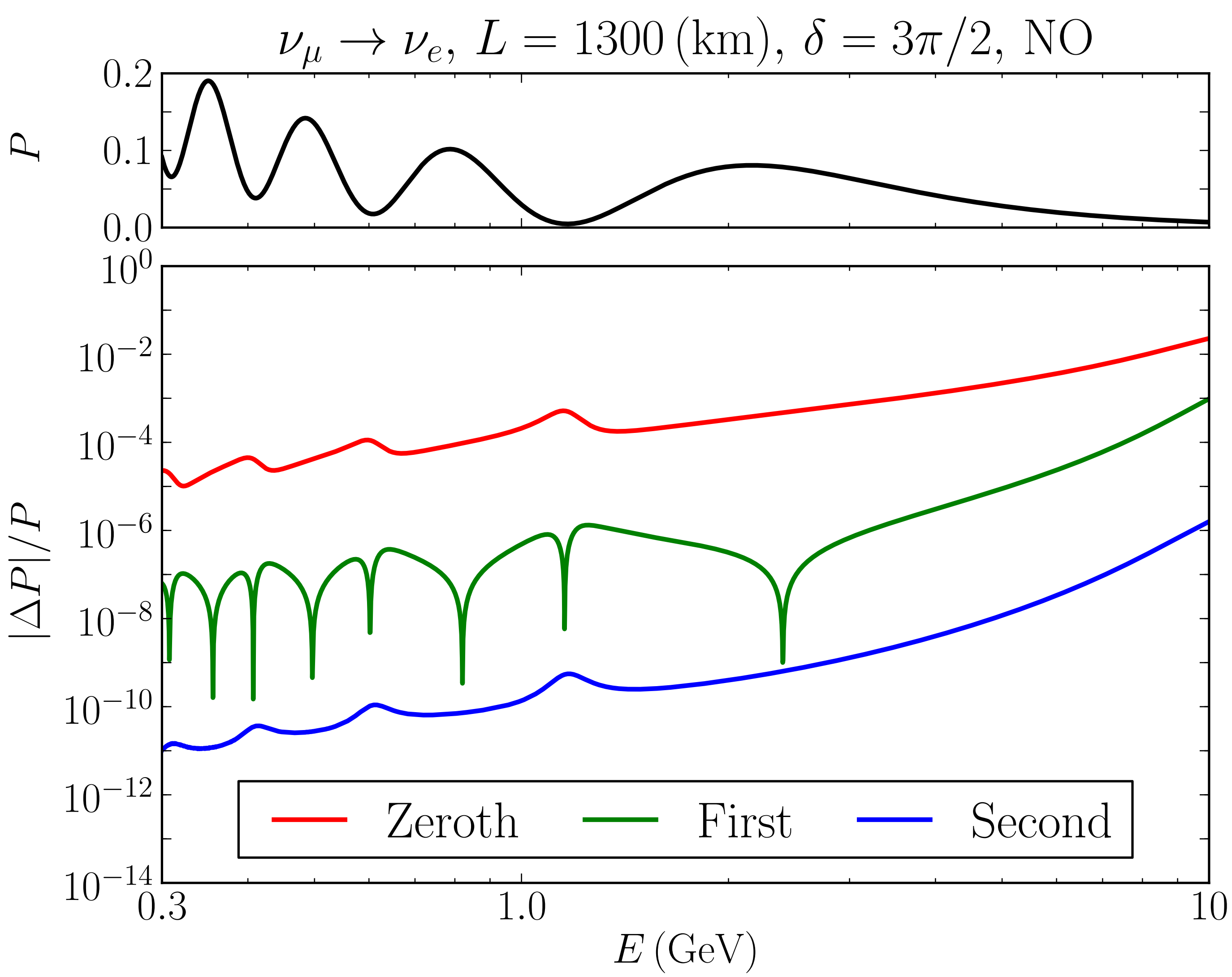
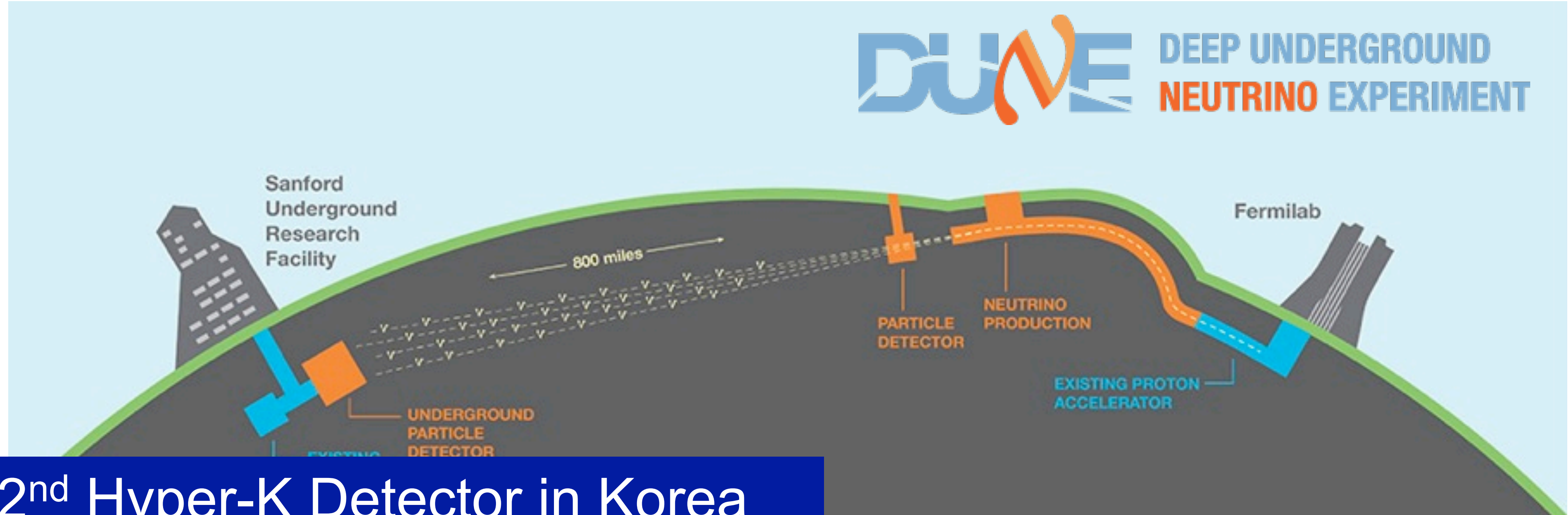
Analytic Neutrino Oscillation Probabilities in Matter Revisited

H. Minakata + SP arXiv:1505.01826
P. Denton + H. Minakata + SP arXiv:1604.08167

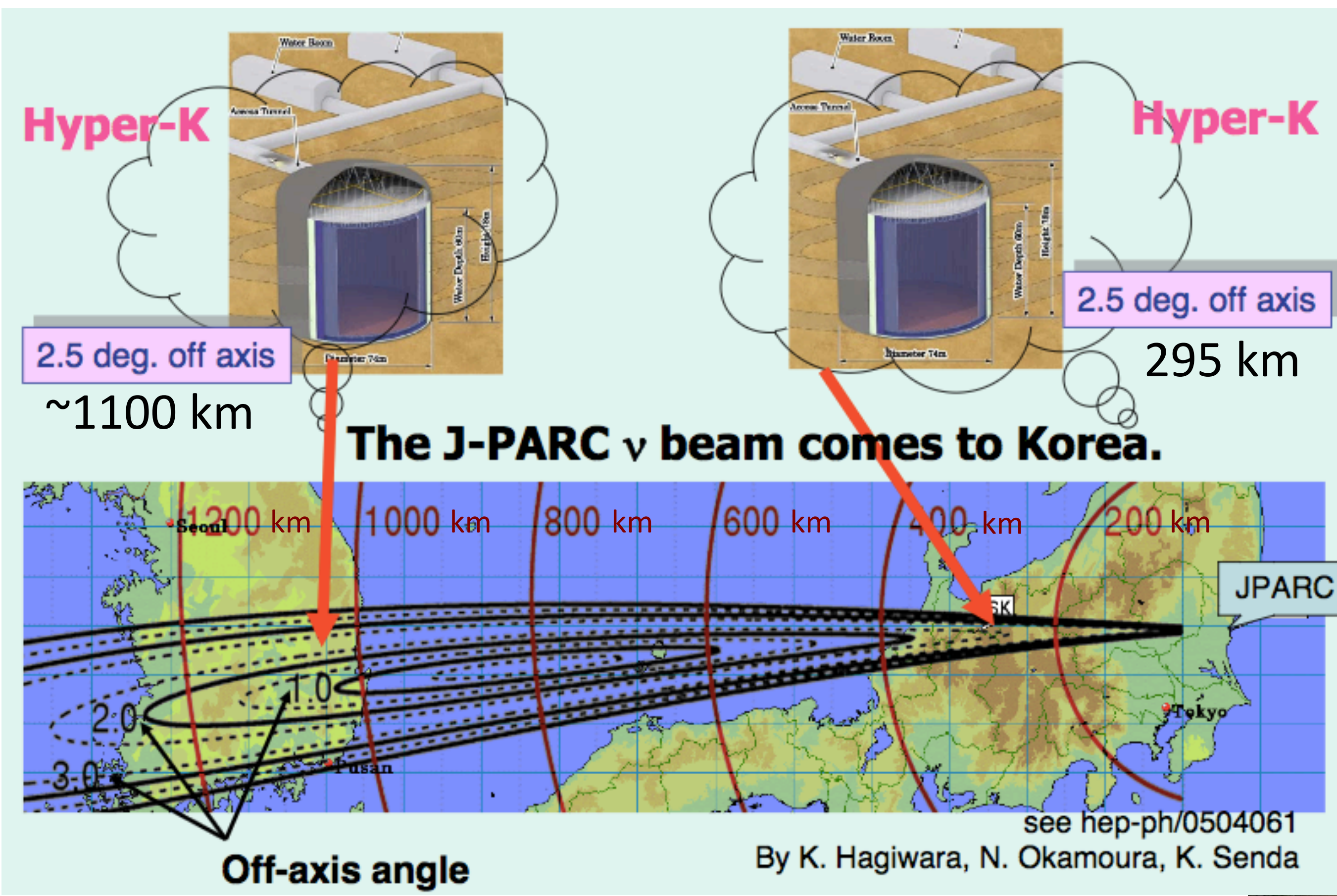


Stephen Parke
Theoretical Physics Dept
Fermilab

DOI: 10.5281/zenodo.1041774



The 2nd Hyper-K Detector in Korea



Neutrino Oscillation Amplitudes in vacuum:

