead to different futures. Therefore, determinism does not apply here. Hence, P1, P2, and P3 are mutually exclusive. ■

To complement this proof, we can examine at the remaining five propositions that are never true (All propositions are summarized in the truth table in 2). Let us try to understand what each of these statements entail and why they are impossible:

~~P4~~. Time is a dimension; determinism; frequency is a dimension—Although time and frequency are two degrees of freedom of the system, its present behavior is completely determined by its past and its future is predetermined. Therefore, it is possible to obtain the frequency at all times from the predetermined time signal using the Fourier integral. But this means that the frequency is not independent of time and hence not a dimension—a contradiction.

~~P5~~. Time is not a dimension; no determinism; frequency is a dimension, *and*,

~~P6~~. Time is not a dimension; determinism; frequency is a dimension—For this proposition and the previous one to be true, it must be possible to continuously vary the frequency f in arbitrary steps δf and observe a respective change in time δt. But any change in frequency must only take place in space, if time itself is not a dimension. However, a spatial change in frequency would make time nonuniform across space, which would contradict its non-dimensionality. Hence, these propositions are incoherent and the question of determinism is moot.

~~P7~~. Time is not a dimension; determinism; frequency is not a dimension—Determinism entails the rigid existence of cause and effect that can be observed

#	$\bar{t}$	D	f	Availability
P1	**Yes**	No	No	**Yes**
P2	No	**Yes**	No	**Yes**
P3	No	No	**Yes**	**Yes**
~~P4~~	No	Yes	Yes	No
~~P5~~	Yes	No	Yes	No
~~P6~~	Yes	Yes	Yes	No
~~P7~~	Yes	Yes	No	No
~~P8~~	No	No	No	No

Table 2 Truth table describing the different available and unavailable combinations of non-dimensional time, determinism, and dimensional frequency. The following symbols are used: $\bar{t}$ – Time is not a dimension; D – Determinism; f – Frequency is a dimension.

over time. But if time is non-dimensional, this notion becomes incoherent—with the present causing the present—or trivial, with a constant, unvarying physics.

~~P8~~. Time is a dimension; no determinism; frequency is not a dimension—Regardless of the degree of knowledge about a time signal, its respective frequency cannot be precisely determined from it. For this to be true, it means that frequency reflects a degree of freedom that is independent of time, which contradicts its non-dimensionality.

Finally, another way to break down this complex statement of Theorem 1 is noting the number of pathways that exist for time, frequency, and determinism to take place or not:

- Time is not a dimension (1 pathway)
- Determinism (1 pathway)
- Frequency is a dimension (1 pathway)
- Time is a dimension (2 pathways)
- No determinism (2 pathways*; see 4.2)
- Frequency is not a dimension (2 pathways)

Theorem 1 was derived by way of deduction and elimination, by contrasting the different definitions of frequency and how they relate to time. While time and determinism are topics that have been often considered in the physics and philosophy literatures, the obligatory addition of frequency into this discussion is novel. The theorem itself, though, may be understood at different levels of abstraction, with more or less metaphysical baggage that is not strictly aimed at in this work. Nonetheless, we shall make a few cautious strides in an attempt to unpack two aspects of the theorem.

4.1 Choice of system: isolated, closed including losses, and open

There are two immediate ways to understand Theorem 1. One way is to take it as an ontological statement, directly pertaining to the entire universe, as long as the three concepts of time, frequency, and determinism are employed as they are in contemporary physics. This understanding would have the universe as fixed on either P1, P2, or P3.

The other way to understand the theorem and its three propositions (or modes) is less rigid and allows for the the propositions to apply in different physical situations, corresponding to the three abstract systems depicted in Fig. 10. P1 corresponds to the situation in Fig. 10 A, in which a system is truly isolated, so we have no access to it as such as long as its boundary is intact. Without being on the inside, the best we can do is to get average quantities that are time invariant, based on what we know that the system contains. Once inside the system, time may appear as a mere parameter. This is another way to say that time is not a true dimension here, because it plays no role in the observed dynamics. P2 is a closed system that includes the whole universe, as is illustrated on Fig. 10 C. It is fully conservative, but it is possible to define loss within the system. If this system corresponds to the whole universe, and it is known to be finite, then it already includes everything by definition: it is impossible to include additional information (addition of forces, loss of energy). Hence, there is complete determinism, as the past dynamics causes the present and future. The inclusion of loss mechanisms (or decoherence in some contexts) that dissipate energy and information within the system gives meaning to time as a dimension as we intuitively know it, as it gives rise to dynamics that can be clearly associated with cause and effect. If we breach the system boundaries (or if the universe is not finite), we would be moving to P3 and the open system of Fig. 10 B. This mode remains in effect as long as the system and its environment retain their identities. If they are merged, or taken as a whole either analytically or in observation, we return to P2 (Fig. 10 C).

4.2 Determinism

Determinism appears in two varieties here—explicit and implicit. Explicitly, there is the low-level determinism that follows from the interdependence of frequency and time. It encompasses every event, movement, and bit of information to have ever existed. There are no real inputs nor outputs to such a closed system we call the universe, because there is nothing external to it, by definition. This inevitably leads to a predetermined future, since the information about it is already contained in the system's past.

Implicitly, another form of determinism emerges from the theorem, which entails an unyielding maintenance of the system boundaries, or relationships with other systems internal or external to it. If it is an isolated system of the form of P2, it is akin to a mini-universe of a finite extent. As long as its boundaries are maintained, implicit determinism is maintained, in the sense that the proposition associated with it becomes an immutable mode of being. However, a play on the definition of the system at hand, by letting its boundaries vary irregularly, can cause a change of mode (i.e., corresponding to P1, P2, or P3), which is an alternative avenue to relax determinism. In other words, this sort of determinism is the adherence to a fixed mode rather than moving between the three modes. In this sense, a closed system that corresponds to P2 and remains as P2 is doubly deterministic.

5 Discussion

The above analysis implies that a five-dimensional reality has merit, based on perceptual, mathematical, physical, and epistemic grounds. Theorem 1 further suggests that a five-dimensional count applies where an open system configuration describes the physical situation best. Several underlying issues regarding the choice of frequency as a dimension and a few further challenges are considered below.

5.1 Is frequency the correct quantity?

The entire argumentation in the above sections revolved around the temporal variation in frequency, which in itself is related to all other fundamental parameters or variables in oscillatory and wave motion. We argue that frequency is the most informative of them all, although it may not be obvious that this is indeed the case. Below are some arguments to favor frequency over other possible choices.

5.1.1 Frequency and not period

The most basic definition of frequency, $f = 1/T$, relates it to the period—the fixed duration of perfectly repetitive oscillatory motion. Just as there is instantaneous frequency, there should be no difficulty to talk about an instantaneous period. Conveniently, it is measured with time units and is easy to understand. Inconveniently, it overlaps the time axis and is often not well distinguished from time when taken to infinity (see 2.4.2 and 2.4.3). Frequency is somewhat more arcane of a concept to explain being a reciprocal quantity, but easy to observe if thought of as a (real) number of cycles per time unit. It is also easier to relate to an arbitrary dimension—namely any spatial dimension—where the meaning of frequency is retained, abstracted from the type of dimension at play (periodicity can apply to space too, but has a connotation of time, typically). Therefore, there may be an advantage to use frequency as a more neutral description of this dimension, which may or may not be described using periodicity equally well.

