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# DYNAMICS OF POLYNOMIAL MAPPINGS OF $\mathbb{C}^2$

By Vincent Guedj

*Abstract.* We study the dynamics of polynomial self mappings $f$ of $\mathbb{C}^2$. We construct, for a large class of mappings, an invariant measure $\mu$ which is mixing and of maximal entropy $h_\mu(f) = \max(\log d_t(f), \log \lambda_1(f))$, where $d_t(f)$ is the topological degree of $f$ and $\lambda_1(f)$ its first dynamical degree. To achieve this, we look at the meromorphic extensions of $f$ to smooth minimal compactifications of $\mathbb{C}^2$. When a good compactification is found, we construct an $f^*$-invariant Green current $T$ which contains many dynamical informations. When $\delta := d_t(f)/\lambda_1(f) > 1$, the measure $\mu$ is obtained as $\mu = dd^c(vT)$, where $v$ is a partial Green function defined on the support of $T$. When $\delta < 1, \mu = T \wedge T^-$ where $T^-$ is a globally defined $f_*$-invariant current.

**1. Introduction.** We study the dynamics of meromorphic self maps $f$: $X \to X$ of a compact Kähler manifold $X$. When $X$ is of general type, a result of Kobayashi and Ochiai [K-O 75] asserts that there exists only a finite number of such maps whose dynamics is henceforth trivial. On the other hand there are plenty of such maps when $X$ is rational, i.e., birationally equivalent to the complex projective space $\mathbb{P}^k$. A general theory has been developed by several authors in the last decade in the case $X = \mathbb{P}^k$; we refer to the survey of Sibony [Si 99] for a general introduction to the subject.

Our main interest here is in the dynamics of polynomial self mappings of $\mathbb{C}^2$. It is natural to consider the meromorphic extension of such maps $f$ to an "adapted" compactification $X$ of $\mathbb{C}^2$. Especially interesting is the case where the extension $\tilde{f}$: $X \to X$ is algebraically stable (see Definition 2.1). Unfortunately, this notion is not preserved under birational conjugacy. Thus one has to consider separately all the possible compactifications of $\mathbb{C}^2$ even if they are birationally equivalent. It was e.g. realized in [Fa-G 99] that $\mathbb{P}^1 \times \mathbb{P}^1$ is the good compactification of a large class of polynomial mappings of $\mathbb{C}^2$. We push further this observation by considering the case of Hirzebruch surfaces $X = \mathbb{F}_a$ (see Section 3). A next step would be to consider nonminimal smooth compactifications of $\mathbb{C}^2$. Indeed a natural question is whether every polynomial self mapping of $\mathbb{C}^2$ can be extended as an algebraically stable meromorphic self-map of some (nonnecessary minimal) compactification of $\mathbb{C}^2$.

There are two numerical data on $f$ which are invariant under birational conjugacy. These are the topological degree $d_t$ of $f$ (i.e., the number of preimages

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of a generic point) and the first dynamical degree $\lambda_1(f)$ defined as

$$\lambda_1(f) = \lim_{j \to +\infty} [\deg (f^j)]^{1/j},$$

where $\deg(f)$ denotes the algebraic degree of $f$, i.e., the degree of the preimage of a generic line $L$ in $\mathbb{C}^2$. They satisfy $1 \leq d_t \leq \lambda_1(f)^2$ (we only consider the case of *dominating* mappings, i.e., we exclude the case $d_t = 0$). Previous works have focused on the two extreme cases $d_t = 1$ (Hénon mappings, birational mappings) and $d_t = \lambda_1(f)^2$ (endomorphisms of $\mathbb{P}^2$)—see references in [Si 99]. Our aim here is to consider the intermediate cases $1 < d_t < \lambda_1(f)^2$. A crucial role is played by the ratio $\delta := d_t/\lambda_1(f)$. We construct an invariant mixing measure of maximal entropy

$$h_\mu(f) = h_{top}(f) = \max(\log d_t, \log \lambda_1(f))$$

for a large class of mappings such that $\delta \neq 1$. Our construction follows closely the tools developed in the study of Hénon mappings when $\delta < 1$ and those from endomorphisms of $\mathbb{P}^2$ when $\delta > 1$. The critical case $\delta = 1$ deserves a special treatment. Simple examples like $f(z, w) = (z^d, w + 1)$ show that the nonwandering set could be empty in $\mathbb{C}^2$.

