CONTINUOUS APPROXIMATION OF QUASIPLURISUBHARMONIC FUNCTIONS

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ABSTRACT. Let X be a compact Kähler manifold and θ a smooth closed (1,1)-real form representing a big cohomology class $\alpha \in H^{1,1}(X,\mathbb{R})$. The purpose of this note is to show, using pluripotential and viscosity techniques, that any θ -plurisubharmonic function φ can be approximated from above by a decreasing sequence of *continuous* θ -plurisubharmonic functions with minimal singularities, assuming that there exists a single such function.

Dedicated to D.H.Phong on the occasion of his 60th birthday

1. Introduction

Let X be a compact Kähler manifold and $\alpha \in H^{1,1}(X,\mathbb{R})$ be a big cohomology class. Recall that a cohomology class $\alpha \in H^{1,1}(X,\mathbb{R})$ is big if it contains a Kähler current, i.e. a positive closed current which dominates a Kähler form.

Fix θ a smooth closed real (1,1)-form representing α . We denote by $PSH(X,\theta)$ the set of all θ -plurisubharmonic functions, i.e. those functions $\varphi: X \to \mathbb{R} \cup \{-\infty\}$ which can be written locally as the sum of a smooth and a plurisubharmonic function and such that the current $\theta + dd^c \varphi$ is a closed positive current, i.e.: $\theta + dd^c \varphi \geq 0$ in the sense of currents. It follows from the $\partial \overline{\partial}$ -lemma that any closed positive current T in α can be written as $T = \theta + dd^c \varphi$ for some $\varphi \in PSH(X, \theta)$. We use the standard normalization

$$d = \partial + \overline{\partial}, \quad d^c := \frac{1}{2i\pi} (\partial - \overline{\partial}) \text{ so that } dd^c = \frac{i}{\pi} \partial \overline{\partial}.$$

In general Kähler currents are too singular, so one usually prefers to work with positive currents in α having minimal singularities. A positive current $T = \theta + dd^c \varphi \in \alpha$ (resp. a θ -plurisubharmonic function φ) has minimal singularities if for every other positive current $S = \theta + dd^c \psi \in \alpha$, there exists $C \in \mathbb{R}$ such that $\psi \leq \varphi + C$ on X. The function

$$V_{\theta} := \sup\{v \mid v \in PSH(X, \theta) \text{ and } \sup_{Y} v \leq 0\}$$

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is an example of θ -psh function with minimal singularities. It satisfies $\sup_X V_{\theta} = 0$. We let

$$P(\alpha) := \{x \in X \mid V_{\theta}(x) = -\infty\} \text{ and } NB(\alpha) := \{x \in X \mid V_{\theta} \notin L^{\infty}_{loc}(\{x\})\}$$

denote respectively the polar locus and the non bounded locus of α . The definitions clearly do not depend on the choice of θ and coincide with the polar (resp. non bounded locus) of any θ -psh function with minimal singularities.

The purpose of this note is to show that if a big cohomology class contains *one* current having minimal singularities and exponentially continuous potentials, then there is actually plenty of such currents:

THEOREM. Let X be a compact Kähler manifold and let $\alpha \in H^{1,1}(X,\mathbb{R})$ be a big cohomology class such that the polar locus $P(\alpha)$ coincides with the non-bounded locus $NB(\alpha)$.

Fix $\theta \in \alpha$ a smooth representative and $T = \theta + dd^c \varphi$ a positive current in α , where $\varphi \in PSH(X, \theta)$. Then there exists $\varphi_j \in PSH(X, \theta)$ a sequence of exponentially continuous θ -plurisubharmonic functions which have minimal singularities and decrease towards φ .

We say here that a θ -psh function is exponentially continuous iff $e^{\varphi}: X \to \mathbb{R}$ is continuous. Observe that if there exists one exponentially continuous θ -psh function with minimal singularities, then $P(\alpha) = NB(\alpha)$.

The technical condition $P(\alpha) = NB(\alpha)$ is thus necessary. It is obviously satisfied when α is semi-positive, or even *bounded* (i.e. there exists a positive closed current in α with bounded potentials, a condition that has become important in complex dynamics recently, see [DG09]), since $P(\alpha) = NB(\alpha) = \emptyset$ in this case. A more subtle example of a big and nef class α with $P(\alpha) = NB(\alpha) \neq \emptyset$ has been given in [BEGZ10, Example 5.4].