$$P(\nu_\mu \to \nu_e) \quad = \quad |\mathcal{A}_{\mu e}|^2$$ "the billion $ process"

$$\mathcal{A}_{\mu e} \equiv (2 s_{23} s_{13} c_{13}) \, [\, c_{12}^2 \, e^{i\Delta_{32}} \, \sin\Delta_{31} + s_{12}^2 \, e^{i\Delta_{31}} \, \sin\Delta_{32} \,]$$

$$+ (2 c_{23} c_{13} s_{12} c_{12}) \, e^{-i\delta} \, \sin\Delta_{21}$$

$\Delta_{ij} = \Delta m_{ij}^2 L/4E$

maintain the symmetry: $m_1^2 \leftrightarrow m_2^2$ with $\theta_{12} \to \pi/2 - \theta_{12}$ and $\delta \to \delta \pm \pi$.

Denton, Minakata, SP arXiv:1604.08167

not unique: $\sin\Delta_{21} \equiv e^{i\Delta_{32}} \, \sin\Delta_{31} - e^{i\Delta_{31}} \, \sin\Delta_{32}$

$$\Delta P_{CP} = 2 \, (2 s_{23} s_{13} c_{13}) \, (2 c_{23} c_{13} s_{12} c_{12}) \, \sin\delta \, \sin\Delta_{21} \sin\Delta_{31} \, \sin\Delta_{32}$$

$\Delta_{32} \approx \Delta_{31}$

$$\mathcal{A}_{\mu e} \approx (2 s_{23} s_{13} c_{13}) \, e^{i\Delta_{31}} \, \sin\Delta_{31} + (2 c_{23} c_{13} s_{12} c_{12}) \, e^{-i\delta} \, \sin\Delta_{21}$$

$\nu_\mu \rightarrow \nu_e$

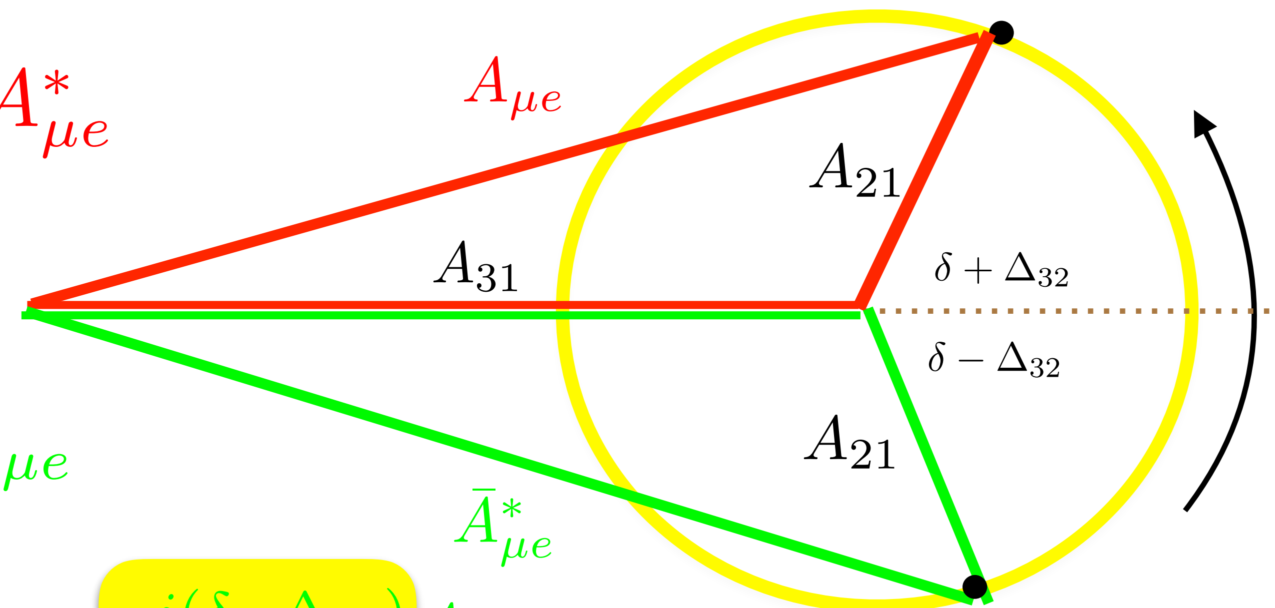
$$A_{31} = 2 s_{23} s_{13} c_{13} \sin \Delta_{31}$$

$$A_{21} = 2 c_{13} c_{23} s_{12} c_{12} \sin \Delta_{21}$$

$$A_{\mu e} = A_{31} + e^{i(\delta+\Delta_{32})} A_{21}$$

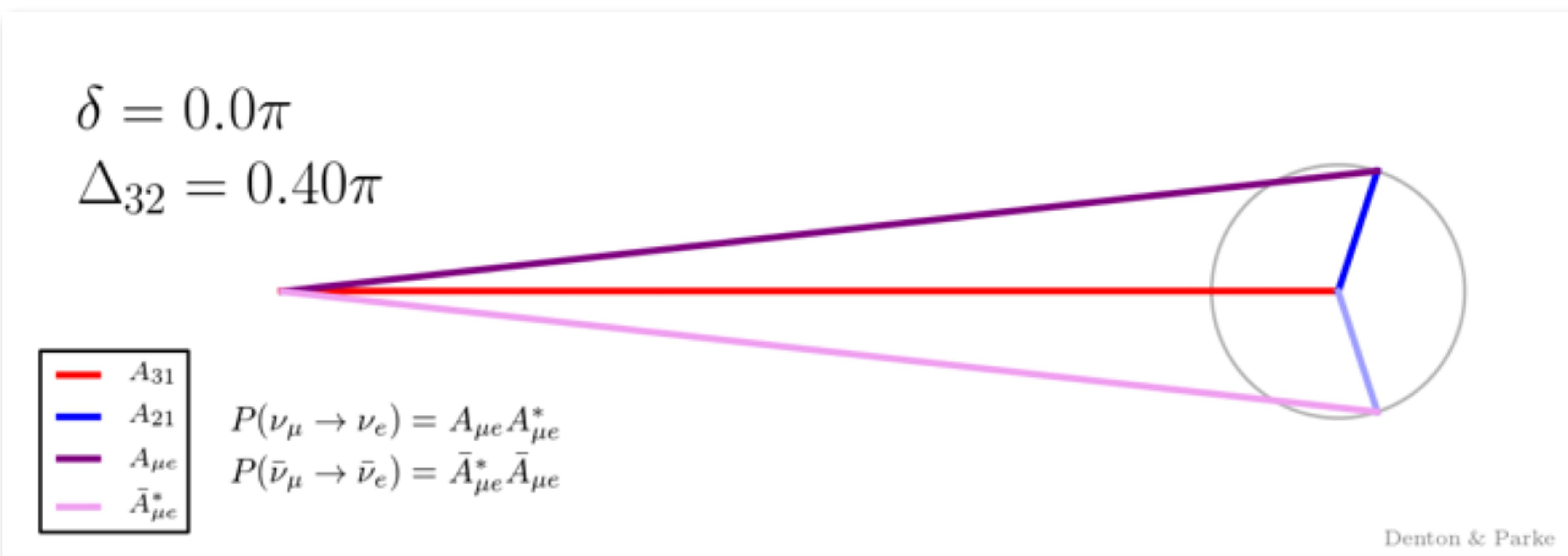
$$\Delta_{ij} = \Delta m^2_{ij} L/4E$$

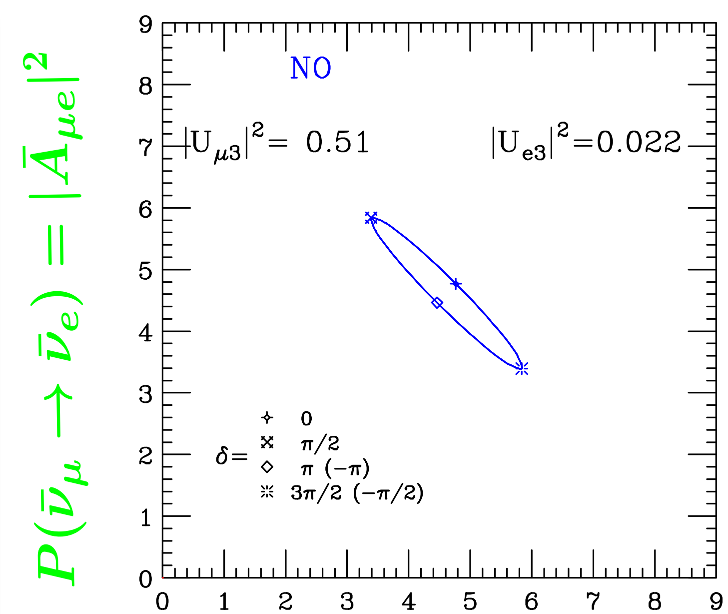
$$P(\nu_\mu \rightarrow \nu_e) = A_{\mu e} A^*_{\mu e}$$