We now describe more precisely the content of the paper. Our first main result (Theorem 2.1) gives a general construction of an $f^*$-invariant "Green current" $T$ for a dominating meromorphic self-map $f\colon X \to X$ on a compact Kähler manifold $X$. We follow the approach of Sibony [Si 99] who solved the case $X = \mathbb{P}^k$. Our proof differs from Sibony's in that it does not depend on the homogeneous representation of $\mathbb{P}^k$ as a quotient of $\mathbb{C}^{k+1}\setminus\{0\}$ under a $\mathbb{C}^*$ action. This construction therefore applies to more general situations such as $K3$-surfaces, where some biholomorphic mappings display interesting dynamics (see [Ca 99]) and shows that the main results in [Ca 99] also hold in the Kähler (nonprojective) case. Moreover our point of view yields very simple proofs of the link between *Supp* $T$ and the Julia set $J_f$ (Theorem 2.2) even in the case $X = \mathbb{P}^k$. We then establish several properties of the Green current, especially extremality properties (Proposition 2.3 and Theorem 2.5) which can be thought of as ergodic properties of $T$. This interpretation should shed some light on the proof of mixing in Section 5.

In Theorem 3.1 we give a description of positive closed currents of bidegree $(1, 1)$ on smooth projective toric varieties (a similar description was given in the author's thesis for homogeneous manifolds of the linear group $GL_m(\mathbb{C})$). This and the description of rational self maps should be useful tools to analyze the dynamics of polynomial self mappings of $\mathbb{C}^k$ which admit a "good" compactification to these manifolds. This is done carefully in case $X = \mathbb{F}_a$ is a smooth minimal compactification of $\mathbb{C}^2$ (see paragraphs 3.3 and 3.4). As a simple consequence, we show that any quadratic polynomial mapping $f\colon \mathbb{C}^2 \to \mathbb{C}^2$ admits an algebraically stable extension either to $\mathbb{P}^2$ or $\mathbb{F}^1$, $\mathbb{F}^2$ (Proposition 3.7).

In Sections 4 and 5 we focus on the case of polynomial mappings of $\mathbb{C}^2$. Under suitable hypotheses, we show that the potential $g$ of the Green current constructed in Theorem 2.1 naturally defines the basin of attraction $\Omega_\infty$ of a superattractive fixed point $q_\infty$ at infinity. It is continuous in $\mathbb{C}^2$ and $(g > 0) = \Omega_\infty$ corresponds to orbits $(f^n(p))_{n\geq 0}$ which grow to infinity with maximal exponential speed of order $\lambda_1(f)$ (Theorem 4.1). When $\delta > 1$, there might be orbits which grow to infinity with lower speed. It is therefore natural to consider a partial Green function, related to the speed of convergence to infinity of these remaining orbits. When the speed order (or growth order of $f$) is optimal, i.e. equals $\delta$, we construct an invariant measure $\mu$ which is mixing and of maximal entropy (Theorem 4.4 and Proposition 4.5). Such a construction was done in [Fa-G 99] in the case of polynomial skew-products of $\mathbb{C}^2$. Here it applies e.g. to mappings of the form $(P(w), Q(z)+R(w))$—see Example 4.1 and Remark 4.2. The mixing property of $\mu$ follows from an equidistribution result of Russakovskii and Shiffman [R-Sh 97] (see also [F-S 95] in the case of endomorphisms) and the crucial fact that $\mu$ does not charge pluripolar sets. This is the latter which motivated our alternative construction (no such information is guaranteed by the general construction given in [R-Sh 97]).

We address the case $\delta = d_t/\lambda_1(f) < 1$ in Section 5. The equidistribution of points does not hold anymore, however there is an analogous result replacing points by truncated positive closed currents (Proposition 5.5). We construct an $f_*$-invariant current $T^-$ (Theorem 5.1) which naturally yields an invariant measure $\mu = T^+ \wedge T^-$ as soon as the wedge product is well defined. The latter is shown to be mixing and of maximal entropy under suitable assumptions (Theorem 5.3).

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## 2. Green Currents.

**2.1. Construction of invariant currents.** Let $f$: $X \to X$ be a meromorphic self-map of a compact Kähler manifold $X$. Denote by $I_f$ the indeterminacy set of $f$, this is an analytic subset of $X$ of codimension greater than 2.

Let $\mathcal{T}(X)$ be the cone of positive closed currents of bidegree $(1,1)$ on $X$. It is possible to define, for every $T \in \mathcal{T}(X)$, the pull-back $f^*T$ of $T$ by $f$: if $V$ is a small open subset of $X\backslash I_f$ and $\varphi$ is a local potential of $T$ in $f(V)$, then we set $f^*T\mid_V := dd^c(\varphi\circ f)$. This definition is easily seen to be independent of the choice of local potentials and yields a current $f^*T \in \mathcal{T}(X\backslash I_f)$. By a result of Harvey-Polking (see [Ha-P 75]), it extends trivially and uniquely as $\widetilde{f^*T}$ a positive *closed* current through $I_f$ since $\operatorname{codim}_{\mathbb{C}} I_f \geq 2$.