5.1.2 Temporal frequency and not spatial frequency

Classical wave motion is defined by the relationship between the spatial and temporal parts of the oscillatory motion. As was seen in 2.1.3, the wavenumber k is also the propagation vector of the wave, so the directional components are three spatial frequencies, k_x, k_y, and k_z which may be independent from one another (Fig. 6). In this sense, a spatial frequency spectrum in which the direction varies dynamically in 3D would be three dimensional as well, whereas the temporal frequency spectrum is only one-dimensional with the three spatial components projected on the time dimension. Therefore, the latter can be thought of as being more parsimonious and geometry agnostic. It also applies directly to oscillators that are not explicitly modeled with respect to spatial variations, and hence, it is more universal.

5.1.3 Frequency and not wavelength

The wavelength is the reciprocal of the wavenumber times a factor of 2π. Thus, the same arguments apply here as for the spatial frequency (5.1.2).

5.1.4 Frequency and not energy

According to quantum mechanics, Planck relation states that the energy of a photon is related to its frequency through a proportionality factor, $h = 6.626 \cdot 10^{-34}$ J/Hz,

$$E = hf \tag{45}$$

or using the angular frequency

$$E = \hbar\omega \tag{46}$$

where $\hbar = h/2\pi$. In the context of a single photon with a sharp spectral line (constant frequency), the energy and the frequency contain the same information. However, in more complex systems, not necessarily macroscopic, energy can take different forms and can be transformed between them, so that this neat relation may no longer have any observational relevance. As was noted in 2.5.2, it is normal to think of the signal spectrum as an energy density function that can be integrated over a frequency interval to obtain the energy for a particular bandwidth. Therefore, except for the simplest quantum systems, there is no apparent point in substituting frequency for energy in the dimensional count.

5.1.5 Frequency and not phase

According to Eq. 38 the instantaneous frequency is the derivative of the time-dependent phase function, in analogy to how angular velocity is the derivative of the phase in central potential problems. Therefore, the phase function can be obtained from the frequency function, up to a constant phase term. However, there are a few reasons for why using the phase is less attractive than frequency. First, it is constantly changing, even in cases where frequency is constant. Therefore, frequency is more parsimonious, as it expresses the same thing using a single number (in the quintessential case of a constant frequency) rather than a function. Second, there are many situations in which the phase function cannot be directly measured. For example, at the light frequency range of the electromagnetic spectrum, incoherent imaging is the standard (and applies to vision), as only the intensity can be detected and not the amplitude or phase of the light waves. Similarly, the "power spectrum" model of hearing, accounts for the insensitivity of the ear to phase changes with many typical stimuli [see references in 30, pp. 113–114]. A similar way to state the same thing is that there are many situations in which we do not care about the exact part of the period where the oscillating system is positioned, but rather, how often it oscillates on average (Eq. 40), which is much easier to estimate. Yet another way to restate it is that for signals that are stochastic by definition there is no valid phase function, and yet their frequency can be validly estimated statistically. Third, following from the second, the instantaneous phase function carries little unique information, because the (wrapped) phase values are bounded on a 2π interval. Therefore, a random sample of the phase function would not give any orientation for the frequency range in which the system oscillates without further calculations.

5.2 Challenges to the 5D view

5.2.1 Possible conflation of different types of frequencies

When translated back to the perceptual domain, the physical 5D reality may lead to conflation between frequencies whose source is macroscopic (e.g., touch and sound) and frequencies whose ultimate source is electromagnetic (e.g., vision, electroreception in fish, infrared thermoreception in snakes and lizards). The macroscopic quantities represent vibrations or other mechanical oscillations, whereas the electromagnetic frequencies stem from quantum processes at the subatomic or molecular level. It is not clear that the frequency dimension is anywhere directly comparable between these senses or others. Also, even though the low modulation-band frequencies of different modalities may become comparable after demodulating the sensory input, they are generally not perceived as equivalent once in the different modality pathways (perhaps with the exception of special cases of synesthesia or a strong sensory binding of stimuli emanating from a common source).

5.2.2 Multiple frequency dimensions

Another challenge is that the frequency dimension may not be singular. Physical objects have multiple frequencies going on simultaneously, which can be understood as independent degrees of freedom. For example, a rigid body has three degrees of rotational freedom—each of which can be taken as one frequency dimension corresponding to one axis. One option is that in perception, all of these frequencies are projected on a single dimension, so that the union of all dimensional frequencies is perceived as the spectrum from a single object. More complicated options may include more than a single frequency dimension in perception either in all or in a subset of the available modalities. Or alternatively, in some cases independent low-frequency information modulates high-frequency carriers and becomes its own dimension after demodulation [for example, see arguments for a two-dimensional spectrum in hearing, 30, p. 123–125].

5.2.3 Trivial addition

In a sense, the addition of frequency as a mandatory dimension to the 4D spacetime is trivial. For example, the universe is mapped through observations at different electromagnetic wavelengths, as there are dedicated instruments for radio, microwave, infrared, light, ultraviolet, x-ray, and gamma radiation astronomy. While the objects on the resultant maps are taken as 3D projections, their existence is only revealed if observed in the appropriate spectral windows. If observed in the wrong wavelength, they are effectively invisible. This is not all that different from the sensory inputs to perception. Objects can be completely invisible (either dark or transparent), unless they produce light in the visual range, sound in the audible range, etc.[27]. Any notion of

[27] However, regardless of their far-field detectability, solid objects seem to be always detectable by touch, as long as they are large enough to actuate some of the receptors on the skin, or to forcefully act on the body. Even here, though, a totally static tactile stimulus—one that never changes (i.e., zero relative velocity between the object and perceiver) can be modeled as being exactly 0 Hz in its Fourier frequency sense (of infinite time support)—is arguably not going to be registered by perception as a valid stimulus (think of the floor under your feet when seated motionless for a very long time).

triviality, though, reflects a parametric approach to frequency, in which time-invariant filtering is sufficient to extract average, coarse spectral information about the object, which is ultimately static in nature. Instead, we emphasized the time-varying nature of spectral information in cases where non-statistical quantities may be required.

5.3 Frequency, time, periodicity: Was anything left out?

While in the parlance of basic harmonic analysis it is customary to refer to time and frequency domain solutions as reciprocal of one another, we saw that time and frequency are not the same thing. In fact, frequency is derived from the periodicity of the system, which can be thought of as a set of rulers in the time domain—fixed yardstick measures that can be arbitrarily positioned along the time axis and that can capture repeating patterns in time. However, even if we remain careful to avoid conflating periodicity and time, we may still ask whether frequency and periodicity contain the same information, and if not, is periodicity yet another dimension that is hidden in plain sight? While the answer seems to be negative (and possibly downright absurd), this question remains unanswered, at present.

5.4 Implications on time as a dimension

Despite the equal weight that is given to time and frequency in the harmonic analyses of numerous physical systems, only time has been nominated as an own dimension (although not without dissent; [110]). Moreover, (once again, to the best knowledge of the author) in the various musings about time, frequency has never been considered as relevant in the discussion, beyond being a necessary component for building clocks—the essential measurement tool to estimate the passage of time. This implicitly assumes that the concept of frequency is not plagued by similarly crippling oddities. The applied mathematical and engineering literature, however, suggests otherwise [99].