$$P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) = \bar{A}^*_{\mu e} \bar{A}_{\mu e}$$

$$\bar{A}^*_{\mu e} = A_{31} + e^{i(\delta-\Delta_{32})} A_{21}$$





$$P(\nu_\mu \rightarrow \nu_e) = |\bar{A}_{\mu e}|^2$$

In Vacuum:

$$P(\nu_\beta \to \nu_\alpha) \;=\; \left| \sum_{i=1}^{3} U_{\alpha i} U^*_{\beta i}\, e^{-i\frac{m_i^2 L}{2E}} \right|^2 \qquad \Delta m^2_{ij} \equiv m^2_i - m^2_j$$

$$= \quad \delta_{\alpha\beta} - 4 \sum_{j>i}^{3} \text{Re}[U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j}] \sin^2 \frac{\Delta m^2_{ij} L}{4E}$$

$$+\, 8\, \text{Im}[U_{\alpha 1} U^*_{\beta 1} U^*_{\alpha 2} U_{\beta 2}]\; \sin \frac{\Delta m^2_{32} L}{4E} \sin \frac{\Delta m^2_{21} L}{4E} \sin \frac{\Delta m^2_{13} L}{4E}$$

3 flavor

$$4\, \sin \frac{\Delta m^2_{32} L}{4E}\, \sin \frac{\Delta m^2_{21} L}{4E}\, \sin \frac{\Delta m^2_{13} L}{4E} = \sin \frac{\Delta m^2_{32} L}{2E} + \sin \frac{\Delta m^2_{21} L}{2E} + \sin \frac{\Delta m^2_{13} L}{2E}$$

CPV: $\sim (L/E)^3$ not $\sim (L/E)^1$

Wronskian is non-vanishing as function of L/E

In Matter:

$$i\frac{d}{dx}\nu = H\nu \qquad \boldsymbol{\nu} \equiv \begin{pmatrix} \boldsymbol{\nu_e} \\ \boldsymbol{\nu_\mu} \\ \boldsymbol{\nu_\tau} \end{pmatrix}$$

$$H = \frac{1}{2E}\left\{ U \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Delta m^2_{21} & 0 \\ 0 & 0 & \Delta m^2_{31} \end{bmatrix} U^\dagger + \begin{bmatrix} a(x) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

$$a = 2\sqrt{2}G_F N_e E \approx 1.52 \times 10^{-4} \left(\frac{Y_e \rho}{\mathrm{g.cm^{-3}}}\right)\left(\frac{E}{\mathrm{GeV}}\right)\mathrm{eV}^2.$$

if $\rho Y_e = 1.5\ \mathrm{g/cm}^3$ and $E = 10$ GeV then $a \approx \Delta m^2_{31}$

$E = 300\ MeV$ then $a \approx \Delta m^2_{21}$

Methods for Solution:

- Numerical Methods:

Yes, FINE for experimental analysis of data
but limited physical understand !

e.g. Magic Baseline

- Analytic Methods:

$$P(\nu_\beta \to \nu_\alpha; L) = |S_{\alpha\beta}|^2. \qquad S = T\exp\left[-i\int_0^L dx H(x)\right]$$

too complicated for arbitrary $a(x)$!

- make simplification that a is constant !

(good approximation for many experiments).

Exact Analytic Solution:

- Solve Cubic Characteristic Eqn.

$$\lambda^3 - \left(a + \Delta m^2_{21} + \Delta m^2_{31}\right)\lambda^2$$
$$+[\Delta m^2_{21}\Delta m^2_{31} + a\left\{(c^2_{12} + s^2_{12}s^2_{13})\Delta m^2_{21} + c^2_{13}\Delta m^2_{31}\right\}]\,\lambda$$
$$- c^2_{12}c^2_{13}a\Delta m^2_{21}\Delta m^2_{31} \qquad = 0$$

See Zaglauer & Schwarzer, Z. Phys. C 1988

$$\lambda_1 = \frac{1}{3}s - \frac{1}{3}\sqrt{s^2-3t}\,[u+\sqrt{3(1-u^2)}],$$

$$\lambda_2 = \frac{1}{3}s - \frac{1}{3}\sqrt{s^2-3t}\,[u-\sqrt{3(1-u^2)}],$$

$$\lambda_3 = \frac{1}{3}s + \frac{2}{3}u\sqrt{s^2-3t},$$

$$s = \Delta_{21} + \Delta_{31} + a,$$

$$t = \Delta_{21}\Delta_{31} + a[\Delta_{21}(1-s^2_{12}c^2_{13}) + \Delta_{31}(1-s^2_{13})],$$

$$u = \cos\left[\frac{1}{3}\cos^{-1}\left(\frac{2s^3 - 9st + 27a\Delta_{21}\Delta_{31}c^2_{12}c^2_{13}}{2(s^2-3t)^{3/2}}\right)\right],$$

here $\Delta_{ij} \equiv \Delta m^2_{ij}$

- then calculate mixing angles in matter or mixing matrix, V:

eg Kimura Takamura & Yokomakura PLB, PRD 2002

then Oscillation Probablities

with λ_i's and $V_{\alpha i}$ in matter then

$$P(\nu_\beta \to \nu_\alpha) = \left| \sum_{i=1}^{3} V_{\alpha i} V_{\beta i}^* \, e^{-i \frac{\lambda_i L}{2E}} \right|^2$$

$$= \delta_{\alpha\beta} - 4 \sum_{j>i}^{3} \mathrm{Re}[V_{\alpha i} V_{\beta i}^* V_{\alpha j}^* V_{\beta j}] \sin^2 \frac{(\lambda_j - \lambda_i)L}{4E}$$

$$+ \, 8 \, \mathrm{Im}[V_{\alpha 1} V_{\beta 1}^* V_{\alpha 2}^* V_{\beta 2}] \, \sin \frac{(\lambda_3 - \lambda_2)L}{4E} \sin \frac{(\lambda_2 - \lambda_1)L}{4E} \sin \frac{(\lambda_1 - \lambda_3)L}{4E}$$

same as VACUUM with $m_i^2 \to \lambda_i$ and $U_{\alpha i} \to V_{\alpha i}$! ! !

Wronskian is nonvanishing,

Are we done ?

Exact Analytic Solution Issue:

- Solve Cubic Characteristic Eqn.

$$\begin{aligned}\lambda^3 - \left(a + \Delta m^2_{21} + \Delta m^2_{31}\right)\lambda^2 \\ +[\Delta m^2_{21}\Delta m^2_{31} + a\left\{(c^2_{12} + s^2_{12}s^2_{13})\Delta m^2_{21} + c^2_{13}\Delta m^2_{31}\right\}]\ \lambda \\ -\ c^2_{12}c^2_{13}a\Delta m^2_{21}\Delta m^2_{31} \qquad = 0\end{aligned}$$

IF

- $a = 0$
- or $\Delta m^2_{21} = 0$
- or $\sin\theta_{12} = 0$
- or $\sin\theta_{13} = 0$

THEN characteristic Eqn

FACTORIZES !

BUT

See Zaglauer & Schwarzer, Z. Phys. C 1988

$$\lambda_1 = \frac{1}{3}s - \frac{1}{3}\sqrt{s^2-3t}\left[u + \sqrt{3(1-u^2)}\right],$$

$$\lambda_2 = \frac{1}{3}s - \frac{1}{3}\sqrt{s^2-3t}\left[u - \sqrt{3(1-u^2)}\right],$$

$$\lambda_3 = \frac{1}{3}s + \frac{2}{3}u\sqrt{s^2-3t},$$

$$s = \Delta_{21} + \Delta_{31} + a,$$

$$t = \Delta_{21}\Delta_{31} + a[\Delta_{21}(1 - s^2_{12}c^2_{13}) + \Delta_{31}(1 - s^2_{13})],$$

$$u = \cos\left[\frac{1}{3}\cos^{-1}\left(\frac{2s^3 - 9st + 27a\Delta_{21}\Delta_{31}c^2_{12}c^2_{13}}{2(s^2-3t)^{3/2}}\right)\right],$$

here $\Delta_{ij} \equiv \Delta m^2_{ij}$

DOES NOT TRIVIALLY SIMPLIFY !