We always assume $f$ is dominating, i.e., generically of maximal rank $\dim_{\mathbb{C}} X$. This insures that the mapping $T \in \mathcal{T}(X) \longmapsto \widetilde{f^*T} \in \mathcal{T}(X)$ is continuous. Moreover cohomology classes are preserved (see [Me 97] or [Si 99]); $f$ therefore induces a linear map

$$\Phi_f\colon\ H^{1,1}(X,\mathbb{R}) \rightarrow H^{1,1}(X,\mathbb{R})$$
$$[T] \mapsto [\widetilde{f^*T}].$$

In general $\Phi_{f^2} \neq \Phi_f \circ \Phi_f$: although $\widetilde{(f^2)^*T}$ and $\widetilde{f^*(\widetilde{f^*T})}$ clearly coincide on $X \backslash I_f \cup f^{-1}(I_f)$, the set $f^{-1}(I_f)$ might contain some hypersurface of $X$. This motivates the following:

*Definition* 2.1. A map $f\colon X \rightarrow X$ is algebraically stable if there is no $j \in \mathbb{N}$ and no complex hypersurface $V$ of $X$ s.t. $f^j(V \backslash I_{f^j}) \subset I_f$. In this case

$$\forall_j \in \mathbb{N},\ \Phi_{f^{j+1}} = \Phi_{f^j} \circ \Phi_f.$$

*Example* 2.1. When $X = \mathbb{P}^k$ is the complex projective space of dimension $k$, any rational self map $f\colon \mathbb{P}^k \rightarrow \mathbb{P}^k$ has the form $f = [P_0 : \cdots : P_k]$, where the $P_j$'s are homogeneous polynomials of the same degree $d$ with no common factor. The integer $d$ is called the algebraic degree of $f$. In this case $H^{1,1}(X,\mathbb{R}) \simeq \mathbb{R}, \Phi_f$ is multiplication by $d$ and the map $f$ is algebraically stable iff the algebraic degree of $f^j$ is $d^j$. This happens if e.g. $f$ is holomorphic, i.e. when $I_f = \emptyset$.

THEOREM 2.1. *Let $X$ be a compact Kähler manifold and $f\colon X \rightarrow X$ a dominating meromorphic self-map which is algebraically stable. Let $\omega \in \mathcal{T}(X)$ with continuous potential and assume $f^*\omega$ is cohomologous to $\lambda\omega$ ($f^*\omega \sim \lambda\omega$ for short), where $\lambda > 1$. Then there exists $T \in \mathcal{T}(X)$ such that*

(1) *$\frac{1}{\lambda^n}(f^n)^*\omega \longrightarrow T$ in the weak sense of currents. When $f$ is holomorphic there is uniform convergence of potentials therefore $T$ admits a continuous potential.*

(2) *$f^*T = \lambda T$ and $T \sim \omega$.*

(3) *If $\omega' \in \mathcal{T}(X)$ is cohomologous to $\omega$ and admits a locally bounded potential, then $\frac{1}{\lambda^n}(f^n)^*\omega' \longrightarrow T$.*

*Proof.* Since $X$ is Kähler, there exists $\psi \in L^1(X)$ s.t. $\frac{1}{\lambda}f^*\omega = \omega + dd^c\psi$. The function $\psi$ is "quasiplurisubharmonic" (see [De 92]), in particular $\psi$ is bounded from above on $X$ hence we can assume $\psi \leq 0$. As $f$ is algebraically stable, we can iterate the previous equation to get

$$\frac{1}{\lambda^n}(f^n)^*\omega = \omega + dd^c\psi_n, \quad \text{where} \quad \psi_n = \sum_{j=0}^{n-1} \frac{1}{\lambda^j}\psi \circ f^j.$$

The sequence $(\psi_n)$ is a decreasing sequence of quasiplurisubharmonic functions whose curvature is uniformly bounded from below by $dd^c\psi_n \geq -\omega$. Its limit $\psi_\infty$ is either identically $-\infty$ or a quasiplurisubharmonic function (see [Hö 83]). We show $\psi_\infty \not\equiv -\infty$. Consider

$$\sigma_n = \frac{1}{n}\sum_{j=1}^{n} \frac{1}{\lambda^j}(f^j)^*\omega.$$

This is a bounded sequence of currents in $\mathcal{T}(X)$ such that $\sigma_n \sim \omega$. It has therefore bounded mass and any cluster point $\sigma$ clearly satisfies $f^*\sigma = \lambda\sigma$ and $\sigma \sim \omega$. Fix $v \in L^1(X)$ s.t. $\sigma = \omega + dd^c v$. The functional equation yields

$$dd^c v = dd^c \psi + \frac{1}{\lambda} dd^c(v \circ f),$$

hence $v - \frac{1}{\lambda} v \circ f = \psi + c$ for some constant $c \in \mathbb{R}$. Replacing $v$ by $v - \frac{\lambda c}{\lambda - 1}$, we can assume $c = 0$. There follows that $\psi_n = v - \frac{1}{\lambda^n} v \circ f^n$, so $v \leq \psi_\infty \not\equiv -\infty$ since $v$ is bounded from above.