The current analysis still gives primacy to time as the first intangible dimension to emerge out of a 3D spatial reality, but it demonopolizes it from being the only one (3.1.8). Because the definition of frequency is intertwined with periodicity, time, and many other specific physical parameters, it may appear to be somewhat less mysterious than time, and perhaps more amenable for a straightforward definition. However, the very mathematical nature of most definitions of frequency, as well as their highly indirect nature that requires some knowledge of physics in order to explain, seem to make it no less abstract than time. The complicated story behind frequency—its earliest definitions, slow mathematical evolution in becoming proficient in working with it, the late development of physical filters and the theory behind them—may render it more suspect than time in its primacy. And yet, this intertwinedness with time may be seen as part of reality that only compounds the underlying mystery of it, rather than alleviates it.

Theorem 1 formalizes the conditions for which time can be elevated to be its own dimension, rather than a parameter. This may have some weight on the age-long controversy about the nature of time and its role in reality—whether it is a real property of the universe, or an emergent one that only serves our perception of it. The ideas and results that were derived here by following the strict definitions and application of frequency that concluded in realities of type P1 and P2 are not unlike those that reached at using much more elaborate physics [110]. The convergence of these markedly different approaches gives credence to the radical and perhaps unintuitive notion that time is an emergent dimension of reality, rather than a fundamental one.

6 Conclusion

This work has attempted to connect a few dots that trace the narrative of frequency as a concept, a physical quantity, a percept, and most speculatively, a dimension of reality. The two other alternatives encapsulated in the theorem—that time is not a fundamental dimension or that the universe is wholly deterministic—are no less mind bending. It is not unlikely that the above argumentation will leave skeptical the prospective readers from the physics, philosophy, applied mathematics, signal processing, and neuroscience communities. Nevertheless, it is the author's hope that the logical reasoning in the above would be robust enough to provoke a discussion and explore the possible implications of these ideas on these different fields.

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Declarations

Not applicable.

References

[1] Triarhou, L.C., Verina, T.: The musical centers of the brain: Vladimir E. Larionov (1857–1929) and the functional neuroanatomy of auditory perception. Journal of Chemical Neuroanatomy **77**, 143–160 (2016)

[2] Pickles, J.O.: An Introduction to the Physiology of Hearing, 4th edn. Emerald Group Publishing Limited, Bingley, United Kingdom (2012)

[3] Ruben, R.J.: The developing concept of tonotopic organization of the inner ear. Journal of the Association for Research in Otolaryngology, 1–20 (2020)

[4] Fishman, R.S.: Gordon Holmes, the cortical retina, and the wounds of war. Documenta Ophthalmologica **93**(1), 9–28 (1997)

[5] Wandell, B.A., Dumoulin, S.O., Brewer, A.A.: Visual field maps in human cortex. Neuron **56**(2), 366–383 (2007)

[6] Penfield, W., Boldrey, E.: Somatic motor and sensory representation in the cerebral cortex of man as studied by electrical stimulation. Brain **60**(4), 389–443 (1937)

[7] Flanders, M.: Functional somatotopy in sensorimotor cortex. Neuroreport **16**(4), 313–316 (2005)

[8] Lee, B.K., Mayhew, E.J., Sanchez-Lengeling, B., Wei, J.N., Qian, W.W., Little, K.A., Andres, M., Nguyen, B.B., Moloy, T., Yasonik, J., Parker, J.K., Gerkin, R.C., Mainland, J.D., Wiltschko, A.B.: A principal odor map unifies diverse tasks in olfactory perception. Science **381**(6661), 999–1006 (2023)

[9] Roskies, A.L.: The binding problem. Neuron **24**(1), 7–9 (1999)

[10] Lackner, J.R., DiZio, P.: Vestibular, proprioceptive, and haptic contributions to spatial orientation. Annual Reviews in Psychology **56**, 115–147 (2005)

[11] Allman, M.J., Teki, S., Griffiths, T.D., Meck, W.H.: Properties of the internal clock: First-and second-order principles of subjective time. Annual Review of Psychology **65**, 743–771 (2014)

[12] Grondin, S.: Timing and time perception: A review of recent behavioral and neuroscience findings and theoretical directions. Attention, Perception, & Psychophysics **72**(3), 561–582 (2010)

[13] Shepard, R.N.: Perceptual-cognitive universals as reflections of the world. Psychonomic Bulletin & Review **1**(1), 2–28 (1994)

[14] Auffarth, B.: Understanding smell—the olfactory stimulus problem. Neuroscience & Biobehavioral Reviews **37**(8), 1667–1679 (2013)

[15] Welchman, A.E.: The human brain in depth: How we see in 3D. Annual Review of Vision Science **2**, 345–376 (2016)

[16] McAdams, S.: The perceptual representation of timbre. In: Siedenburg, K., Saitis, C., McAdams, S., Fay, A.N., Popper, R.R. (eds.) The Human Auditory Cortex vol. 69, pp. 23–57. Springer, Cham, Switzerland (2019)

[17] Chen, Y.-C., Chang, A., Rosenberg, M.D., Feng, D., Scholl, B.J., Trainor, L.J.: “taste typicality” is a foundational and multi-modal dimension of ordinary aesthetic experience. Current Biology **32**, 1–6 (2022)

[18] Mollon, J.D.: The origins of modern color science. In: Shevell, S.K. (ed.) The Science of Color, 2nd edn., pp. 1–39. Elsevier, The Optical Society of America, Oxford, United Kingdom (2003)

[19] Fletcher, H.: Auditory patterns. Reviews of Modern Physics **12**(1), 47–65 (1940)

[20] Talbot, W.H., Darian-Smith, I., Kornhuber, H.H., Mountcastle, V.B.: The sense of flutter-vibration: comparison of the human capacity with response patterns of mechanoreceptive afferents from the monkey hand. Journal of Neurophysiology **31**(2), 301–334 (1968)

[21] Johansson, R.S., Landström, U., Lundström, R.: Responses of mechanoreceptive afferent units in the glabrous skin of the human hand to sinusoidal skin displacements. Brain Research **244**(1), 17–25 (1982)

[22] Bolanowski Jr, S.J., Gescheider, G.A., Verrillo, R.T., Checkosky, C.M.: Four channels mediate the mechanical aspects of touch. The Journal of the Acoustical society of America **84**(5), 1680–1694 (1988)

[23] Todd, N.P.M., Rosengren, S.M., Colebatch, J.G.: Tuning and sensitivity of the human vestibular system to low-frequency vibration. Neuroscience Letters **444**(1), 36–41 (2008)

[24] Malcolm Dyson, G.: The scientific basis of odour. Chemistry and Industry **57**(28), 647–651 (1938)

[25] Wright, R.H.: Odor and molecular vibration: Neural coding of olfactory information. Journal of Theoretical Biology **64**(3), 473–502 (1977)

[26] Turin, L.: A spectroscopic mechanism for primary olfactory reception. Chemical Senses **21**(6), 773–791 (1996)

[27] Keller, A., Vosshall, L.B.: A psychophysical test of the vibration theory of olfaction. Nature Neuroscience **7**(4), 337–338 (2004)