2 flavor mixing in matter

$$ax^2 + bx + c = 0$$

simple, intuitive, useful

3 flavor mixing in matter

$$ax^3 + bx^2 + cx + d = 0$$

complicated, counter intuitive, ...

need more of a physicists approach: Perturbation Theory

- $\sin\theta_{13} \sim 0.15$

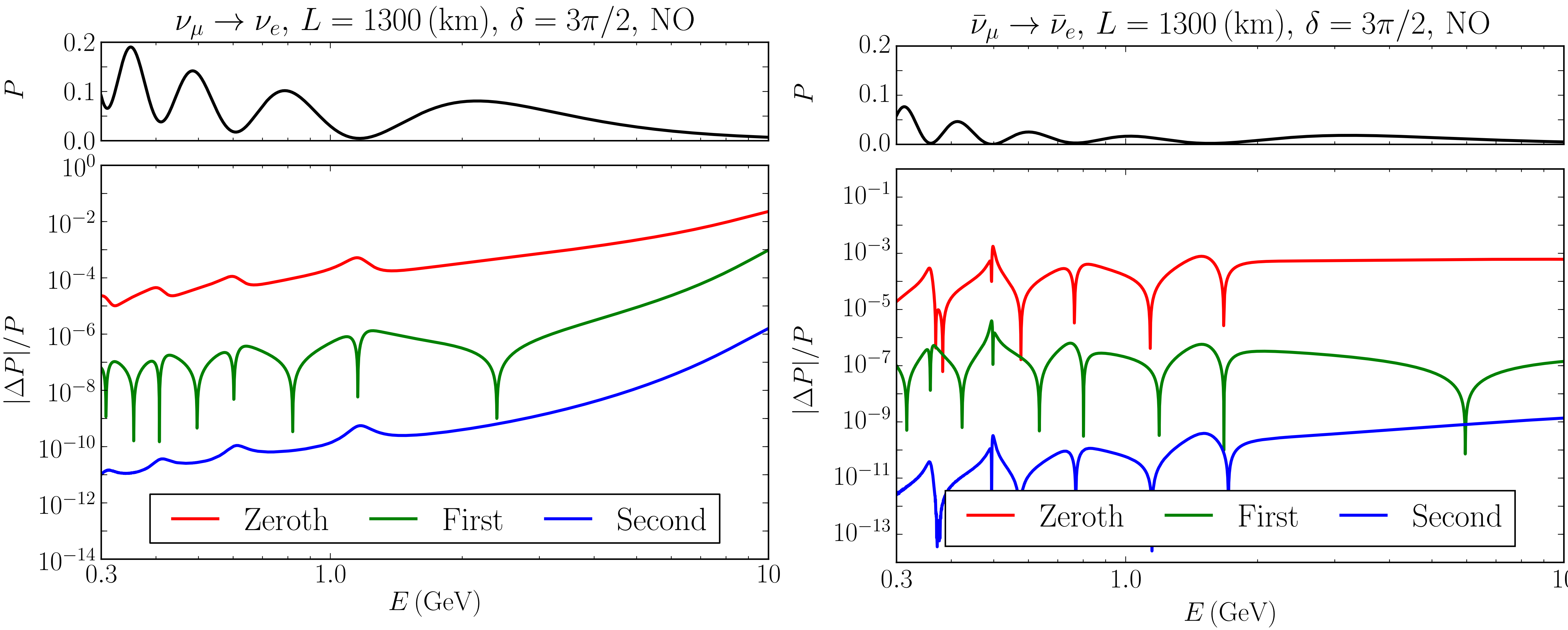
- $\Delta m^2_{21}/\Delta m^2_{31} \sim 0.03$

for Long Baseline Experiments using $\Delta m^2_{21}/\Delta m^2_{31}$ is more appropriate.

- Treat $\boldsymbol{\theta_{13}}$ exactly first, then do perturbation theory in

$$\epsilon \; s_{12}c_{12} \equiv s_{12}c_{12} \; \Delta m^2_{21}/\Delta m^2_{ee} \approx 0.015$$

New Perturbation Theory for Osc. Probabilities



systematic expansion

Hamiltonian:

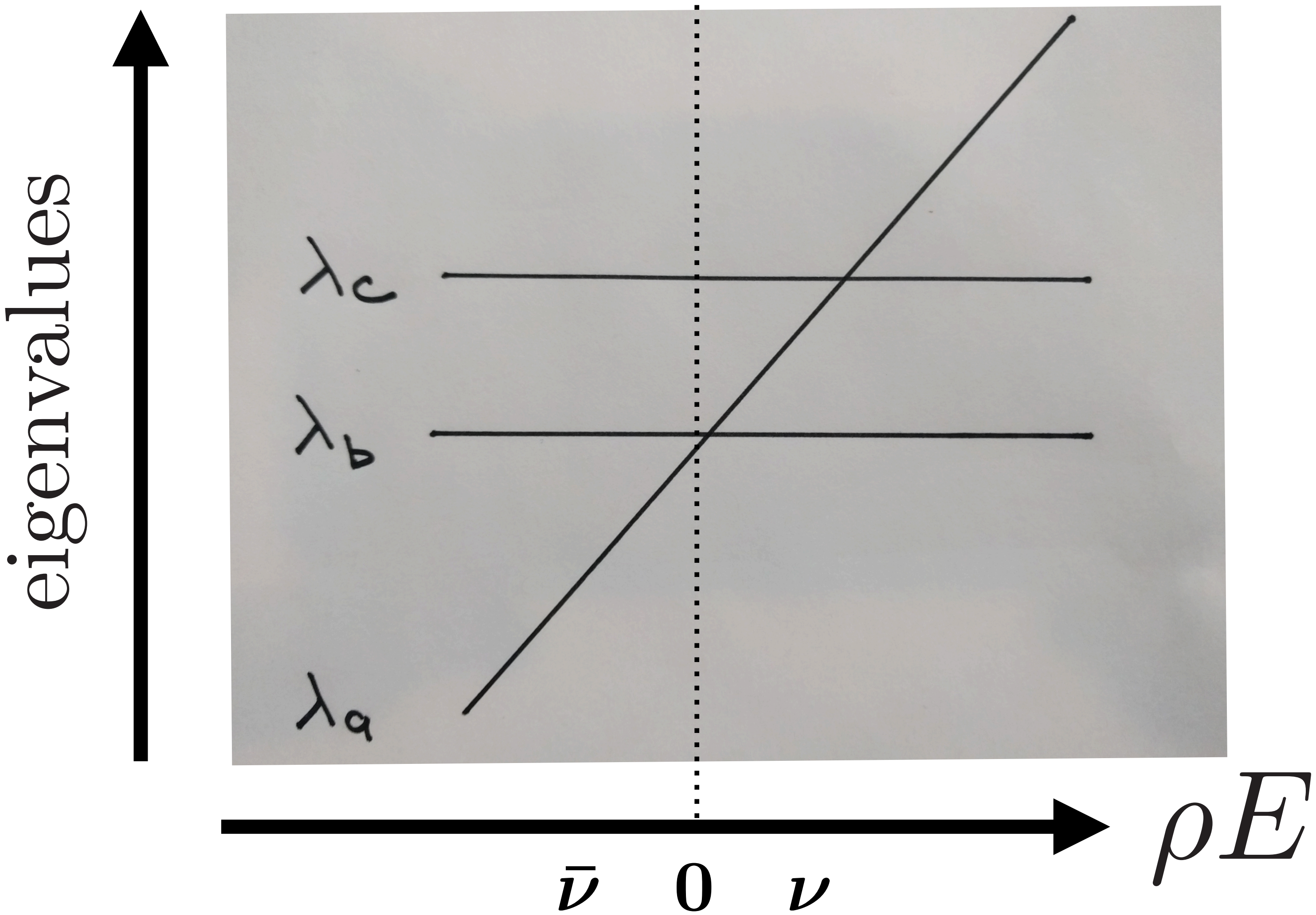
$$H = \frac{1}{2E}\left\{U\begin{bmatrix} 0 & 0 & 0 \\ 0 & \Delta m^2_{21} & 0 \\ 0 & 0 & \Delta m^2_{31} \end{bmatrix}U^\dagger + \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}$$

Rewrite as $\boldsymbol{H = H_0 + H_1}$

where H_0 is diagonal

and H_1 is off-diagonal.

$\theta_{13}, \ \theta_{12}, \theta_{23}, \ \delta$



$$H_0 = \frac{1}{2E}\begin{bmatrix} \lambda_a & & \\ & \lambda_b & \\ & & \lambda_c \end{bmatrix} = \frac{1}{2E}\,\mathrm{diag}(\lambda_a, \lambda_b, \lambda_c)$$

$$\lambda_a = a + (s_{13}^2 + \epsilon s_{12}^2)\Delta m_{ee}^2\,,$$

$$\lambda_b = \epsilon c_{12}^2 \Delta m_{ee}^2\,,$$

$$\lambda_c = (c_{13}^2 + \epsilon s_{12}^2)\Delta m_{ee}^2\,,$$

$$H_1 = s_{13}c_{13}\frac{\Delta m_{ee}^2}{2E}\begin{bmatrix} & & 1 \\ & 0 & \\ 1 & & \end{bmatrix} + \epsilon\ s_{12}c_{12}\frac{\Delta m_{ee}^2}{2E}\begin{bmatrix} & c_{13} & \\ c_{13} & & -s_{13} \\ & -s_{13} & \end{bmatrix}$$