It is easy to construct θ -plurisubharmonic functions ψ with $P(\psi) \subseteq NB(\psi)$, however we do not know of a single example of a big class α for which $P(\alpha)$ is strictly smaller than $NB(\alpha)$.

Despite the relative modesty of its conclusion, this result relies on three important tools:

- -the regularization techniques of Demailly as used in [BD12],
- -the resolution of degenerate complex Monge-Ampère equations in big cohomology classes, as developed in [BEGZ10],
 - -and the viscosity approach to complex Monge-Ampère equations [EGZ11].

The latter was developed in the case where α is both big and semi-positive, hence we need here to extend this technique so as to cover the general setting of big classes. We stress that the global viscosity comparison principle holds for a big cohomology class α if and only if $P(\alpha) = NB(\alpha)$ (Theorem 3.4).

Our approximation result is new even when α is both big and semipositive. Let us stress that continuous θ -plurisubharmonic functions are easy to regularize by using Richberg's technique [R68]. As a consequence we obtain the following:

COROLLARY. Let (V, ω_V) be a compact normal Kähler space and let φ be a ω_V -plurisubharmonic function on V. Then there exists a sequence (φ_j) of smooth ω_V -plurisubharmonic functions which decrease towards φ .

When ω_V is a Hodge form, this regularization can be seen as a consequence of the extension result of [CGZ13].

Approximation from above by regular objects is of central use in the theory of complex Monge-Ampère operators, as the latter are continuous along (and even defined through) such monotone sequences [BT82], while they are not continuous with respect to the weaker L^1 -topology [Ceg83].

Plan of the note. We first establish our main result when the underlying cohomology class is also semi-positive (section 2), as the viscosity technology is already available [EGZ11]; the corollary follows then easily by using Richberg's regularization result. We then (section 3) adapt the techniques of [EGZ11] to the general context of big cohomology classes. The technical condition $P(\alpha) = NB(\alpha)$ naturally shows up as it is necessary for the global viscosity comparison principle to hold. We finally (section 4) use recent stability estimates for big cohomology classes [GZ12] to obtain continuous solutions of slightly more general Monge-Ampère equations, which allow us to prove our main result.

Dédicace. C'est un plaisir de contribuer à ce volume en l'honneur de Duong Hong Phong, dont nous apprécions la générosité, la vision et le bon goût, tant mathématique que gastronomique!

2. The case of semi-positive classes

We fix once and for all (X, ω_X) a compact Kähler manifold of complex dimension $n, \alpha \in H^{1,1}(X,\mathbb{R})$ a big cohomology class and θ a smooth closed (1,1) form representing α .

2.1. Minimal vs analytic singularities. Recall that α is semi-positive if θ can be chosen to be a semi-positive form. In this case a θ -psh function has minimal singularities if and only if its is bounded. The easiest example of θ -psh functions with minimal singularities are constant functions which are indeed θ -psh iff θ is semipositive.

For more general α , θ -psh function with minimal singularities can be constructed as enveloppes, e.g.:

$$V_{\theta} := \sup\{v \mid v \in PSH(X, \theta) \text{ and } \sup_{X} v \le 0\}.$$

Note that if $V \in PSH(X, \theta)$ is another function with minimal singularities, then $V - V_{\theta}$ is globally bounded on X. Also if $\theta' = \theta + dd^{c}\rho$ is another smooth form representing α , then $PSH(X, \theta') = PSH(X, \theta) - \rho$ where $\rho \in \mathcal{C}^{\infty}(X, \mathbb{R})$ hence $V_{\theta} - V_{\theta'}$ is also globally bounded on X.

Definition 2.1. The polar locus of α is

$$P(\alpha) := \{ x \in X \mid V_{\theta}(x) = -\infty \}.$$

The non-bounded locus of α is

$$NB(\alpha) := \{ x \mid V_{\theta} \notin L^{\infty}_{loc}(\{x\}) \}$$

The observations above show that these definitions only depend on α . Clearly $P(\alpha) \subset NB(\alpha)$ and $NB(\alpha)$ is closed. We shall assume in the sequel that $P(\alpha) = NB(\alpha)$ which, as will turn out, is equivalent to saying that there exists one exponentially continuous θ -psh function with minimal singularities.