Set $T = \omega + dd^c\psi_\infty$. Then $T \sim \omega$ and $f^*T = \lambda T$. When $f$ is holomorphic, $\psi$ is also bounded from below hence $(\psi_j)$ uniformly converges towards $\psi_\infty$ which is therefore continuous.

Let $\omega' \in \mathcal{T}(X)$ be cohomologous to $\omega$. If $\omega'$ admits a locally bounded potential we can find a bounded function $\varphi$ on $X$ so that $\omega' = \omega + dd^c\varphi$. There follows that $\lambda^{-n}\varphi \circ f^n$ uniformly converges to 0 thus

$$\frac{1}{\lambda^n}(f^n)\omega' = \frac{1}{\lambda^n}(f^n)^*\omega + \frac{1}{\lambda^n} dd^c(\varphi \circ f^n) \longrightarrow T. \qquad \square$$

*Remark* 2.1. Similar convergence results have been previously established. When $f$ is holomorphic, the case $X = \mathbb{P}^k$ is due to Fornaess-Sibony [F-S 94] and Hubbard-Papadopol [H-P 94]. In an arithmetical context, Zhang [Z 95] considers the case where $[\omega] = c_1(L)$ is the first Chern class of a positive holomorphic line bundle.

When $f$ is merely meromorphic, such a construction was done by Hubbard [H 86] and Bedford-Sibony (see [B-Sm 91]) in case $f$ is a Hénon mapping. Sibony solved the case of a general rational selfmap of $\mathbb{P}^k$ in [Si 99] and a similar construction was done in [Fa-G 99] for multiprojective spaces $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_p}$.

**2.2. Dynamical interpretation.** We first recall some standard definitions from complex dynamics.

*Definition* 2.2. Let $f$: $X \to X$ be a dominating meromorphic self-map of a compact Kähler manifold $X$. We assume $f$ is algebraically stable.

- A point $x$ belongs to the Fatou set $\mathcal{F}_f$ of $f$ if there exists a neighborhood $U$ of $x$ such that $(f^n_{|U})$ is equicontinuous. The Julia set if $J_f = X \backslash \mathcal{F}_f$.
- A point $x$ is normal if there exists a neighborhood $U$ of $x$ and a neighborhood $V$ of the indeterminacy set $I_f$ such that $f^n(U) \cap V = \emptyset$ for all $n \in \mathbb{N}$. We denote by $\mathcal{N}_f$ the set of normal points.
- The map $f$ is said to be normal if $\mathcal{N}_f = X \backslash E_f$, where $E_f = \overline{\cup_{j\geq 1} I_{f^j}}$.

It follows from the definitions that $\mathcal{F}_f$ is an open set, $I_f \subset E_f \subset J_f$ and $\mathcal{N}_f$ is an open subset of $X \backslash E_f$. Note that holomorphic mappings are normal.

THEOREM 2.2. *Let $f, X, \omega, T$ be as in Theorem 2.1. Assume further that $\omega$ is a Kähler form. Then*
(1) *Supp $T \subset J_f$.*
(2) *$\mathcal{N}_f \backslash$SuppT $\subset \mathcal{F}_f$.*
*In particular if $f$ is normal, then $J_f$ = Supp T has a positive $2(k-1)$-Hausdorff dimensional measure (here $k = \dim_{\mathbb{C}} X \geq 2$).*

*Remark* 2.2. It follows from the proof of Theorem 2.1 that $T$ admits a continuous potential in $\mathcal{N}_f$. One can actually show (see [Br-D 99]) that $T$ admits Hölder-continuous potential of exponent $\alpha > 0$ in $\mathcal{N}_f$. It follows by standard arguments (see [Si 99]) that *Supp T* has positive $H_{2(k-1)+\alpha^-}$ measure.

*Proof.* Let $U$ be a small open subset of $\mathcal{F}_f$. We can assume $(f^{n_i})$ converges to some holomorphic mapping $h$ in $U$, hence $f^{n_i}(U) \subset U'$ for $i$ large enough. Since $\omega$ is Kähler, we can find $\omega'$ a smooth closed positive $(1,1)$-form such that $\omega' = 0$ in $U'$ and $\omega' \sim \omega$. By Theorem 2.1 we get

$$T = \lim \frac{1}{\lambda^{n_i}} (f^{n_i})^* \omega' = 0 \text{ in } U.$$

Conversely let $U$ be an open subset of $\mathcal{N}_f$ s.t. $\bar{U} \subset\subset \mathcal{N}_f \backslash Supp\, T$. Using the notations of the proof of Theorem 2.1, we have $T = \omega + dd^c \psi_\infty$ and $\lambda^{-n}(f^n)^*\omega = \omega + dd^c \psi_n$, therefore

$$(f^n)^*\omega = \lambda^n \left[ \frac{1}{\lambda^n} (f^n)^* \omega - T \right] = dd^c(\lambda^n[\psi_n - \psi_\infty]) \text{ in } U.$$

Now $\lambda^n |\psi_n - \psi_\infty| \leq C_U$ in $U$, therefore $(f^n)^*\omega$ admits a uniformly bounded potential. Since $\omega$ is Kähler, it follows from Chern-Levine-Nirenberg inequalities that the $L^2$-norm of the derivatives of $(f^n)$ is uniformly bounded in $U$. So is the $L^\infty$-norm by subharmonicity, hence $(f^n)$ is equicontinuous, i.e. $U \cap J_f = \emptyset$.