$s_{13}c_{13}$ ↑ **0.15**

ϵ ↑ **0.015**

Diagonization:

$$\begin{pmatrix} \lambda_\sigma & \\ & \lambda_\rho \end{pmatrix} = U(\phi)^\dagger \begin{pmatrix} \lambda_a & \lambda_x \\ \lambda_x & \lambda_c \end{pmatrix} U(\phi)$$

$$\text{Eigenvalues}: \qquad \lambda_{\rho,\sigma} = \frac{1}{2}\left[(\lambda_a + \lambda_c) \pm \sqrt{(\lambda_a - \lambda_c)^2 + 4\lambda_x^2}\right]$$

$$U(\phi) \equiv \begin{pmatrix} c_\phi & s_\phi \\ -s_\phi & c_\phi \end{pmatrix} : \quad \sin(2\phi) = \frac{4\lambda_x}{\lambda_\rho - \lambda_\sigma} \quad \text{and} \quad \cos(2\phi) = \frac{\lambda_c - \lambda_a}{\lambda_\rho - \lambda_\sigma}$$

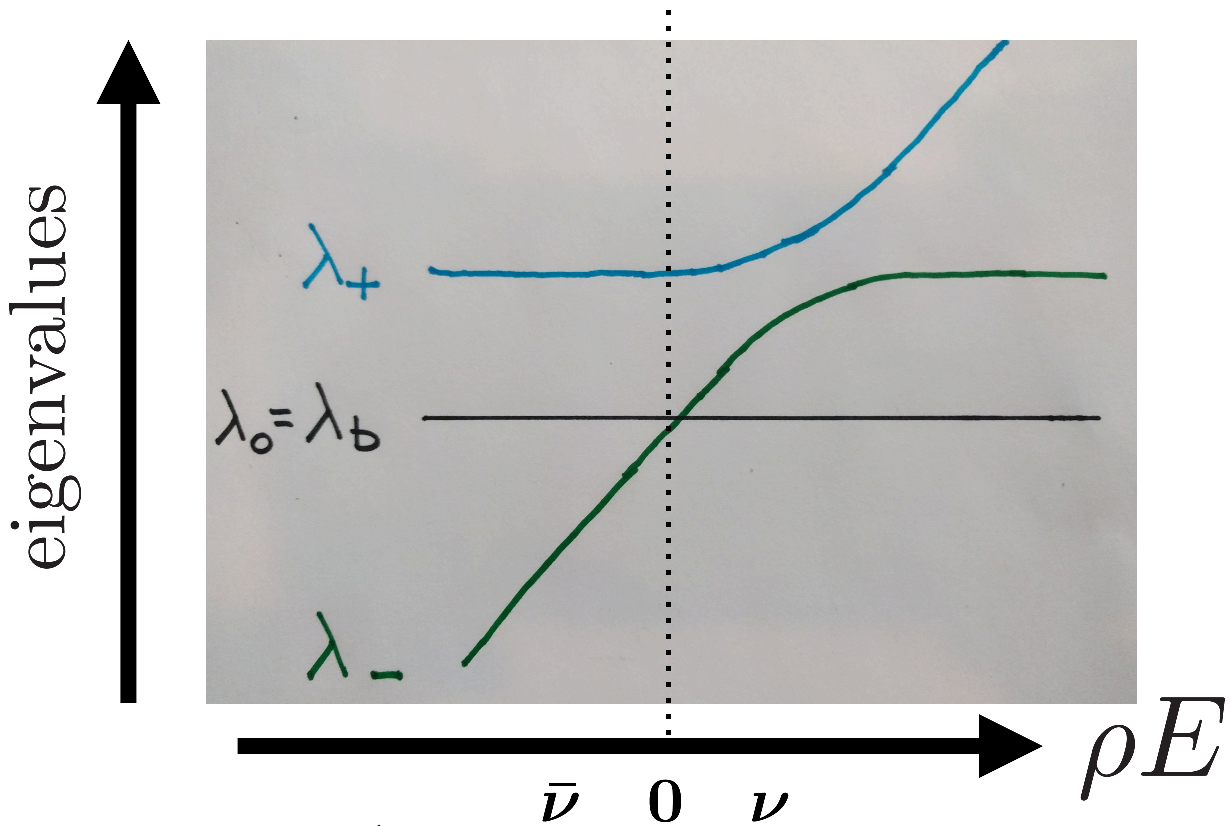
OR

$$c_\phi^2 = \frac{\lambda_\rho - \lambda_a}{\lambda_\rho - \lambda_\sigma} = \frac{\lambda_c - \lambda_\sigma}{\lambda_\rho - \lambda_\sigma}$$

$$s_\phi^2 = \frac{\lambda_\rho - \lambda_c}{\lambda_\rho - \lambda_\sigma} = \frac{\lambda_a - \lambda_\sigma}{\lambda_\rho - \lambda_\sigma}$$

1-3 Rotation

ϕ is θ_{13} in matter



$$\boldsymbol{\phi,\ \theta_{12}, \theta_{23},\ \delta}$$

$$s_\phi c_\phi = \frac{s_{13}c_{13}\Delta m^2_{ee}}{\lambda_+ - \lambda_-},$$

$$\lambda_{\mp} = \frac{1}{2}\left[(\lambda_a + \lambda_c) \mp \text{sign}(\Delta m^2_{ee})\sqrt{(\lambda_c - \lambda_a)^2 + 4(s_{13}c_{13}\Delta m^2_{ee})^2}\right],$$

$$\lambda_0 = \lambda_b = \epsilon c^2_{12}\Delta m^2_{ee},$$

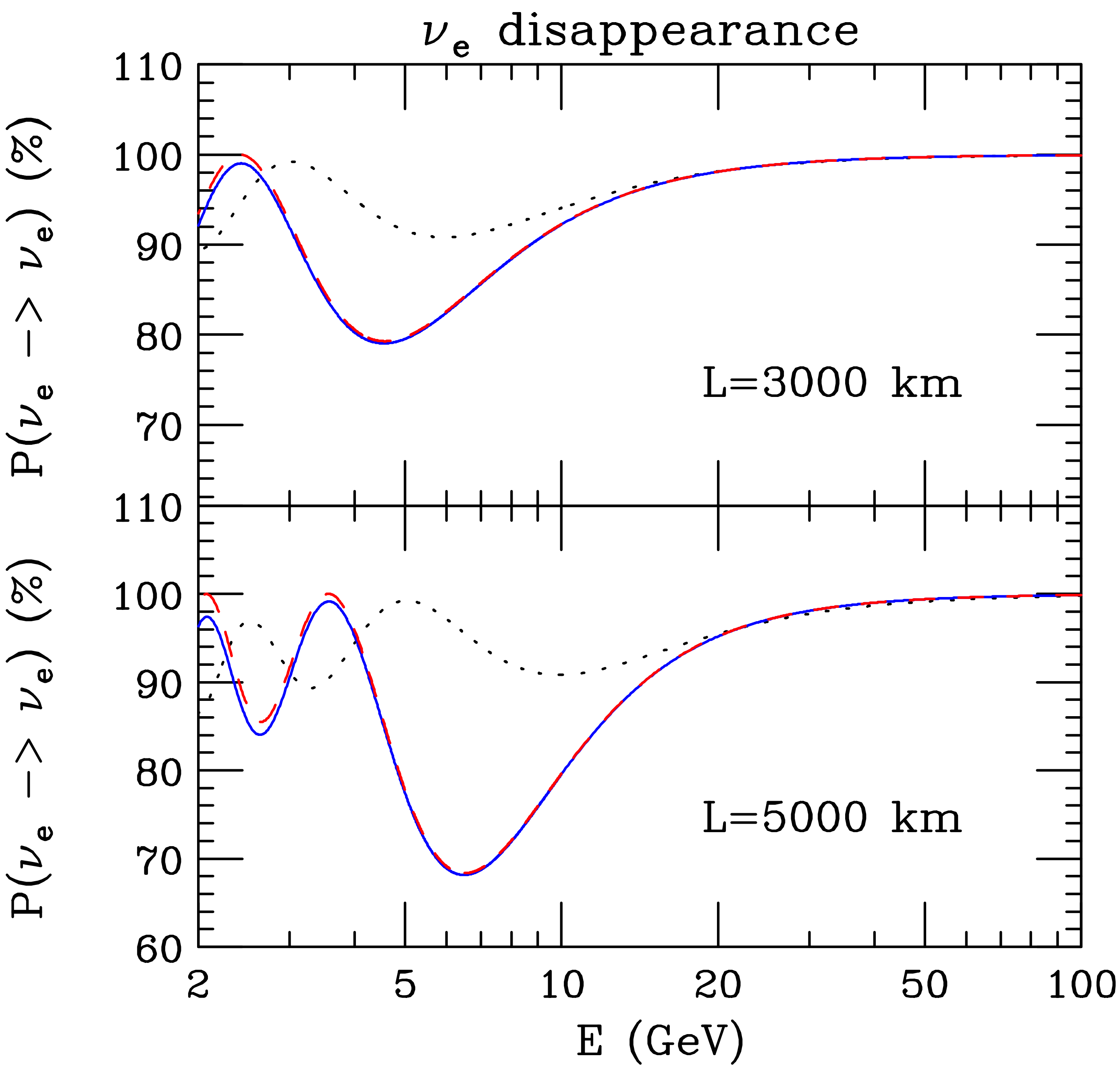
$$H_0 = \tfrac{1}{2E}\ \text{diag}(\lambda_-, \lambda_0, \lambda_+)$$

$$H_1 = \epsilon\ s_{12}c_{12}\frac{\Delta m^2_{ee}}{2E}\begin{bmatrix} & c_{(\phi-\theta_{13})} & \\ c_{(\phi-\theta_{13})} & & s_{(\phi-\theta_{13})} \\ & s_{(\phi-\theta_{13})} & \end{bmatrix}$$