[28] Block, E., Jang, S., Matsunami, H., Sekharan, S., Dethier, B., Ertem, M.Z., Gundala, S., Pan, Y., Li, S., Li, Z., Lodge, S.N., Ozbil, M., Jiang, H., Penalba, S.F., Batista, V.S., Zhuang, H.: Implausibility of the vibrational theory of olfaction. Proceedings of the National Academy of Sciences **112**(21), 2766–2774 (2015)

[29] Rhodes, J.E.: Microscope imagery as carrier communication. Journal of the Optical Society of America **43**(10), 848–852 (1953)

[30] Weisser, A.: Treatise on hearing: The temporal auditory imaging theory inspired by optics and communication. arXiv preprint arXiv:2111.04338 (2021)

[31] Wurm, L.H., Legge, G.E., Isenberg, L.M., Luebker, A.: Color improves object recognition in normal and low vision. Journal of Experimental Psychology: Human perception and performance **19**(4), 899 (1993)

[32] French, N.R., Steinberg, J.C.: Factors governing the intelligibility of speech sounds. The Journal of the Acoustical Society of America **19**(1), 90–119 (1947)

[33] Kasturi, K., Loizou, P.C., Dorman, M., Spahr, T.: The intelligibility of speech with "holes" in the spectrum. The Journal of the Acoustical Society of America **112**(3), 1102–1111 (2002)

[34] Trevarthen, C.B.: Two mechanisms of vision in primates. Psychologische Forschung **31**(4), 299–337 (1968)

[35] Schneider, G.E.: Two visual systems. Science **163**(3870), 895–902 (1969)

[36] Cajori, F.: Origins of fourth dimension concepts. The American Mathematical Monthly **33**(8), 397–406 (1926)

[37] Nordström, G.: Uber die moglichkeit, das elektromagnetische feld und das gravitationsfeld zu vereinigen. Physik. Zeitschr. XV, 504–506 (1914). Translated by Frank Borg, as "On the possibility of unifying the electromagnetic and gravitational fields", arXiv preprint: arXiv:physics/0702221v1

[38] Kaluza, T.: Zum unitätsproblem der physik. Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin, 966–972 (1921). Revised translation by V. T. Toth, based in part on translation by T. Muta, HUPD-8401, March 1984, Dep. Phys. Univ. Hiroshima.

[39] Klein, O.: Quantentheorie und fünfdimensionale relativitätstheorie. Zeitschrift für Physik **37**(12), 895–906 (1926)

[40] Gonzalez-Ayala, J., Cordero, R., Angulo-Brown, F.: Is the (3+ 1)-d nature of the universe a thermodynamic necessity? Europhysics Letters **113**(4), 40006 (2016)

[41] Green, M.B., Schwarz, J.H., Witten, E.: Superstring Theory: Volume 1, Introduction. Cambridge University Press, Cambridge, UK (1987)

[42] Tegmark, M.: On the dimensionality of spacetime. Classical and Quantum Gravity **14**(4), 69 (1997)

[43] Witten, E.: Reflections on the fate of spacetime. Physics Today **49**(4), 24–30 (1996)

[44] Cole, K.: Time, space obsolete in new view of universe. Los Angeles Times (1999). Nov 16

[45] Arkani-Hamed, N.: Quantum Mechanics and Spacetime in the 21st Century. https://www.youtube.com/watch?v=U47kyV4TMnE. Lecture given at the Perimeter Institute for Theoretical Physics, on Nov. 6, 2014 (2014)

[46] Hoffman, D.D., Singh, M., Prakash, C.: The interface theory of perception. Psychonomic Bulletin & Review **22**(6), 1480–1506 (2015)

[47] Sommerfeld, A.: Partial Differential Equations in Physics. Academic Press Inc.,

Publishers, New York, N.Y. (1949). translated by Ernst G. Straus

[48] Dostrovsky, S.: Early vibration theory: Physics and music in the seventeenth century. Archive for History of Exact Sciences, 169–218 (1975)

[49] Campbell, G.A.: Electric wave-filter. US Patent 1,227,113; Application filed July 15, 1915 (1917)

[50] Campbell, G.A.: Physical theory of the electric wave-filter. The Bell System Technical Journal **1**(2), 1–32 (1922)

[51] Oppenheim, A.V., Schafer, R.W.: Discrete-Time Signal Processing, 3rd edn. Pearson Higher Education, Inc., Upper Saddle River, NJ (2009)

[52] Dekker, H.: Classical and quantum mechanics of the damped harmonic oscillator. Physics Reports **80**(1), 1–110 (1981)

[53] Um, C.-I., Yeon, K.-H., George, T.F.: The quantum damped harmonic oscillator. Physics Reports **362**(2-3), 63–192 (2002)

[54] Morse, P.M., Ingard, K.U.: Theoretical Acoustics. Princeton University Press, Princeton, NJ (1968)

[55] Goldstein, H., Poole Jr., C.P., Safko, J.L.: Classical Mechanics, 3rd edn. Pearson Education Limited, Harlow, UK (2014)

[56] Goodman, J.W.: Introduction to Fourier Optics, 4th edn. W. H. Freeman and Company, New York, NY (2017)

[57] Whithman, G.B.: Linear and Nonlinear Waves. John Wiley & Sons, Inc., New York, NY (1999). First published in 1974

[58] Mersenne, M.: Harmonie Universelle, Contenant la Théorie et la Pratique de la Musique. Sebastien Cramoisy, Paris, France (1636)

[59] Galileo, G.: Dialogues Concerning Two New Sciences. Dover Publishing, New York, NY (1638 / 1914). Translated by Henry Crew & Alfonso de Salvio

[60] Bode, H.W.: Network Analysis and Feedback Amplifier Design. D. Van Nostrand Company, Inc., Princeton, NJ (1945). 12th printing, 1957

[61] Fourier, J.B.J.: The Analytical Theory of Heat. Cambridge University Press, New York (2009 / 1822). Translated from the French, "Théorie analytique de la chaleur", by Alexander Freeman and originally appeared in English in 1878.

[62] Heisenberg, W.: Quantum-theoretical re-interpretation of kinematic and mechanical relations. Z. Physik **33**, 879–893 (1925)

[63] Heisenberg, W.: Über den anschaulichen inhalt der quantentheoretischen Kinematik und Mechanik. Z. Physik **43**, 172–198 (1927)

[64] Ewald, P.P.: Die berechnung optischer und elektrostatischer gitterpotentiale. Annalen der Physik **369**(3), 253–287 (1921)

[65] Gabor, D.: Theory of communication. Part 1: The analysis of information. Journal of the Institution of Electrical Engineers-Part III: Radio and Communication Engineering **93**(26), 429–457 (1946)

[66] Blinchikoff, H.J., Zverev, A.I.: Filtering in the Time and Frequency Domains. Scitech Publishing Inc., Rayleigh, NC (2001)

[67] Titchmarsh, E.C.: Introduction to the Theory of Fourier Integrals, 2nd edn. Oxford University Press, London, UK (1948)

[68] Shannon, C.E.: A mathematical theory of communication. The Bell System Technical Journal **27**(3), 379–423623656 (1948)

[69] Shannon, C.E.: Communication in the presence of noise. Proceedings of the IRE **37**(1), 10–21 (1949)

[70] Slepian, D.: On bandwidth. Proceedings of the IEEE **64**(3), 292–300 (1976)

[71] Slepian, D.: Some comments on Fourier analysis, uncertainty and modeling. SIAM review **25**(3), 379–393 (1983)