When $f$ is normal this yields $J_f$ = *Supp T*. It follows from the support theorem of Federer (see [Fe 69]) that *Supp T* has positive $2(k-1)$-Hausdorff measure. □

As will become clear in the forthcoming sections, the extremality properties of the Green current $T$ are related to the ergodic properties of certain invariant measures. This motivates the following:

PROPOSITION 2.1. *Let $f, X, \omega, T$ be as in Theorem 2.1. Then $T$ is an extremal point of the closed convex cone*

$$\mathcal{K}_{f^*}^{[\omega]} = \{S \in \mathcal{T}(X) / f^*S = \lambda S \text{ and } S \sim \omega\}.$$

*Remark* 2.3. When the $\Phi_f$-eigenspace associated to $\lambda$ is one-dimensional, any current $S$ satisfying $f^*S = \lambda S$ is cohomologous to $\omega$ and $T$ is extremal among those currents. This will be the case when $X$ is e.g. a Hirzebruch surface (see Section 3).

*Proof.* Consider $S \in \mathcal{K}_{f^*}^{[\omega]}$ and fix $v$ a potential for $S$, i.e. $S = \omega + dd^c v$. We have $T = \omega + dd^c\psi_\infty$, where $\psi_\infty$ is the potential defined in the proof of Theorem 2.1 by

$$\psi_\infty = \sum_{j\geq 0} \frac{1}{\lambda^j}\psi \circ f^j.$$

Since $f^*S = \lambda S$, we can assume $v - \lambda^{-1}v \circ f = \psi$. Composing with $f^j$, this yields $v \leq \psi_\infty$. Now if $S'$ is another current in $\mathcal{K}_{f^*}^{[\omega]}$ such that $T = (S+S')/2$, we can find $v' \in L^1(X)$ such that $v' - \lambda^{-1}v' \circ f = \psi$ and $S' = \omega + dd^c v'$. Therefore $u = (v+v')/2$ is another potential for $T$. It differs from $\psi_\infty$ by a constant which has to be 0 since $u - \lambda^{-1}u \circ f = \psi$. On the other hand $u \leq \psi_\infty$, therefore $v = v' = \psi_\infty$ hence $S = S' = T$, so $T$ is extremal. □

THEOREM 2.3. *Let $f, X, \omega, T$ be as in Theorem 2.1. Assume moreover that the $\Phi_f$ eigenspace associated to $\lambda$ is one-dimensional.*

*Then $T$ does not charge any complex hypersurface of $X$.*

*Remark* 2.4. This result is due to Sibony [Si 99] in the case $X = \mathbb{P}^k$ and we follow his approach. Our hypothesis on $\Phi_f$ is purely technical (and could be omitted with some more work, see [Fa 99]). Note however that it is satisfied when e.g. $X$ is a Hirzebruch surface (see Section 3).

*Proof.* The basic idea of the proof is as follows: if $T$ charges some irreducible hypersurface $V$ then its potential satisfies $\psi_{\infty|V} \equiv -\infty$. On the other hand, the invariance $f^*T = \lambda T$ implies $V$ (or some component of $f^{-j}(V)$ for some integer $j$) is invariant under $f$ (or some iterate of $f$), say $f(V\backslash I_f) \subset V$. If $f_{|V}$ is dominating (i.e. $\overline{f(V\backslash I_f)} = V$), then one can construct an invariant current on $V$ whose potential minorates $\psi_{\infty|V}$, contradicting $\psi_{\infty|V} \equiv -\infty$. We now make this more precise.

Our assumption on $\Phi_f$ insures $T$ is an extremal point in the cone of currents $S \in \mathcal{T}(X)$ satisfying $f^*S = \lambda S$ (see Remark 2.3).

By a theorem of Siu [Siu 74], $T$ can be decomposed as $T = T_1 + T_2$, where $T_1 \in \mathcal{T}(X)$ does not charge any hypersurface of $X$ and $T_2 = \sum c_j[V_j]$, where the $c_j$'s are nonnegative constants and the $V_j$'s are irreducible divisors of $X$. The invariance $f^*T = \lambda T$ yields $T_1 \leq \lambda^{-1}f^*T_1 \leq T$. Set

$$R_N = \frac{1}{N}\sum_{j=1}^{N} \frac{1}{\lambda^j}(f^j)^*T_1$$

and let $R$ be a cluster point of $(R_N)$. Then $f^*R = \lambda R$ and $T_1 \leq R \leq T$. By extremality of $T$ it follows that $R = cT$ for some constant $c \in [0, 1]$. Therefore either $c = 0$ and $T = T_2$ or $c = 1$ and $T = T_1$ does not charge any hypersurface.