By definition α is big if it contains a Kähler current, i.e. there is a (singular) positive current $T \in \alpha$ and $\varepsilon > 0$ such that $T \geq \varepsilon \omega_X$. It follows from the regularization techniques of Demailly (see [Dem92]) that one can further assume that T has analytic singularities:

Definition 2.2. A positive closed current T has analytic singularities if it can be locally written $T = dd^c u$, with

$$u = \frac{c}{2} \log \left[\sum_{j=1}^{s} |f_j|^2 \right] + v,$$

where c > 0, v is smooth and the f_i 's are holomorphic functions.

We let $Amp(\alpha)$ denote the *ample locus* of α , i.e. the Zariski open subset of all points $x \in X$ for which there exists a Kähler current in α with analytic singularities which is smooth in a neighborhood of x.

It follows from the work of Boucksom [Bou04] that one can find a single Kähler current $T_0 = \theta + dd^c \psi_0$ with analytic singularities in α such that

$$Amp(\alpha) = X \setminus Sing T_0.$$

In the sequel we fix such a Kähler current T_0 and assume for simplicity that

$$T_0 > \omega_X$$
.

Observe that ψ_0 is exponentially continuous, however ψ_0 does not have minimal singularities unless α is Kähler (see [Bou04]).

Bounded vs continuous approximations. Fix $\varphi \in PSH(X, \theta)$ a θ -psh function. It is easy to approximate φ from above by a decreasing sequence of θ -psh functions with minimal singularities. Indeed we can set

$$\varphi_j := \max(\varphi, V_\theta - j) \in PSH(X, \theta).$$

The latter have minimal singularities and decrease to φ as $j \nearrow +\infty$. This construction needs however to be refined to get exponentially continuous θ -psh approximations with minimal singularities. The mere existence of exponentially continuous θ -psh functions ψ with minimal singularities is actually not obvious.

2.2. Continuous approximations in the semi-positive case. We show here our main result in the simpler case when α is both big and semi-positive.

Theorem 2.3. Assume $\alpha \in H^{1,1}(X,\mathbb{R})$ is big and semi-positive and let $\varphi \in PSH(X,\theta)$ be a θ -plurisubharmonic function.

Then there exists a sequence of continuous θ -plurisubharmonic functions which decrease towards φ .

Proof. Fix h_j a sequence of smooth functions decreasing to φ (recall that φ is upper semi-continuous) and set

$$\varphi_j = P(h_j) := \sup\{u \mid u \in PSH(X, \theta) \text{ and } u \le h_j\}.$$

Observe that $\varphi_i \in PSH(X, \theta)$ and $\varphi_i \leq h_i$ hence

$$\varphi \leq \varphi_{j+1} \leq \varphi_j$$
.

We claim that $\varphi = \lim_{\searrow} \varphi_j$. Indeed set $\psi := \lim_{\searrow} \varphi_j \ge \varphi$. Then $\psi \le h_j$ for all j and $\psi \in PSH(X, \theta)$, hence $\psi \le \varphi$, so that $\psi = \varphi$ as claimed.

It thus suffices to check that φ_j is continuous. It follows from the work of Berman and Demailly [BD12] that φ_j has locally bounded Laplacian on the ample locus $Amp(\alpha)$ of α , with

$$(\theta + dd^c \varphi_j)^n = \mathbf{1}_{\{P(h_j) = h_j\}} (\theta + dd^c h_j)^n \text{ in Amp}(\alpha).$$

The measure on the right hand side is absolutely continuous with respect to Lebesgue measure, with bounded density. It therefore follows from [EGZ11, Theorem C] that $\varphi_i = P(h_i)$ is continuous.

Remark 2.4. Observe that a key point in the proof above is that if h is a smooth function on X, then its θ -plurisubharmonic projection P(h) is a continuous θ -plurisubharmonic function with minimal singularities.

Although the proof is quite short in appearance, it uses several important tools: Demailly's regularization technique (which is heavily used in [BD12]), and the viscosity approach for degenerate complex Monge-Ampère equations developed in [EGZ11].

The proof of our main theorem follows exactly the same lines: the result of Berman-Demailly applies for general big classes, while [BEGZ10] produces solutions of complex Monge-Ampère equations with minimal singularities in big cohomology classes. It thus remains to extend the viscosity approach of [EGZ11] to the setting of big cohomology classes, which is the contents of the next section.

Since Richberg's regularization technique [R68] applies in a singular setting, we obtain the following interesting consequence:

Corollary 2.5. Let (V, ω_V) be a compact normal Kähler space and let φ be a ω_V -plurisubharmonic function on V. Then there exists a sequence (φ_j) of smooth ω_V -plurisubharmonic functions which decrease towards φ .