[72] Marburger, J.H.: A historical derivation of Heisenberg's uncertainty relation is flawed. American Journal of Physics **76**(6), 585–587 (2008)

[73] Ozawa, M.: Heisenberg's original derivation of the uncertainty principle and its universally valid reformulations. Current Science, 2006–2016 (2015)

[74] Kennard, E.H.: Zur quantenmechanik einfacher bewegungstypen. Zeitschrift für Physik **44**(4), 326–352 (1927)

[75] Weyl, H.: The Theory of Groups and Quantum Mechanics. Dover Publications, Inc., NY (1950 / 1928). Translated from the 2nd German edition by H.P. Robertson; Originally published in 1928

[76] Robertson, H.P.: The uncertainty principle. Physical Review **34**(1), 163 (1929)

[77] Küpfmüller, K.: Über einschwingvorgange in wellenfiltern (transient phenomena in wave filters). Elektrische Nachrichten-Technik **1**, 141–152 (1924)

[78] Wiener, N.: I Am a Mathematician: The Later Life of a Prodigy. The M.I.T. Press, Cambridge, MA (1956)

[79] Folland, G.B., Sitaram, A.: The uncertainty principle: A mathematical survey. Journal of Fourier Analysis and Applications **3**, 207–238 (1997)

[80] Tao, R., Zhao, J.: Uncertainty principles and the linear canonical transform. In: Healy, J.J., Kutay, M.A., Ozaktas, H.M., Sheridan, J.T. (eds.) Linear Canonical Transforms: Theory and Applications vol. 198, pp. 97–111. Springer, New York (2016)

[81] Cohen, L.: Time-Frequency Analysis. Prentice Hall PTR, Upper Saddle River, NJ (1995)

[82] Debnath, L.: Wavelet Transforms and Their Applications. Birkhäuser, Boston, MA (2002)

[83] Klug, A., Grothe, B.: Ethological stimuli. In: Rees, A., Palmer, A.R. (eds.) The Oxford Handbook of Auditory Science: The Auditory Brain vol. 2, pp. 173–192. Oxford University Press, New York, USA (2010)

[84] Van Der Pol, B.: Frequency modulation. Proceedings of the Institute of Radio Engineers **18**(7), 1194–1205 (1930)

[85] Carson, J.R., Fry, T.C.: Variable frequency electric circuit theory with application to the theory of frequency-modulation. Bell System Technical Journal **16**(4), 513–540 (1937)

[86] Klauder, J.R., Price, A., Darlington, S., Albersheim, W.J.: The theory and design of chirp radars. Bell System Technical Journal **39**(4), 745–808 (1960)

[87] Middleton, D.: An Introduction to Statistical Communication Theory. IEEE Press, Piscataway, NJ (1996). 2nd reprint edition; originally published in 1960

[88] Rayleigh, L.: XCII. Remarks concerning Fourier's theorem as applied to physical problems. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science **24**(144), 864–869 (1912)

[89] Potter, R.K.: Visible patterns of sound. Science **102**(2654), 463–470 (1945)

[90] Cohen, L.: Time-frequency distributions—A review. Proceedings of the IEEE **77**(7), 941–981 (1989)

[91] Page, C.H.: Instantaneous power spectra. Journal of Applied Physics **23**(1), 103–106 (1952)

[92] Carson, J.R.: Notes on the theory of modulation. Proceedings of the Institute of Radio Engineers **10**(1), 57–64 (1922)

[93] Mandel, L., Wolf, E.: Optical Coherence and Quantum Optics. Cambridge University Press, Cambridge, United Kingdom (1995)

[94] Vakman, D., Vaĭnshteĭn, L.: Amplitude, phase, frequency—fundamental concepts of oscillation theory. Soviet Physics Uspekhi **20**(12), 1002 (1977)

[95] Huang, N.E., Shen, Z., Long, S.R., Wu, M.C., Shih, H.H., Zheng, Q., Yen, N.-C., Tung, C.C., Liu, H.H.: The empirical mode decomposition and the Hilbert spectrum for nonlinear and non-stationary time series analysis. Proceedings of the Royal Society of London A **454**(1971), 903–995 (1998)

[96] Pol, B.: The fundamental principles of frequency modulation. Proc. IEE **93**(111), 153–158 (1946)

[97] Shekel, J.: Instantaneous frequency. Proceedings of the Institute of Radio Engineers **41**(4), 548–548 (1953)

[98] Mandel, L.: Interpretation of instantaneous frequencies. American Journal of Physics **42**(10), 840–846 (1974)

[99] Boashash, B.: Estimating and interpreting the instantaneous frequency of a signal. I. fundamentals. Proceedings of the IEEE **80**(4), 520–538 (1992)

[100] Sandoval, S., De Leon, P.L.: The instantaneous spectrum: A general framework for time-frequency analysis. IEEE Transactions on Signal Processing **66**(21), 5679–5693 (2018)

[101] Huang, N.E., Shen, Z., Long, S.R.: A new view of nonlinear water waves: The Hilbert spectrum. Annual Review of Fluid Mechanics **31**(1), 417–457 (1999)

[102] Flandrin, P.: Explorations in Time-Frequency Analysis. Cambridge University Press, Cambridge, United Kingdom (2018)

[103] Tsang, M., Shapiro, J.H., Lloyd, S.: Quantum theory of optical temporal phase and instantaneous frequency. Physical Review A **78**(5), 053820 (2008)

[104] Landauer, R.: The physical nature of information. Physics Letters A **217**(4-5), 188–193 (1996)

[105] Einstein, A., Podolsky, B., Rosen, N.: Can quantum-mechanical description of physical reality be considered complete? Physical Review **47**(10), 777 (1935)

[106] Bohm, D., Aharonov, Y.: Discussion of experimental proof for the paradox of Einstein, Rosen, and Podolsky. Physical Review **108**(4), 1070 (1957)

[107] Bell, J.S.: On the Einstein Podolsky Rosen paradox. Physics Physique Fizika **1**(3), 195 (1964)

[108] Clauser, J.F., Horne, M.A., Shimony, A., Holt, R.A.: Proposed experiment to test local hidden-variable theories. Physical Review Letters **23**(15), 880 (1969)

[109] Hensen, B., Bernien, H., Dréau, A.E., Reiserer, A., Kalb, N., Blok, M.S., Ruitenberg, J., Vermeulen, R.F., Schouten, R.N., Abellán, C., Amaya, W., Pruneri, V., Mitchell, M.W., Markham, M., Twitchen, D.J., Elkouss, D., Wehner, S., Taminiau, T.H., Hanson, R.: Loophole-free bell inequality violation using electron spins separated by 1.3 kilometres. Nature **526**(7575), 682–686 (2015)

[110] Rovelli, C.: The Order of Time. Penguin Random House, UK (2019)

[111] Lineweaver, C.H., Davis, T.M.: Misconceptions about the big bang. Scientific American **292**(3), 36–45 (2005)

[112] Einstein, A.: Relativity and the problem of space. In: Ideas and Opinions, pp. 360–377. Bonanza Books, New York, NY (1954)

[113] DeWitt, B.S.: Quantum theory of gravity. I. The canonical theory. Physical Review **160**(5), 1113 (1967)

[114] Sakurai, J.J.: Advanced Quantum Mechanics. Addison-Wesley Publishing Company, Inc., Boston, MA (1967)