H. Minakata + SP arXiv:1505.01826

ν_e Survival Probability:

$$P(\nu_e \to \nu_e) = 1 - \sin^2 2\phi \ \sin^2 \frac{(\lambda_+ - \lambda_-)L}{4E}$$



$$|\lambda_+ - \lambda_-| = \sqrt{(\Delta m^2_{ee} - a)^2 + 4 s^2_{13} a \Delta m^2_{ee}}$$

depth of first minimum

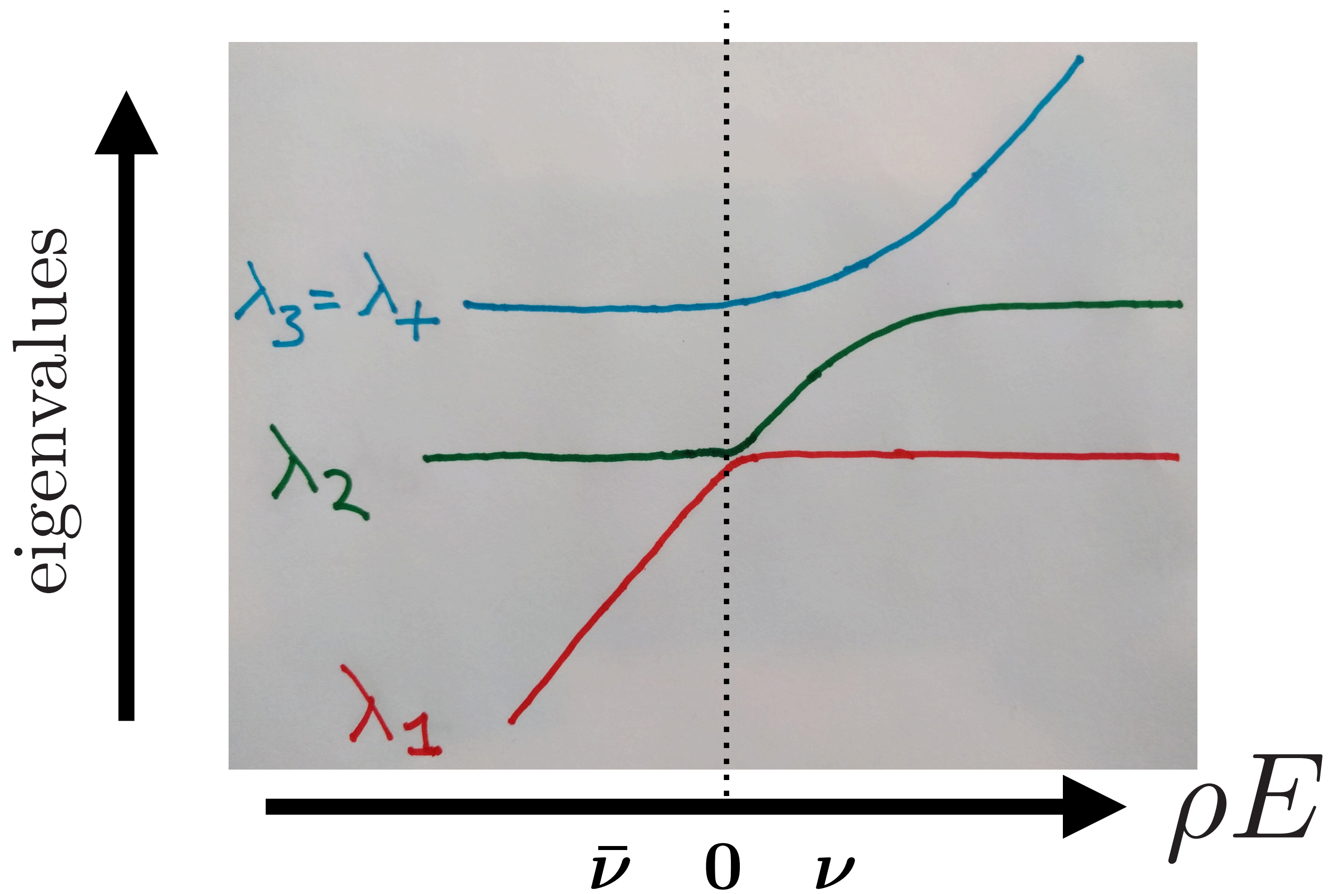
$$\sin^2 2\theta_{13} \quad \to \quad \left(\frac{\Delta m^2_{ee}}{\lambda_+ - \lambda_-}\right)^2 \sin^2 2\theta_{13}$$

energy at first minimum

$$\frac{\Delta m^2_{ee} L}{2\pi} \quad \to \quad \frac{(\lambda_+ - \lambda_-)L}{2\pi}.$$

exact - approx - vaccum

then 1-2 Rotation



ψ is θ_{12} in matter

$$\phi,\ \psi,\ \theta_{23},\ \delta$$

$$s_\psi c_\psi = \frac{\epsilon c_{(\phi-\theta_{13})} s_{12} c_{12} \Delta m^2_{ee}}{\Delta\lambda_{21}}$$

$$\lambda_{1,2} = \frac{1}{2}\left[(\lambda_0+\lambda_-) \mp \sqrt{(\lambda_0-\lambda_-)^2 + 4(\epsilon c_{(\phi-\theta_{13})} c_{12} s_{12} \Delta m^2_{ee})^2}\right],$$
$$\lambda_3 = \lambda_+ .$$

$$H_0 = \frac{1}{2E}\,\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)$$

$$H_1 = \epsilon\ s_{(\phi-\theta_{13})} s_{12} c_{12} \frac{\Delta m^2_{ee}}{2E} \begin{bmatrix} & & -s_\psi \\ & & c_\psi \\ -s_\psi & c_\psi & \end{bmatrix}$$

ZERO in vacuum !!!

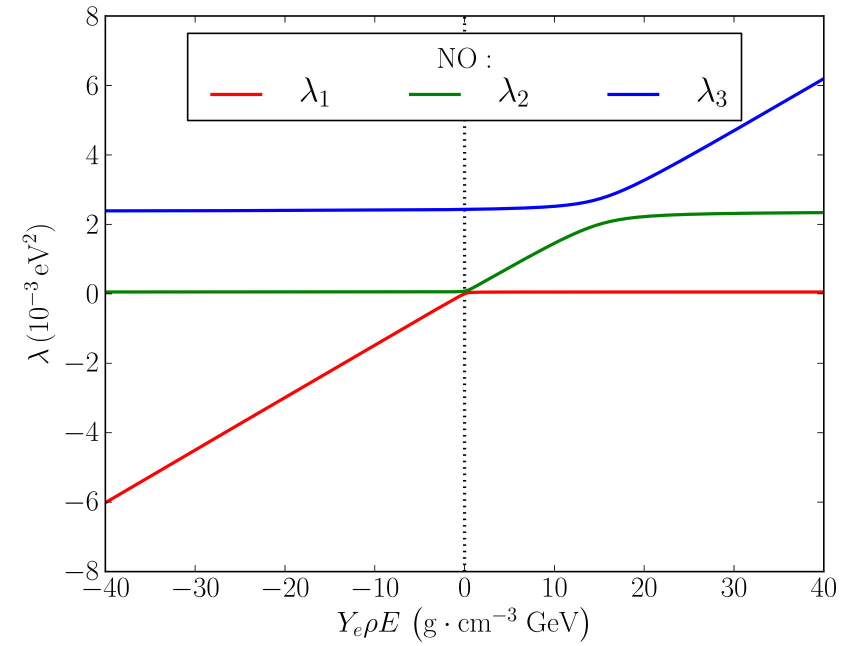
ϕ is θ_{13} in matter

P. Denton + H. Minakata + SP arXiv:1604.08167

Mixing Angles and Masses in Matter:

In matter $\theta_{13} \to \phi$ and $\theta_{12} \to \psi$ and

$$\Delta_{jk} \equiv \Delta m^2_{jk} L/4E \to (\lambda_j - \lambda_k) L/4E$$



$$\mathcal{A}_{\mu e} \equiv (2 s_{23} s_{13} c_{13})\ [\ c^2_{12}\ e^{i\Delta_{32}}\ \sin\Delta_{31} + s^2_{12}\ e^{i\Delta_{31}}\ \sin\Delta_{32}\ ]$$

$$+ (2 c_{23} c_{13} s_{12} c_{12})\ e^{-i\delta}\ \sin\Delta_{21}$$

$$P(\nu_\mu \to \nu_e) = |\mathcal{A}_{31} + \mathcal{A}_{32} + \mathcal{A}_{21}|^2$$

$$= 4\ s^2_{23} s^2_{13} c^2_{13}\ \ [c^4_{12} \sin^2\Delta_{31} + s^4_{12} \sin^2\Delta_{31}$$