Proof. Fix $\varphi \in PSH(X, \omega_V)$. We can assume without loss of generality that $\varphi < 0$ on V. Let $\pi : X \to V$ be a desingularization of V and set $\theta := \pi^* \omega_V$. Then $\psi := \varphi \circ \pi \in PSH(X, \theta)$.

Since $\pi^*\{\omega_V\} \in H^{1,1}(X,\mathbb{R})$ is big, it follows from our previous result that we can find continuous θ -psh functions $\psi_j < 0$ which decrease towards ψ on X. Since π has connected fibers, one easily checks that

$$PSH(X, \theta) = \pi^* PSH(V, \omega_V),$$

in particular there exists $\varphi_j \in PSH(V, \omega_V) \cap \mathcal{C}^0(V)$ such that $\psi_j := \varphi_j \circ \pi$, with φ_j decreasing to φ .

We can now invoke Richberg's regularization result [R68] (see also [Dem92]): using local convolutions and patching, one can find smooth functions $(\varphi_{j,k})$ on V which decrease to φ_j as $k \to +\infty$ and such that $\varphi_{j,k} \in PSH(V, (1 + \varepsilon_k)\omega_V)$ with $\varepsilon_k \searrow 0$. We can also assume that $\varphi_{j,k} < 0$ on V. Set finally

$$u_j := \frac{1}{(1 + \varepsilon_j)} \varphi_{j,j} \in PSH(V, \omega_V) \cap \mathcal{C}^{\infty}(V).$$

We let the reader check that (u_i) still decreases to φ .

Remark 2.6. When ω_V has integer class, i.e. when it represents the first Chern class of an ample line bundle on V, the above result was obtained in [CGZ13] as a consequence of an extension result of ω_V -psh functions.

3. Viscosity approach in a big setting

We set here the basic frame for the viscosity approach to the equation

$$(DMA_v^{\varepsilon}) \qquad (\theta + dd^c \varphi)^n = e^{\varepsilon \varphi} v$$

where v is a volume form with nonnegative *continuous* density and $\varepsilon > 0$ is a real parameter.

3.1. Viscosity sub/super-solutions for big cohomology classes. To fit in with the viscosity point of view, we rewrite the Monge-Ampère equation as

$$(DMA_v^{\varepsilon}) \qquad e^{\varepsilon \varphi} v - (\theta + dd^c \varphi)^n = 0$$

Let $x \in X$. If $\kappa \in \Lambda^{1,1}T_xX$ we define κ_+^n to be κ^n if $\kappa \geq 0$ and 0 otherwise. For a technical reason, we will also consider a slight variant of (DMA_n^{ε}) ,

$$(DMA_v^{\varepsilon})_+ \qquad \qquad e^{\varepsilon \varphi} v - (\theta + dd^c \varphi)_+^n = 0$$

If $\varphi_x^{(2)}$ is the 2-jet at $x \in X$ of a \mathcal{C}^2 real valued function φ we set

$$F(\varphi_x^{(2)}) = F_v^{\varepsilon}(\varphi_x) = \begin{cases} e^{\varepsilon \varphi(x)} v_x - (\theta_x + dd^c \varphi_x)^n & \text{if } \theta + dd^c \varphi_x \ge 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Recall the following definition from [CIL92], [EGZ11, Definition 2.3]:

Definition 3.1. A subsolution of (DMA_v^{ε}) is an upper semi-continuous function $\varphi: X \to \mathbb{R} \cup \{-\infty\}$ such that $\varphi \not\equiv -\infty$ and the following property is satisfied: if $x_0 \in X$ and $q \in C^2$, defined in a neighborhood of x_0 , is such that $\varphi(x_0) = q(x_0)$ and

 $\varphi - q$ has a local maximum at x_0 ,

then $F(q_{x_0}^{(2)}) \leq 0$.

We say that φ has minimal singularities if there exists C>0 such that

$$V_{\theta} - C \le \varphi \le V_{\theta} + C$$
 on X .

It has been shown in [EGZ11, Corollary 2.6] that a viscosity subsolution φ of (DMA_v^{ε}) is a θ -psh function which satisfies $(\theta + dd^c \varphi)^n \geq e^{\varepsilon \varphi}v$ in the pluripotential sense of [BT82, BEGZ10].