[115] Travers, S.P., Norgren, R.: The time course of solitary nucleus gustatory responses: Influence of stimulus and site of application. Chemical Senses **14**(1), 55–74 (1989)

[116] Iannilli, E., Noennig, N., Hummel, T., Schoenfeld, A.M.: Spatio-temporal correlates of taste processing in the human primary gustatory cortex. Neuroscience **273**, 92–99 (2014)

[117] Morris, M.S., Thorne, K.S., Yurtsever, U.: Wormholes, time machines, and the weak energy condition. Physical Review Letters **61**(13), 1446 (1988)

[118] Gleick, J.: Time Travel: A History. 4th Estate, London, UK (2016)

[119] Wells, H.G.: The Time Machine. William Heinemann, London (1895)

[120] Poincaré, H.: The measure of time. In: Čapek, M. (ed.) The Concepts of Space and Time: Their Structure and Their Development, pp. 317–327. Springer, Dordrecht, Holland (1913 / 1976). Originally published in 1913

[121] Wiener, N., Struik, D.: The fifth dimension in relativistic quantum theory. Proceedings of the National Academy of Sciences of the United States of America **14**(3), 262 (1928)

[122] Vigran, T.E.: Building Acoustics. Taylor & Francis, New York, NY (2009)

[123] Brillouin, L.: Wave Propagation and Group Velocity. Academic Press, Inc., New York, NY (1960)

[124] Planck, M.: Über irreversible Strahlungsvorgänge. fünfte Mittheilung. Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin, 440–480 (1899)

[125] Peres, A., Rosen, N.: Quantum limitations on the measurement of gravitational fields. Physical Review **118**(1), 335 (1960)

[126] Mead, C.A.: Possible connection between gravitation and fundamental length. Physical Review **135**(3B), 849 (1964)

[127] Tomilin, K.A.: Natural systems of units. To the centenary anniversary of the Planck system. In: Proceedings Of The XXII Workshop On High Energy Physics And Field Theory (1999)

[128] Aertsen, A., Olders, J., Johannesma, P.: Spectro-temporal receptive fields of auditory neurons in the grassfrog. I. Characterization of tonal and natural stimuli. Biological Cybernetics **38**(3), 223–234 (1980)

[129] Aertsen, A., Johannesma, P.I., Hermes, D.: Spectro-temporal receptive fields of auditory neurons in the grassfrog. II. Analysis of the stimulus-event relation for tonal stimuli. Biological Cybernetics **38**(4), 235–248 (1980)

[130] Land, E.H., McCann, J.J.: Lightness and retinex theory. Journal of the Optical Society of America **61**(1), 1–11 (1971)

[131] Land, E.H.: The retinex theory of color vision. Scientific American **237**(6), 108–129 (1977)

[132] Born, M., Wolf, E., Bhatia, A.B., Clemmow, P.C., Gabor, D., Stokes, A.R., Taylor, A.M., Wayman, P.A., Wilcock, W.L.: Principles of Optics, 7th (expanded) edn. Cambridge University Press, Cambridge, United Kingdom (2003)

[133] Morse, P.M., Bolt, R.H.: Sound waves in rooms. Reviews of Modern Physics **16**(2), 69 (1944)

[134] Attneave, F., Olson, R.K.: Pitch as a medium: A new approach to psychophysical scaling. The American Journal of Psychology, 147–166 (1971)

[135] Pressnitzer, D., Meddis, R., Delahaye, R., Winter, I.M.: Physiological correlates of comodulation masking release in the mammalian ventral cochlear nucleus. Journal of Neuroscience **21**(16), 6377–6386 (2001)

[136] Fiore, T., Pellerito, C.: Infrared absorption spectroscopy. In: Agnello, S. (ed.) Spectroscopy for Materials Characterization, pp. 129–167. John Wiley & Sons, Inc., Hoboken, NJ (2021)

[137] Schraffenberger, H., Heide, E.: Sonically tangible objects. In: xCoAx 2015: Proceedings of the Third Conference on Computation, Communication, Aesthetics

and X., Glasgow, Scotland, pp. 233–248 (2015)

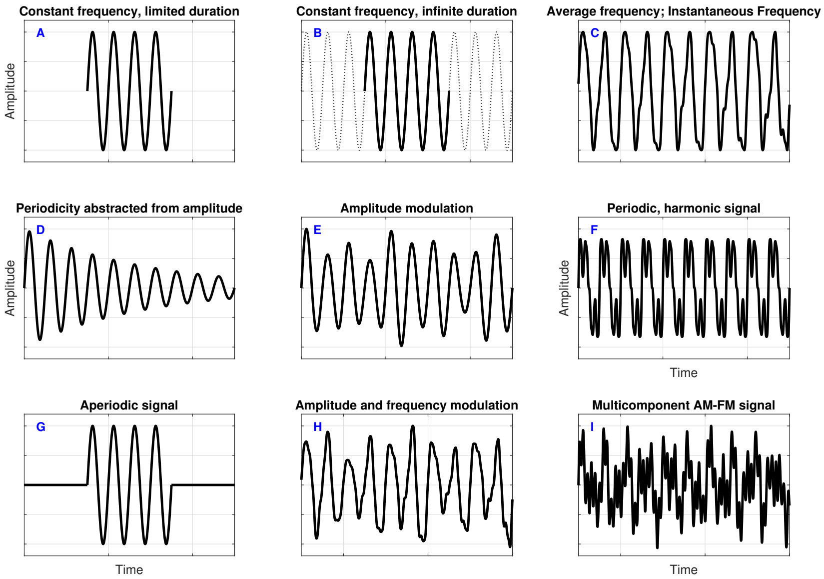

Fig. 2 Visualization of the escalation of the concept of frequency with time signals of increasing complexity. **A**. Frequency is a parameter that is computed from the limited duration measurement of simple periodic motion. **B**. Frequency is a parameter, whose instantiation implies constant (inertial) periodic motion in the remote past and future. **C**. Frequency is an average of imperfect periods, either due to the measurement errors or to instability in the oscillation. The average, nevertheless, produces the same long-term periodic motion as the simple harmonic oscillator. A different way to relate to this waveform is to define a time-dependent, instantaneous frequency. **D**. When loss of energy is included, a strict definition of periodic motion would emphasize that each period is slightly different in amplitude than its neighbors. Nevertheless, using a small correction, a constant frequency can still describe the oscillatory part of the motion, separate from a damping term that takes care of the decreasing amplitude (Eq. 18.) **E**. A periodic force of slower frequency than the oscillation drives the system, or modulates the signal in amplitude, further blurring the aspect of simple harmonic motion periodicity. **F**. Complex (non-sinusoidal) signals that are still periodic can be modeled using a Fourier series decomposition of the waveform to a sum of sinusoids whose frequencies are integer multiples (harmonics) of the fundamental frequency. As in B, the periodicity is extended to the remote past and future. **G**. An extension of the Fourier series period to infinity results in Fourier transform, which allows for modeling of aperiodic signals that comprise of a continuum of sinusoids with constant frequencies. **H**. Combining the slow variations in amplitude (E) with the variations in frequency (C) gives rise to a so called AM-FM signal, whose only constant may be the average frequency. **I**. A broadband signal can be decomposed to many narrowband AM-FM signals of the form of H. This form produces complex waveforms that do not necessarily disclose a clear periodicity, unless the components are separated using band-pass filtering.