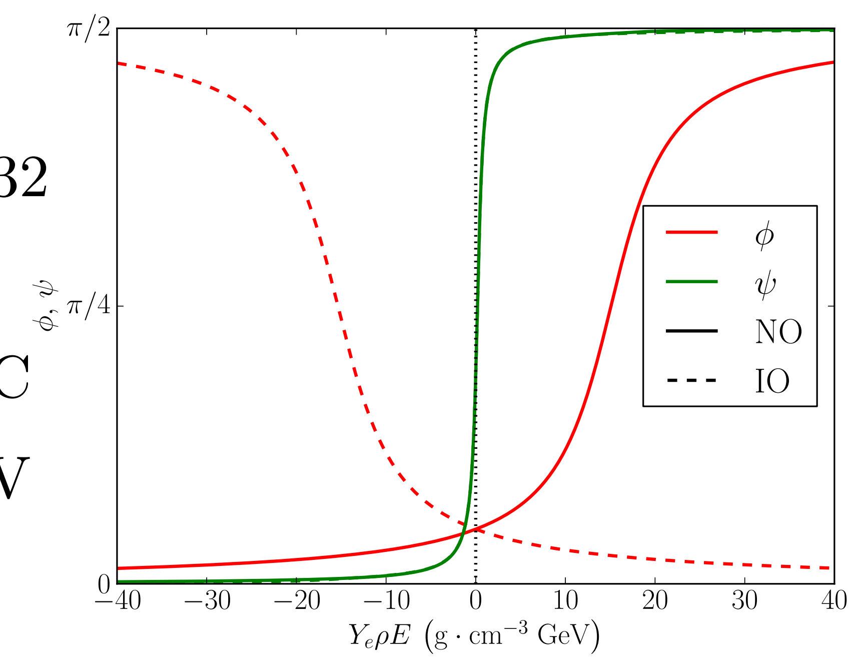
$$+ 2 s^2_{12} c^2_{12} \sin\Delta_{31} \sin\Delta_{32} \cos\Delta_{21}] \qquad 31/32$$

$$+ 8\ J_r \cos\delta\ \ \sin\Delta_{21}$$

$$[c^2_{12} \sin\Delta_{31} \cos\Delta_{32} + s^2_{12} \sin\Delta_{32} \cos\Delta_{31}] \qquad \mathrm{CPC}$$

$$-\ 8\ J_r \sin\delta\ \ \sin\Delta_{21} \sin\Delta_{31} \sin\Delta_{32} \qquad \mathrm{CPV}$$

$$+ 4\ c^2_{23} c^2_{13} s^2_{12} c^2_{12}\ \ \sin^2\Delta_{21} \qquad 21$$



Intuitive and Analytically simple !

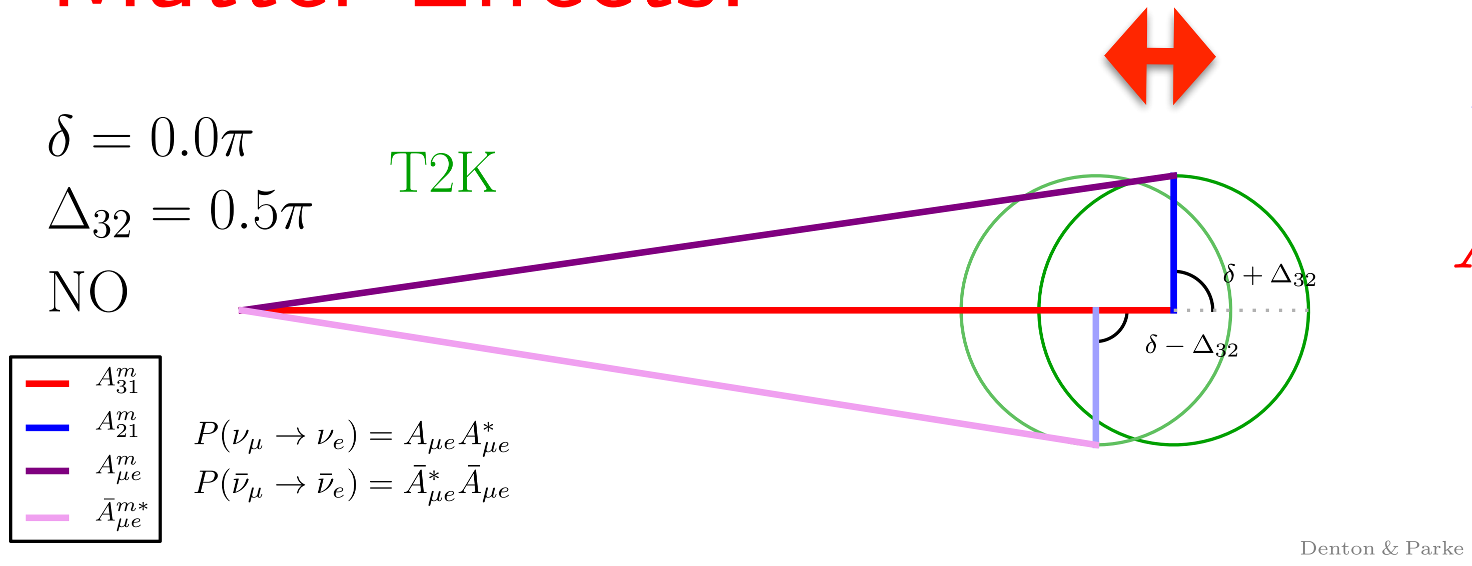
Matter Effects:

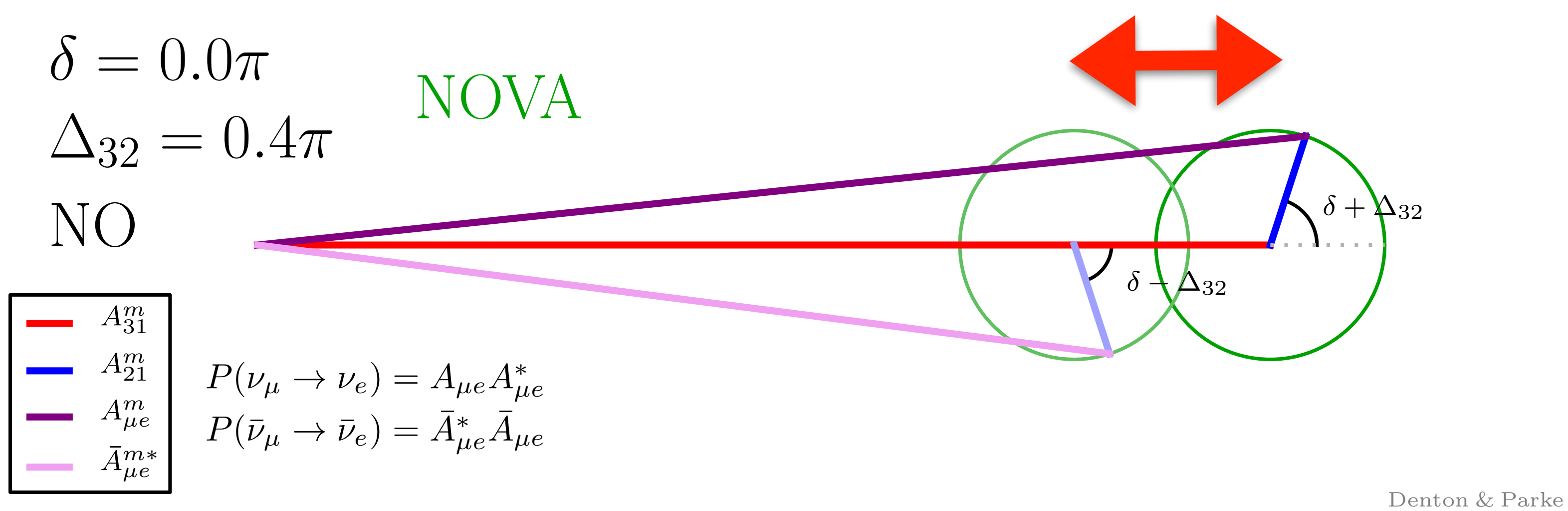
$$A_{31} + e^{i(\Delta_{32} \pm \delta)} A_{21}$$

$$A_{31} = 2 s_{23} s_{13} c_{13} \frac{\sin(\Delta_{31} \mp aL)}{(\Delta_{31} \mp aL)} \Delta_{31}$$

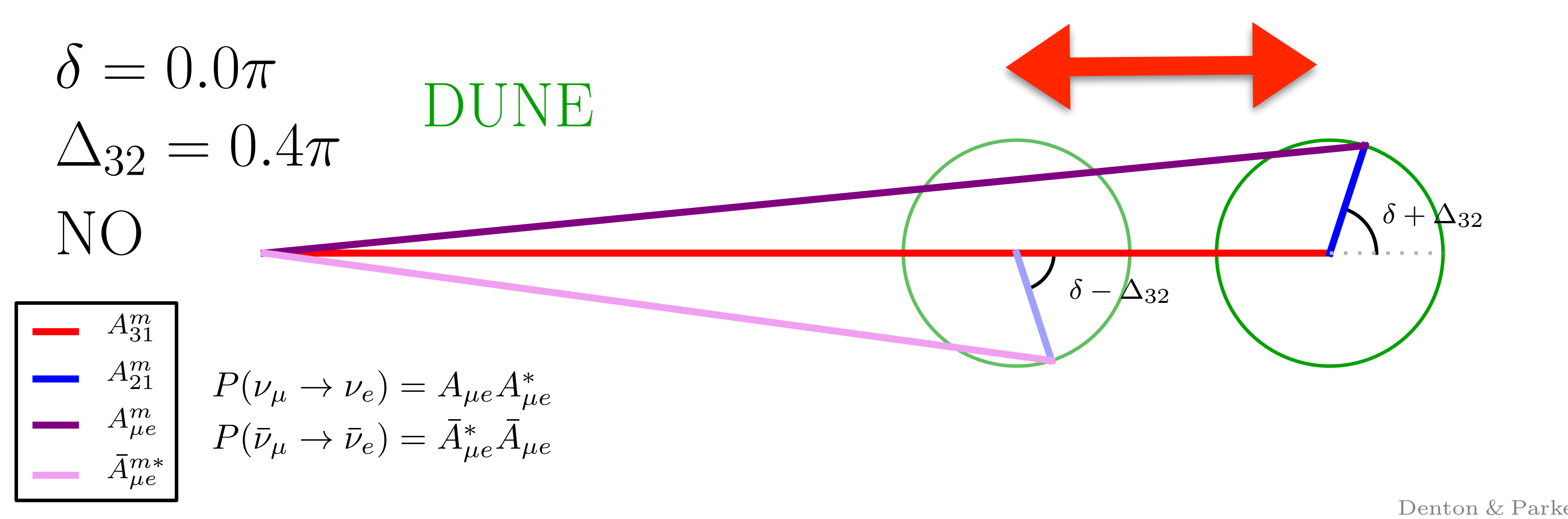
$$A_{21} = 2 c_{13} c_{23} s_{12} c_{12} \frac{\sin(aL)}{(aL)} \Delta_{21}$$