We now slightly extend the concept of supersolution, so as to allow a supersolution to take $-\infty$ values:

Definition 3.2. A supersolution of (DMA_v^{ε}) is a supersolution of $(DMA_v^{\varepsilon})_+$, that is, a function $\varphi: X \to \mathbb{R} \cup \{\pm \infty\}$ such that e^{φ} is lower semicontinuous, $\varphi \not\equiv +\infty$, $\varphi \not\equiv -\infty$ and the following property is satisfied: if $x_0 \in X$ and $q \in C^2$, defined in a neighborhood of x_0 , is such that $\varphi(x_0) = q(x_0)$ and

$$\varphi - q$$
 has a local minimum at x_0 ,

then $F_+(q_{x_0}^{(2)}) \geq 0$.

We say that φ has minimal singularities if there exists C > 0 such that

$$(V_{\theta})_* - C \le \varphi \le (V_{\theta})_* + C \text{ on } X.$$

Here $(V_{\theta})_*$ denotes the lower semi-continuous regularization of V_{θ} . It is important to allow $-\infty$ values since we are trying to build a θ -psh viscosity solution of (DMA_v^{ε}) : in general such a function will be infinite at polar points $x_0 \in P(\alpha)$. Note that we don't impose any condition at such points.

Definition 3.3. A viscosity solution of (DMA_v^{ε}) is a function that is both a sub-and a supersolution. In particular a viscosity solution φ is automatically an exponentially continuous θ -plurisubharmonic function.

By comparison, a pluripotential solution of (DMA_v^{ε}) is an usc function $\varphi \in L_{loc}^{\infty}(\mathrm{Amp}(\alpha)) \cap \mathrm{PSH}(\mathrm{X}, \omega)$ such that

$$(\theta + dd^c \varphi)_{BT}^n = e^{\varepsilon \varphi} v$$
 in $\text{Amp}(\alpha)$

in the sense of Bedford-Taylor [BT82] (see [BEGZ10] for the slightly more general notion of non-pluripolar products): it follows from [BEGZ10] that such a pluripotential solution automatically has minimal singularities, however there is no continuity information, especially at points in $X \setminus \text{Amp}(\alpha)$, as this set is pluripolar hence invisible from the pluripotential point of view.

3.2. The big viscosity comparison principle.

Theorem 3.4. Let $\alpha \in H^{1,1}(X,\mathbb{R})$ be a big cohomology class and assume $\varepsilon > 0$ and v > 0. Let φ (resp. ψ) be a subsolution (resp. supersolution) of (DMA_v^{ε}) with minimal singularities, then

$$\varphi \leq \psi$$
 in Amp(α).

Moreover $\varphi \leq \psi$ on X if and only if $P(\alpha) = NB(\alpha)$.

Proof. We can assume $\varepsilon = 1$ without loss of generality.

We let $x_0 \in X$ denote a point that realizes the maximum of $e^{\varphi} - e^{\psi}$ on X. If $x_0 \in P(\alpha)$, then we conclude trivially: $\varphi(x_0) = -\infty$, hence $\max_X (e^{\varphi} - e^{\psi}) \leq 0$.

Assume now $x_0 \notin NB(\alpha)$. Then φ and ψ are locally bounded near x_0 . Since $NB(\alpha)$ is closed, we can choose complex coordinates (z^1, \ldots, z^n) near x_0 defining a biholomorphism identifying an open neighborhood of x_0 in $X - NB(\alpha)$ to the complex ball $B(0,5) \subset \mathbb{C}^n$ of radius 5 sending x_0 to the origin.

We define $h \in \mathcal{C}^2(\overline{B(0,5)},\mathbb{R})$ to be a local potential smooth up to the boundary for θ and extend it smoothly to X. In particular $dd^ch = \theta$ and $w_- := \varphi + h$ is a bounded viscosity subsolution of the equation

$$(dd^c w)^n = e^w W$$
 in $B(0,5)$

with W a positive and continuous volume form. On the other hand $w^+ = \psi + h$ is a bounded viscosity supersolution of the same equation.