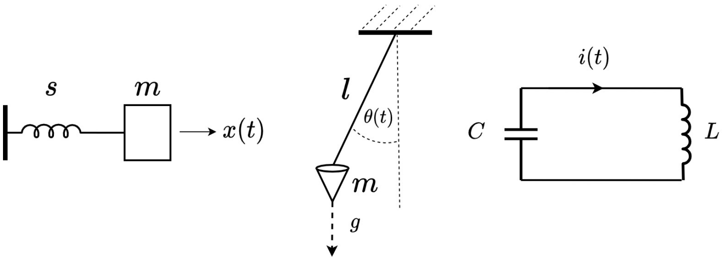

Fig. 3 Three types of ideal harmonic oscillators: spring-mass (left), pendulum (middle), inductor-capacitor (LC) circuit (right). All three systems are described by the same ordinary differential equation (5), where frequency is a parameter defined as the reciprocal of the period, which is itself determined by the various constants of the system.

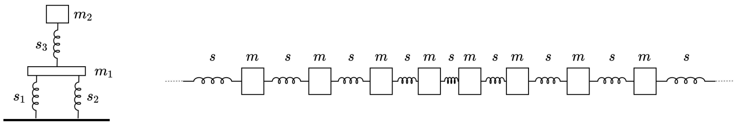

Fig. 4 **Left**: a mass-spring system with three springs and two masses. **Right**: approximation of wave motion using identical spring-mass building-block model.

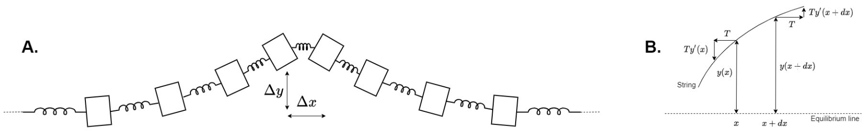

Fig. 5 **A.**: approximation of a string using identical spring-mass building blocks. **B.** Construction for the derivation of the string equation, using tension forces in two dimensions (drawn after [54, p. 98]).

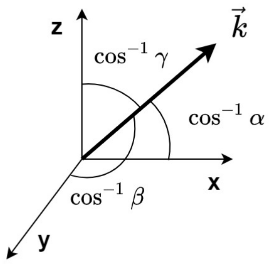

Fig. 6 The wave propagation vector $\vec{k}$, whose magnitude is the wavenumber and its direction can be expressed by three angles, which together produce up to three spatial frequencies.

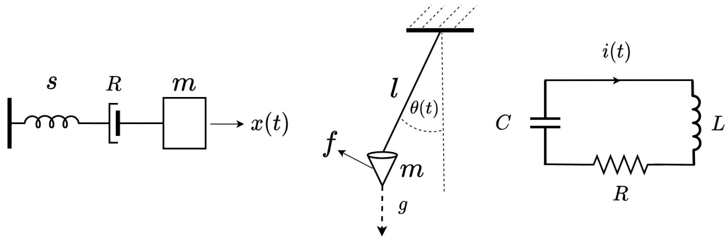

Fig. 7 Three types of harmonic oscillators with damping: spring-mass-damper (left), pendulum with friction from air (middle), and capacitor-inductor-resistor (RLC) circuit (right).

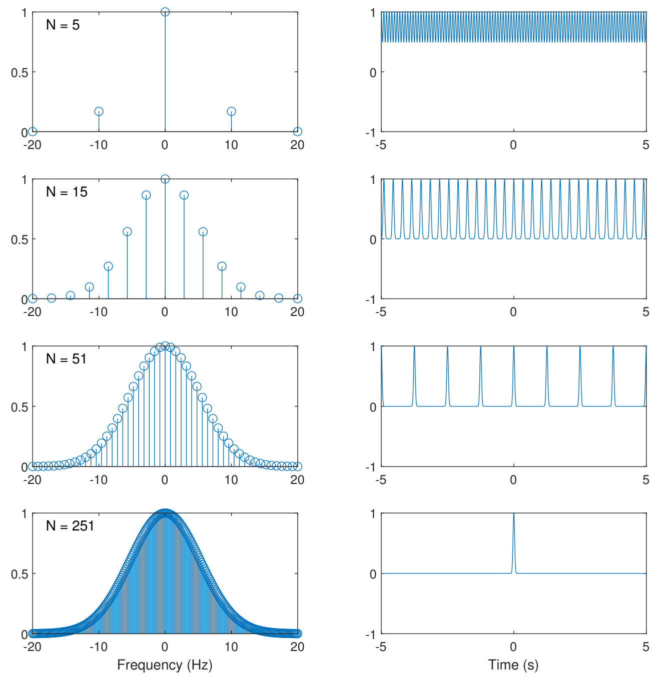

Fig. 8 An example of the reconstruction of an aperiodic signal—a narrow Gaussian (bottom right)—from infinitesimally many (odd number of) periodic functions using the Fourier transform. The Fourier transform itself is based on the Fourier series, which is a formulation of periodic functions with period T as series of simple periodic functions. As the period T is made larger, a bigger portion of the time domain is being captured by the Fourier series, until in the limit of the Fourier transform, it captures all of the time axis. In the example of the figure, the exact spectrum of the Gaussian (i.e., its Fourier transform, which is a Gaussian as well) is sampled on the frequency axis in diminishing intervals (left column). Each sample corresponds to a single sinusoidal component in the Fourier series, which can be then summed to reproduce an approximation of the original time signal (right column). With a growing number of components, $\delta\omega$ gets closer to zero, T covers more of the time axis, and the approximation gets better. The signal aperiodicity is captured in the approximation, as the increased number of components pushes the inevitable periodic parts of the summation (aliasing) away from the main lobe of the spectrum. In the Fourier transform limit of infinitely many dense components, there are no aliases and the aperiodic nature of the signal is perfectly retained.

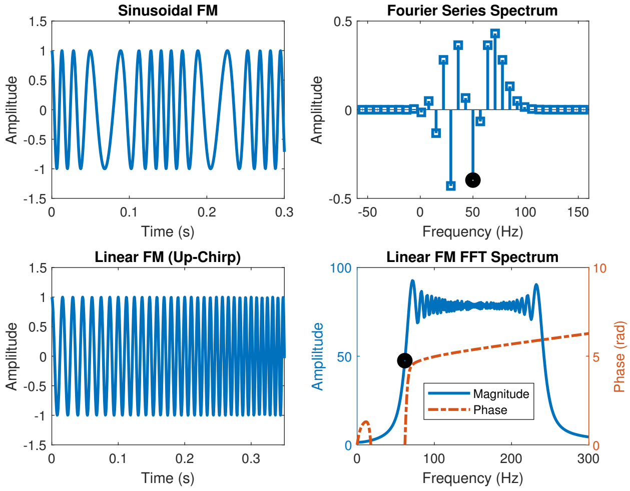

Fig. 9 Two common frequency modulation (FM) time signals and their respective spectra. On the top is a sinusoidal FM $x(t) = \cos[2\pi \cdot 50t + 4\sin(2\pi \cdot 7t)]$ with its Fourier series components according to Eq. 36 on the top right. On the bottom is a linear FM signal, a rising (or up-) chirp, with the equation $x(t) = \cos(2\pi \cdot 200t + 60\pi t^2)$. The spectrum of the signal was computed using the fast Fourier transform (FFT), whose magnitude and unwrapped phase are displayed on the bottom right. In both spectral plots, the carrier is marked with a black circle ($f_c = 50$ Hz for the sinusoidal FM and $f_c = 200$ Hz for the linear FM). The amplitude is in arbitrary units.