$$a = G_F N_e/\sqrt{2} = (4000\ km)^{-1},$$





$$\propto \rho L \ \sin^2 \theta_{23}$$



then Oscillation Probablities

with λ_i's and $V_{\alpha i}$ in matter then

$$P(\nu_\beta \to \nu_\alpha) = \left| \sum_{i=1}^{3} V_{\alpha i} V^*_{\beta i} \, e^{-i\frac{\lambda_i L}{2E}} \right|^2$$

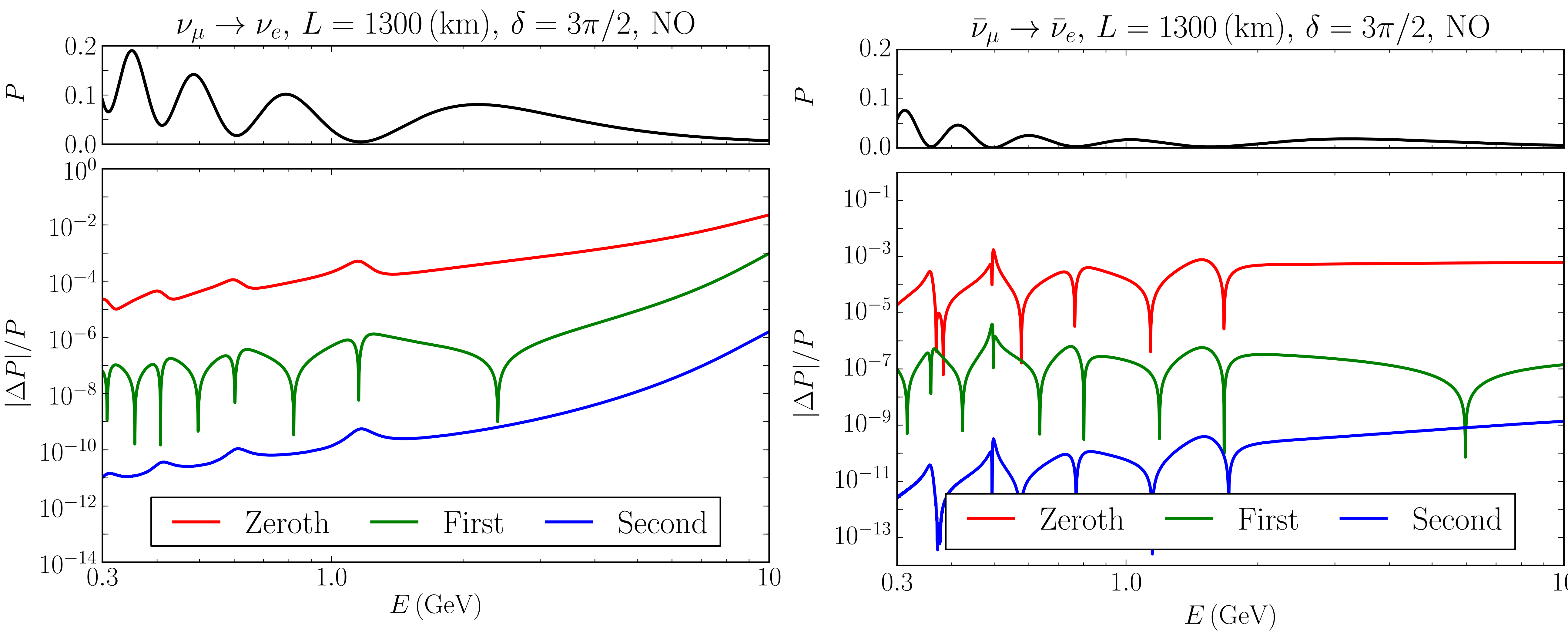
$$= \delta_{\alpha\beta} - 4 \sum_{j>i}^{3} \mathrm{Re}[V_{\alpha i} V^*_{\beta i} V^*_{\alpha j} V_{\beta j}] \sin^2 \frac{(\lambda_j - \lambda_i)L}{4E}$$

$$+ 8 \, \mathrm{Im}[V_{\alpha 1} V^*_{\beta 1} V^*_{\alpha 2} V_{\beta 2}] \; \sin \frac{(\lambda_3 - \lambda_2)L}{4E} \sin \frac{(\lambda_2 - \lambda_1)L}{4E} \sin \frac{(\lambda_1 - \lambda_3)L}{4E}$$

same as VACUUM with $m_i^2 \to \lambda_i$ and $U_{\alpha i} \to V_{\alpha i}$! ! !

Wronskian is nonvanishing,

New Perturbation Theory for Osc. Probabilities



systematic expansion

Conclusions:

- Harmony between

Perturbation Theory & General Expression

$$P(\nu_\beta \to \nu_\alpha) \;=\; \delta_{\alpha\beta} - 4\sum_{j>i}^{3} \mathrm{Re}[V_{\alpha i}V^*_{\beta i}V^*_{\alpha j}V_{\beta j}]\sin^2\frac{(\lambda_j-\lambda_i)L}{4E}$$

$$+\,8\ \mathrm{Im}[V_{\alpha 1}V^*_{\beta 1}V^*_{\alpha 2}V_{\beta 2}]\ \sin\frac{(\lambda_3-\lambda_2)L}{4E}\sin\frac{(\lambda_2-\lambda_1)L}{4E}\sin\frac{(\lambda_1-\lambda_3)L}{4E}$$

- New Perturbation Theory reveals

Structure, Simplicity and Universal Form
of Oscillation Probabilities in Matter

- Provides Advanced Understanding of

Neutrino Amplitudes in Matter

backup

Neutrino Oscillation Amplitudes

$$P(\nu_\alpha \to \nu_\beta) \quad = \quad |\mathcal{A}_{\alpha\beta}|^2$$

Two Flavors:

$$\Delta \equiv \Delta m^2 L/4E$$

$$\mathcal{A}_{\alpha\alpha} = 1 + (2i)\ s_\theta^2\ e^{+i\Delta}\ \sin\Delta$$

and $\mathcal{A}_{\alpha\beta} = (2i)\ s_\theta c_\theta\ e^{-i\Delta}\ \sin\Delta$

overall phase of amplitude arbitrary

Summary:

$$\Delta m^2_{ee} \equiv \Delta m^2_{31} - s^2_{12}\Delta m^2_{21}\,,$$

$$\epsilon \equiv \Delta m^2_{21}/\Delta m^2_{ee}\,.$$

$$\lambda_a = a + (s^2_{13} + \epsilon s^2_{12})\Delta m^2_{ee}\,,$$
$$\lambda_b = \epsilon c^2_{12}\Delta m^2_{ee}\,,$$
$$\lambda_c = (c^2_{13} + \epsilon s^2_{12})\Delta m^2_{ee}\,,$$

ϕ is θ_{13} in matter

$$\lambda_3 = \lambda_+$$

$$\lambda_{\mp} = \frac{1}{2}\left[(\lambda_a + \lambda_c) \mp \mathrm{sign}(\Delta m^2_{ee})\sqrt{(\lambda_c - \lambda_a)^2 + 4(s_{13}c_{13}\Delta m^2_{ee})^2}\right]$$
$$\lambda_0 = \lambda_b = \epsilon c^2_{12}\Delta m^2_{ee}\,,$$

$$s_\phi = \sqrt{\frac{\lambda_+ - \lambda_c}{\lambda_+ - \lambda_-}}\,, \qquad c_\phi = \sqrt{\frac{\lambda_c - \lambda_-}{\lambda_+ - \lambda_-}}$$

ψ is θ_{12} in matter

$$\lambda_{1,2}$$

$$\lambda_{1,2} = \frac{1}{2}\left[(\lambda_0 + \lambda_-) \mp \sqrt{(\lambda_0 - \lambda_-)^2 + 4(\epsilon c_{(\phi-\theta_{13})}c_{12}s_{12}\Delta m^2_{ee})^2}\right]$$
$$\lambda_3 = \lambda_+\,.$$

$$s_\psi = \sqrt{\frac{\lambda_2 - \lambda_0}{\Delta\lambda_{21}}}\,, \qquad c_\psi = \mathrm{sign}(\Delta\lambda_{21})\sqrt{\frac{\lambda_0 - \lambda_1}{\Delta\lambda_{21}}}\,.$$

θ_{23}, δ unchanged