Now choose C>0 such that $\sup_{x\in\overline{B(0,4)}}\max(|\varphi(x)|,|\psi(x)|)\leq C/1000$ and $\sup_{x\in\overline{B(0,4)}}|h(x)|\leq C/10$. With this constant C>0, construct as in [EGZ11, p. 1076] a smooth auxiliary function φ_3 on $\overline{B(0,4)}^2$. Using the same notations as in [EGZ11, p. 1077], fix $\beta>0$ and consider $(x_\beta,y_\beta)\in\overline{B(0,4)}^2$ such that:

$$M_{\beta} = \sup_{(x,y)\in \overline{B(0,4)}^2} w_{-}(x) - w^{+}(y) - \varphi_{3}(x,y) - \frac{\beta}{2}d^{2}(x,y)$$
$$= w_{-}(x_{\beta}) - w^{+}(y_{\beta}) - \varphi_{3}(x_{\beta},y_{\beta}) - \frac{\beta}{2}d^{2}(x_{\beta},y_{\beta}).$$

By construction, φ_3 is big enough outside $B(0,2)^2$ to ensure that the sup is achieved at some point $(x_{\beta}, y_{\beta}) \in B(0,2)^2$. Limit points (x,y) of (x_{β}, y_{β}) satisfy x = y and the construction forces φ_3 to vanishes to high order at such a limit point. Then, the argument of [EGZ11, p. 1077-1078] based on Ishii's version of the maximum principle (see [CIL92]) applies verbatim to prove that

$$\lim_{\beta \to 0} \sup w^+(x_\beta) - w^-(y_\beta) \ge 0$$

and enables us to conclude that $\varphi \leq \psi$.

However, if $x_0 \in NB(\alpha) \setminus P(\alpha)$, $\varphi(x_0) > -\infty$ since ϕ has minimal singularities, while $\psi(x_0) = -\infty$ since ψ has minimal singularities in the sense

of definition 3.2 and $x_0 \in NB(\alpha)$ implies that $(V_\theta)_*(x_0) = -\infty$. The global comparison principle thus fails if $P(\alpha) \neq NB(\alpha)$.

Let us now justify that in general we do have $\varphi \leq \psi$ on $Amp(\alpha)$. Let $T_0 = \theta + dd^c \psi_0$ be a Kähler current such that $Amp(\alpha) = X \setminus Sing(T_0)$, $\psi_0 \leq \psi$ and $\psi_0 \leq 0$. Fix $\delta > 0$ and consider $\varphi_\delta := (1 - \delta)\varphi + \delta\psi_0 + n\log(1 - \delta)$. We claim that φ_δ is again a subsolution of (DMA_v^{ε}) . Indeed there is nothing to test on $Sing(T_0)$, while in $Amp(\alpha)$

$$(\theta + dd^c \varphi_\delta)^n \ge (1 - \delta)^n (\theta + dd^c \varphi)^n \ge (1 - \delta)^n e^{\varphi} v \ge e^{\varphi_\delta} v,$$

as follows easily by interpreting these inequalities in the pluripotential sense (see [EGZ11, Proposition 1.11]).

Let x_{δ} be a point where the upper semi-continuous function $e^{\varphi_{\delta}} - e^{\psi}$ attains its maximum. If $x_{\delta} \in Sinq(T_0)$, then $e^{\varphi_{\delta}(x_{\delta})} = 0$ hence

$$e^{\varphi_{\delta}} \leq e^{\psi} \Rightarrow \varphi_{\delta} \leq \psi \text{ on } X.$$

If $x_{\delta} \in \text{Amp}(\alpha)$, then both φ_{δ} and ψ are locally bounded near x_{δ} and the argument above leads to the conclusion that $e^{\varphi_{\delta}(x_{\delta})} \leq e^{\psi(x_{\delta})}$ hence $\varphi_{\delta} \leq \psi$ on X. Letting δ decrease to zero, we infer that $\varphi \leq \psi$ in $\text{Amp}(\alpha)$.

3.3. Continuous solutions of big Monge-Ampère equations.

Theorem 3.5. Let $\alpha \in H^{1,1}(X,\mathbb{R})$ be a big cohomology class and assume $\varepsilon > 0$ and v > 0 is a continuous positive density. Then there exists a unique pluripotential solution φ of (DMA_v^{ε}) on X, such that

- (1) φ is a θ -plurisubharmonic function with minimal singularities,
- (2) φ is a viscosity solution in Amp(α) hence continuous there,
- (3) Its lower semicontinuous regularisation φ_* is a viscosity supersolution.

If $P(\alpha) = NB(\alpha)$ then φ is a viscosity solution of (DMA_v^{ε}) on X, hence e^{φ} is continuous on X.

Proof. We can always assume that $\varepsilon = 1$. Since the comparison principle holds on $Amp(\alpha)$, we can use Perron's method by considering the upper enveloppe of subsolutions.