There remains to show that $T \neq T_2$. Assuming $T = T_2 = \sum c_j[V_j]$, we infer again from the invariance and the extremality of $T$ that there exists $l \in \mathbb{N}^*$ with $V_0 \subset f^{-1}(V_0)$, otherwise the currents $R'_N = N^{-1} \sum_{j=1}^N \lambda^{-j}(f^j)^*T$ would not charge $V_0$. Assume $l = 1$ for simplicity. Since $f(V_0 \backslash I_f) \subset V_0$, we can define a decreasing sequence of analytic subsets of $X$

$$W_1 = \overline{f(V_0 \backslash I_f)}, \ldots, W_j = \overline{f(W_{j-1} \backslash I_f)}.$$

The analytic subset $W = \cap_j W_j$ is nonempty since $f$ is algebraically stable. Thus $W$ is an irreducible analytic subset of $X$ such that $\overline{f(W \backslash I_f)} = W$, i.e., $f_{|W}$ is a dominating self-map of $W$. If $W$ is reduced to a point, then it is a fixed point for $f$ which does not belong to $I_f$. Thus $\psi_\infty(p) > -\infty$ contradicting $\psi_{\infty|V_0} \equiv -\infty$.

Assume now $W$ has positive dimension. Set

$$\sigma_N = \frac{1}{N} \sum_{j=0}^{N-1} \frac{1}{\lambda^j} (f^j_{|W})^*(\omega_{|W}).$$

Then $(\sigma_N)$ is a bounded sequence of currents in $\mathcal{T}(W)$ which are cohomologous to $\omega_{|W}$. Let $\sigma$ be a cluster point of $(\sigma_N)$, then $(f_{|W})^*\sigma = \lambda\sigma$. We can argue as in the proof of Proposition 2.1 and find a potential $v \in L^1(W)$ for $\sigma$ on $W$ ($\sigma = \omega_{|W} + dd^c v$) such that $v \leq \psi_{\infty|W}$. Therefore $\psi_{\infty|W} \not\equiv -\infty$ and this contradicts $T_{|V_0} = c_0[V_0]$. □

We now show that the Green current is extremal in $\mathcal{T}(X)$ when $f$ is bimeromorphic, i.e., when there exists a meromorphic map $f^{-1}\colon X \to X$ such that $f^{-1} \circ f = f \circ f^{-1}$ is the identity outside some complex hypersurface. A similar result also appears in [G-S 00] for $X = \mathbb{P}^k$.

THEOREM 2.4. *Let $f, X, \omega, T$ be as in Theorem 2.1. Assume $\lambda > 1$ is the spectral radius of $\Phi_f$ and the corresponding $\Phi_f$-eigenspace is one-dimensional. Assume moreover $f$ is bimeromorphic.*

*Then $T$ is extremal in $\mathcal{T}(X)$.*

*Proof.* Let $S \in \mathcal{T}(X)$ be such that $0 \leq S \leq T$. We need to show that $S$ is proportional to $T$. Observe that $\lambda^j(f^{-j})^*T = T$ outside some critical hypersurface $H_j$. We define $S_j := \widetilde{\lambda^j(f^{-j})^*S}$ the trivial extension of $\lambda^j(f^{-j})^*S$ through $H_j$. By construction we get $0 \leq S_j \leq T$. Consider now $S'_j = \lambda^{-j}(f^j)^*S_j$. The invariance of $T$ yields again $0 \leq S'_j \leq T$. Therefore $S, S'_j$ do not charge any complex hypersurface and, since they coincide outside the critical set of $f^j$, it follows that

$$S = S'_j = \frac{1}{\lambda^j}(f^j)^* S_j.$$

Note that $S_j$ is cohomologous to $c_j\omega + \theta_j$, so

$$\frac{1}{\lambda^j}(f^j)^* S_j \sim c_j\omega + \frac{1}{\lambda^j}(f^j)^*\theta_j \sim S.$$

With our assumptions on the cohomology class of $\omega$, this insures $\theta_j = 0$ and $c_j = c \in [0,1]$ is independent of $j$. Therefore $S_j \sim c\omega$ and we now show that $\lambda^{-j}(f^j)^* S_j$ converges (in the weak sense of currents) towards $cT$ as $j$ goes to infinity. This will prove that $S = cT$.