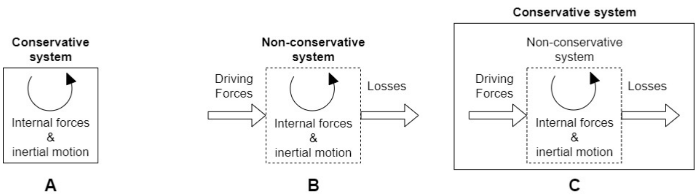

Fig. 10 Three ways to model the energy balance in a dynamical system: **A**. By assuming no losses and no energy inputs that add energy to the system. This assumes that all energy sources are self-contained and / or that the motion is inertial. This is a classical conservative system that appears stationary. **B**. By including losses and energy inputs in the models, but not as an integral part of the system, which is instead modeled as non-conservative. **C**. By including all losses and inputs to the system as part of yet a larger system (the environment plus the system), which is itself conservative, in the sense that the total energy is accounted for and remains within the modeled system.

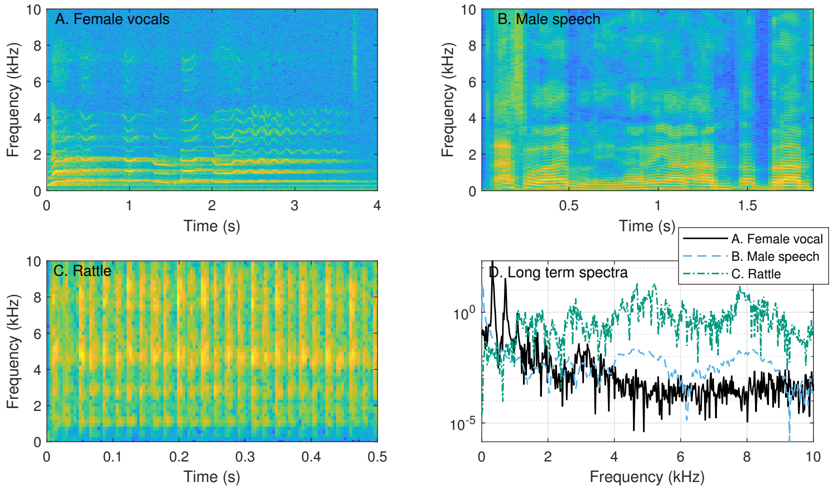

Fig. 11 Examples of three nonstationary acoustic signals and their spectrograms, which are a visualization of the short-time Fourier transform. The hotter the color of the time-frequency bin is, the more energy it has. The number of frequency bins and the Hann time-window overlap between processed signal frames was optimized for enhanced overall time-frequency resolution. **A**. Female vocals singing a long “love”. Each frame comprised $N = 2048$ samples with 50% overlap between the samples of consecutive frames. The timbre of the voice is determined (also) by the fundamental (the lowest curve) and its harmonics (the parallel curves above it), which move together up and down the musical scale to produce the melody. **B**. Male speech saying “That’s what I believe, I mean, I am... but I’m...” ($N = 2048$ samples; 50% overlap). Here, the fundamental frequency is lower and has many more harmonics, some of which are emphasized by the natural filtering (formants) of the larynx and mouth cavity. At high frequencies, the sound production tends to be noise-like (turbulent) and a deterministic frequency may not exist—only a stochastic description of the signal. **C**. A vibraslap sound—a musical rattle that produces a periodic noise-like sound. The periods can be seen distinctly along the time axis, whereas any “pitchiness” of the instrument is much less distinct, as it has very faint harmonic structure along the y-axis ($N = 256$ samples; 25% overlap). **D**. The long-term Fourier transforms of the three signals, computed using the fast-Fourier transform (FFT) with $N = 2048$. It is evident that the temporal structure of the signals is not visible in this way, although the same information should be contained in these spectra, ideally.

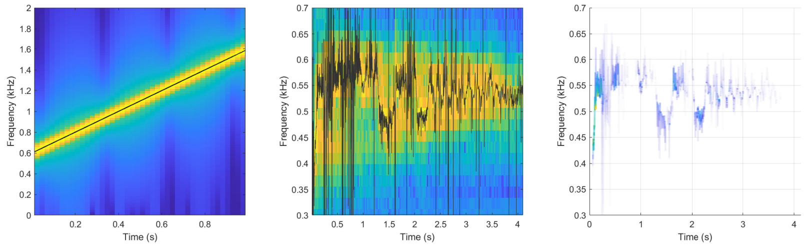

Fig. 12 Examples the estimated instantaneous frequencies of two acoustic signals. The first signal (left) is a linear frequency modulation chirp, of the form of $s(t) = \exp(2\pi f t + \frac{d}{\pi}t^2)$ with $f = 600$ Hz and slope $d = 1000$ Hz/s. Using a spectrogram ($N = 2048$, 50% overlap), its instantaneous frequency is blurred both in time and in frequency. The direct estimation of the instantaneous frequency using the Hilbert transform produces a sharp curve (in black), which directly overlaps the spectrogram. The second signal (middle and right) is taken from the female vocals on Fig. 11 A, where the fundamental was roughly picked using a band-pass filter (fourth-order Butterworth bandpass filter centered at 500 Hz with quality factor $Q = 3.33$). Once again, the spectrogram shows the smeared trend, whose center corresponds to the instantaneous frequency, which was also calculated using the Hilbert transform. However, the latter produces very rapid excursions from the mean, which makes it difficult to interpret and be certain of. On the right, an alternative employment of the Hilbert transform is applied to the same filtered signal using the Hilbert-Huang transform—a popular applied method to compute the instantaneous frequency of different modes in arbitrary broadband signals [95]. Unlike the standard Hilbert transform, the instantaneous frequency in this plot is also weighted by the instantaneous amplitude, so the effect of the extremities, as seen in the middle plot, is significantly reduced.

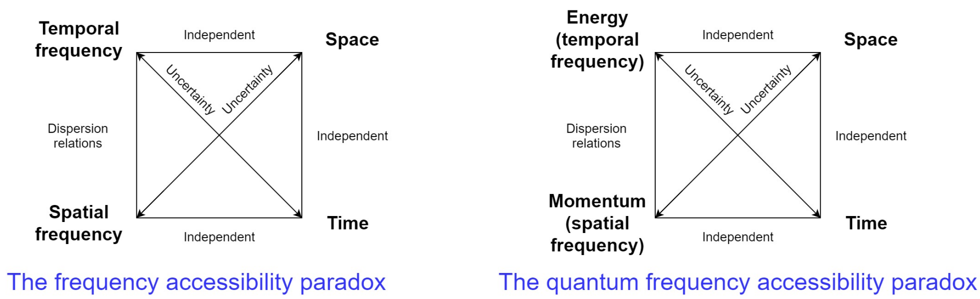

Fig. 13 The frequency accessibility paradox in classical and quantum systems illustrates the minimal constraints that apply to the five dimensions in all physical systems. More constraints may apply to further reduce the degrees of freedom.