It follows from [BEGZ10] that the equation (DMA_v^1) has a pluripotential solution ϕ_0 which is a θ -plurisubharmonic function on X satisfying the equation $(\theta + dd^c\phi_0)^n = e^{\phi_0}v$ weakly on X. Since the right hand side is a bounded volume form, it follows from the big version of Kolodziej's uniform estimates that φ_0 has minimal singularities.

Moreover from the definition of a subsolution in the big case and [EGZ11, Corollary 2.6], it follows that φ_0 is a viscosity subsolution to the equation (DMA_v^1) .

On the other hand, since by [BD12], V_{θ} satisfies the equation $(\theta + dd^{c}V_{\theta})^{n} = \mathbf{1}_{\{V_{\theta}=0\}}\theta^{n}$ in the pluripotential sense and the right hand side is a bounded volume form, it follows that for some constant C >> 1 the function $\phi_{1} := V_{\theta} + C$ satisfies the inequality $(\theta + dd^{c}\phi_{1})^{n} \leq e^{\phi_{1}}v$ in pluripotential sense. It

follows therefore from the proof of [EGZ11, Lemma 4.7(1)] that $\psi_1 := (\phi_1)_*$ is a (viscosity) supersolution to the equation (DMA_v^1) .

We can now consider the upper envelope of (viscosity) subsolutions,

$$\varphi := \sup \{ \psi : \psi \text{ viscosity subsolution}, \ \phi_0 \leq \psi \leq \psi_1 \},$$

which is a subsolution with minimal singularities to the equation (DMA_v^1) . Using the bump construction ([CIL92], [EGZ11]) we can show that the lower semi-continuous regularization φ_* of φ is a (viscosity) supersolution with minimal singularities to the equation (DMA_v^1) .

Therefore since the comparison principle holds on $Amp(\alpha)$, it follows that $\varphi \leq \varphi_*$ on $Amp(\alpha)$, hence $\varphi = \varphi_*$ on $Amp(\alpha)$ is a viscosity solution to the equation (DMA_v^1) on $Amp(\alpha)$.

If moreover $P(\alpha) = NB(\alpha)$ then we conclude by Theorem 3.4 that $\varphi = \varphi_*$ on X is a viscosity solution to the equation (DMA_v^1) on X, hence e^{φ} is continuous on X.

4. Pluripotential tools

4.1. Stability inequalities for big classes. The following result is the main stability inequality established in [GZ12]:

Theorem 4.1. Assume $(\theta + dd^c \varphi_{\mu})^n = f_{\mu} \omega_X^n$, $(\theta + dd^c \varphi_{\nu})^n = f_{\nu} \omega_X^n$, where the densities $0 \leq f_{\mu}$, f_{ν} are in $L^p(\omega_X^n)$ for some p > 1 and $\varphi_{\mu}, \varphi_{\nu} \in PSH(X,\theta)$ are normalized by $\sup_X \varphi_{\mu} = \sup_X \varphi_{\nu} = 0$. Then

$$\|\varphi_{\mu} - \varphi_{\nu}\|_{L^{\infty}(X)} \le M_{\tau} \|f_{\mu} - f_{\nu}\|_{L^{1}(X)}^{\tau},$$

where $M_{\tau} > 0$ only depends on upper bounds for the L^p norms of f_{μ} , f_{ν} and

$$0<\tau<\frac{1}{2^n(nq+1)-1},\quad \frac{1}{p}+\frac{1}{q}=1.$$

Corollary 4.2. Let $\alpha \in H^{1,1}(X,\mathbb{R})$ be a big cohomology class and assume $\varepsilon > 0$ and $v \geq 0$ is a probability measure with L^p -density with respect to Lebesgue measure, where p > 1. Then there exists a unique θ -plurisubharmonic function φ with minimal singularities which is a pluripotential solution of (DMA_v^{ε}) on X

Moreover φ is continuous on $Amp(\alpha)$ and if $P(\alpha) = NB(\alpha)$ then e^{φ} is also continuous on X.

Proof. The first part follows from [BEGZ10] and we get a unique pluripotential solution φ with minimal singularities. Let f denote the density of $v=f\omega_X^n$. We can approximate f by continuous and positive densities by using convolutions, locally $f_\delta:=f\star\chi_\delta+\delta,\,\delta>0$. Theorem 3.5 insures that there exists a unique $\varphi_\delta\in PSH(X,\theta)$ solution to

$$(\theta + dd^c \varphi_{\delta})^n = e^{\varepsilon \varphi_{\delta}} f_{\delta} \omega_{\mathbf{Y}}^n$$

which has minimal singularities and is continuous in Amp(α). It follows moreover from [BEGZ10] Theorem 4.1 that the functions $\varphi_{\delta} - V_{\theta}$ are uniformly bounded as $\delta \to 0^+$.