Let $v_j, w_j \in L^1(X)$ be potentials for $S_j$ and $R_j := T - S_j$, in other words

$$S_j = c\omega + dd^c v_j \text{ and } R_j = (1-c)\omega + dd^c w_j.$$

We can assume without loss of generality that $w_j \leq 0$ and $v_j + w_j = \psi_\infty$, where $\psi_\infty$ denotes the potential of $T$ defined in the proof of Theorem 2.1. Since $\lambda^{-j}\psi_\infty \circ f^j \to 0$ and $(v_j)$ is bounded from above, we get $\lambda^{-j} v_j \circ f^j \to 0$ hence

$$S = \frac{1}{\lambda^j}(f^j)^* S_j = c\frac{1}{\lambda^j}(f^j)^*\omega + dd^c\left(\frac{1}{\lambda^j} v_j \circ f^j\right) \longrightarrow cT. \qquad \square$$

*Remark* 2.5. (i) It is an interesting problem to describe the cone $\mathcal{K}_{f^*}$ of $f^*$-invariant currents. A complete answer was given in [Fa-G 99] and [Fa 99] in case $f$ is a bimeromorphic self-mapping of a compact Kählerian surface $X$. It seems that invariant measures of maximal entropy should arise from such currents.

(ii) It was recently shown by Diller and Favre [D-Fa 00] that the $\Phi_f$ eigenspace associated to $\lambda_1(f)$ is always 1-dimensional if $d_t(f) < \lambda_1(f)^2$. Thus our cohomological assumption is automatically satisfied here.

**3. Algebraically stable mappings on rational surfaces.** When $X = \mathbb{CP}^k$, there is a useful description of rational self maps, using "homogeneous coordinates" (see e.g. Theorem 2.1 in [F-S 94]). These coordinates can be used to describe the cone $\mathcal{T}(\mathbb{P}^k)$. Such homogeneous coordinates exist for a broad class of toric varieties (see [Cox 95]). We use them to describe the cone $\mathcal{T}(X)$ in Section 3.1 and consider the particular case of Hirzebruch surfaces $\mathbb{F}_a$ in Section 3.2. Homogeneous representation of rational self maps of the $\mathbb{F}_a$'s are then explored in Sections 3.3 and 3.4.

**3.1. The cone $\mathcal{T}(X)$ on toric varieties.** Let $X$ be a smooth compact projective toric variety. According to [Cox 95], $X$ can be realized as a geometric quotient

$$X = \mathbb{C}^N \backslash Z / G$$

where $Z$ is an analytic subset of $\mathbb{C}^N$ of codimension greater than 2 and $G = Hom_{\mathbb{Z}}(Pic(X), \mathbb{C}^*) \simeq (\mathbb{C}^*)^r$ acts on $\mathbb{C}^N \backslash Z$ via

$$\lambda = (\lambda_1, \dots, \lambda_r) \in (\mathbb{C}^*)^r \longmapsto (\lambda^{a^1} z_1, \dots, \lambda^{a^N} z_N),$$

where $a^i = (a_{i1}, \dots, a_{ir}) \in \mathbb{N}^r$ are fixed and $\lambda^{a^i} := \lambda_1^{a^{i1}} \cdots \lambda_r^{a^{ir}}$. We denote by $\pi$: $\mathbb{C}^N \backslash Z \longrightarrow X$ the canonical projection. For $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{R}^r$, we set

$$\mathcal{P}_\alpha := \{\psi \in PSH(\mathbb{C}^N)/sup_B \Psi = 0 \text{ and } \Psi \text{ satisfies } (*)_\alpha\},$$

where $B$ denotes the unit ball of $\mathbb{C}^N$ and

$$(*)_\alpha \quad \Psi(\lambda^{a^1} z_1, \dots, \lambda^{a^N} z_N) = \sum_{i=1}^r \alpha_i \log |\lambda_i| + \Psi(z_1, \dots, z_N)$$

for all $(z, \lambda) \in \mathbb{C}^N \times (\mathbb{C}^*)^r$.

THEOREM 3.1. *Let $X$ be a smooth compact projective toric variety. There is a unique isomorphism $\pounds$ between $\mathcal{P} := \cup_{\alpha \in \mathbb{R}_+^r} \mathcal{P}_\alpha$, with the $L^1_{loc}$ topology, and $\mathcal{T}(X)$, endowed with the weak topology of currents, which satisfies the relation*