The family $\{\varphi_{\delta}\}_{\delta>0}$ is compact in the L^1 topology hence we can extract a sequence $(\delta_k)_k$ such that φ_{δ_k} converges almost everywhere and $\sup_X \varphi_{\delta_k}$ converges. We can now apply the stability inequality (Theorem 4.1) to $\tilde{\varphi}_{\delta} := \varphi_{\delta} - \sup_X \varphi_{\delta}$ to check that the functions $\tilde{\varphi}_{\delta_k}$ form a Cauchy sequence, so that the functions φ_{δ_k} form a Cauchy sequence too. The uniform limit $\phi = \lim \varphi_{\delta_k}$ has minimal singularities and satisfies $(\theta + dd^c \phi)^n = e^{\varepsilon \phi} f \omega_X^n$ in the pluripotential sense, hence coincides with φ .

4.2. The flat setting.

Theorem 4.3. Let $\alpha \in H^{1,1}(X,\mathbb{R})$ be a big cohomology class and assume $v \geq 0$ is a volume form with non-negative L^p -density with respect to Lebesgue measure such that $\int_X v = \operatorname{Vol}(\alpha)$ and p > 1. Then there exists a unique θ -plurisubharmonic function φ with minimal singularities which is a pluripotential solution of (DMA_v^0) on X and such that $\int_X \varphi dv = 0$.

Moreover φ is continuous on $Amp(\alpha)$ and if $P(\alpha) = NB(\alpha)$ then e^{φ} is also continuous on X.

Proof. Fix $\varepsilon > 0$ and let φ_{ε} be the unique solution of (DMA_v^{ε}) given by Corollary 4.2. It follows from [BEGZ10] that the functions $\varphi_{\varepsilon} - V_{\theta}$ are uniformly bounded, hence the argument of Corollary 4.2 enables to extract a Cauchy sequence whose limit ψ is a solution of (DMA_v^0) which satisfies

$$\int \psi \, dv = \lim_{\varepsilon \to 0} \int \left[\frac{e^{\varepsilon \varphi_{\varepsilon}} - 1}{\varepsilon} \right] \, dv = \int \varphi dv = 0,$$

hence $\varphi = \psi$ (by the uniqueness result proved in [BEGZ10]) has all required properties.

Corollary 4.4. The function V_{θ} is continuous if and only if $P(\alpha) = NB(\alpha)$.

Proof. Observe that if $e^{V_{\theta}}$ is continuous then $P(\alpha) = NB(\alpha)$, as points in $NB(\alpha) \setminus P(\alpha)$ correspond to points where V_{θ} is finite but not locally finite. Conversely it follows from the work of Berman-Demailly [BD12] that V_{θ}

$$(\theta + dd^c V_{\theta})^n = \mathbf{1}_{\{V_{\theta} = 0\}} \theta^n \text{ in Amp}(\alpha).$$

has locally bounded Laplacian in $Amp(\alpha)$ and satisfies

Since θ^n is smooth and the density $\mathbf{1}_{\{V_{\theta}=0\}}$ is bounded, it follows from previous theorem that V_{θ} is continuous on X if $P(\alpha) = NB(\alpha)$.

4.3. **Conclusion.** The proof of the main theorem proceeds now exactly as in that of Theorem 2.3: if φ is a given θ -psh function, we approximate it from above by a decreasing sequence of smooth functions h_j and set

$$\varphi_i := P(h_i) \in PSH(X, \theta).$$

These functions have minimal singularities, decrease to φ and solve the complex Monge-Ampère equation

$$(\theta + dd^c \varphi_j)^n = \mathbf{1}_{\{\varphi_j = h_j\}} (\theta + dd^c h_j)^n = f_j \omega_X^n,$$

where f_j is bounded. It follows therefore from Theorem 4.3 that φ_j is continuous if $P(\alpha) = NB(\alpha)$.

Let us conclude by mentionning that we don't know any example of a compact Kähler manifold X and a big cohomology class $\alpha \in H^{1,1}(X,\mathbb{R})$ such that $P(\alpha)$ does not coincide with $NB(\alpha)$.

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