$$\pi^* \pounds\,(\psi) = dd^c \psi, \quad \forall \psi \in \mathcal{P}.$$

*Proof.* Let $\psi \in \mathcal{P}$. Given $s = (s_1, \dots, s_N)$: $U \to \mathbb{C}^N \backslash Z$ a local holomorphic section of $\pi$, we can define a positive closed current of bidegree $(1, 1)$ in $U$ setting $T_s := dd^c(\psi \circ s)$. If $s'$ is another section of $\pi$ in $U$, then $s' = (h^{a^1} s_1, \dots, s_N h^{a^N})$, where $h = (h_1, \dots, h_r)$: $U \to (\mathbb{C}^*)^r$ is holomorphic. Thus it follows from $(*)_\alpha$ that $T_s = T_{s'}$ since each $\log |h_i|$ is pluriharmonic in $U$. This shows that the definition is independent of the choice of a local section, hence $T$ defined in $U$ by $T_s$ is actually a globally well defined positive closed current of bidegree $(1, 1)$ on $X$ which we denote by $\mathcal{L}(\psi)$. Observe that $\pi^* \mathcal{L}(\psi) = dd^c \psi$ by construction. So $\mathcal{L}(\psi) = \mathcal{L}(\varphi)$ implies $\psi - \varphi$ is pluriharmonic with logarithmic growth in $\mathbb{C}^N$. Thus it is constant and the normalization yields $\psi \equiv \varphi$, that is $\mathcal{L}$ is injective.

We now show $\mathcal{L}$ is surjective. given $T \in \mathcal{T}(X)$, we can consider $\pi^* T \in \mathcal{T}(\mathbb{C}^N \backslash Z)$ which admits a trivial extension through $Z$ since $\text{codim}_{\mathbb{C}} Z \geq 2$ (see [Ha-P 75]). Since $H^1(\mathbb{C}^N, \mathcal{O}) = H^2_{dR}(\mathbb{C}^N, \mathbb{R}) = 0$, we can find $u \in PSH(\mathbb{C}^N)$ *s.t.* $\pi^* T = dd^c u$. Consider

$$v(z) := \int_{G_{\mathbb{R}}} u(g \cdot z)\, dg,$$

where $G_{\mathbb{R}} \simeq (\mathbb{R}^*)^k$ is the maximal compact subgroup of $G$ and $dg$ denotes the Haar measure of $G_{\mathbb{R}}$. Since $\pi^*T$ is invariant under the action of $G_{\mathbb{R}}$, we infer $\pi^*T = dd^c v$.

Given $g \in G$, the function $w_g$: $z \in \mathbb{C}^N \mapsto v(g \cdot z) - v(z)$ is pluriharmonic in $\mathbb{C}^N$ and invariant under the rotations of $G_{\mathbb{R}}$, therefore it is constant: $v(g \cdot z) = c(g) + v(z)$. The map $c$: $g = (\lambda_1, \dots, \lambda_r) \in G = (\mathbb{C}^*)^r \to c(g) \in \mathbb{R}$ satisfies $c(g \cdot g') = c(g) + c(g')$ and $c(g) = 0$ if $g \in G_{\mathbb{R}}$. Moreover $c(g) \geq 0$ if $g = (\lambda_1, \dots, \lambda_r)$ is such that $| \lambda_i | \geq 1$ for all $i$. This follows from convexity properties of psh functions (see [K 91]). Thus we end up with a group morphism

$$h\colon (\mathbb{R}^r, +) \longrightarrow (\mathbb{R}, +)$$
$$(t_1, \dots, t_r) \longmapsto c(e^{t1}, \dots, e^{tr}).$$

The increasing properties of $c$ insure $h$ is continuous, hence there exists $\alpha_1, \dots, \alpha_r \geq 0$ such that $h(t_1, \dots, t_r) = \sum_{i=1}^r \alpha_i t_i$. The function $\Psi := v - K$ belongs to $\mathcal{P}_\alpha$ for an appropriate choice of the constant $K \in \mathbb{R}$ and it satisfies $\pi^*T = dd^c\Psi$.

Note that $\mathcal{L}$ is obviously continuous by construction. We can extend naturally $\mathcal{L}$ as a one-to-one linear mapping $\mathcal{L}$: $\mathcal{P} \otimes \mathbb{R} \to \mathcal{T}(X) \otimes \mathbb{R}$. Note that $\mathcal{P} \otimes \mathbb{R}$ is a closed subspace of $L^1_{loc}(\mathbb{C}^N)$ and $\mathcal{T}(X) \otimes \mathbb{R}$ is also closed in the space of closed currents of bidegree $(1, 1)$. It follows therefore from the open mapping theorem that $\mathcal{L}^{-1}$ is continuous.

Finally let $s$: $U \to \mathbb{C}^N \backslash Z$ be a local section of $\pi$. If $\pi^*T = dd^c\psi$, then $s^*(\pi^*T) = T_{|U} = dd^c(\psi \circ s)$. This shows $T$ (hence $\mathcal{L}$) is uniquely determined by the relation $\pi^*T = \pi^*\mathcal{L}(\psi) = dd^c\psi$. □

**3.2. Compactifications of $\mathbb{C}^2$.** Any smooth minimal compactification $X$ of $\mathbb{C}^2$ is a smooth projective toric surface, indeed it is either the projective space $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_a := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(0) \otimes \mathcal{O}_{\mathbb{P}^1}(a))$, $a \in \mathbb{N}\backslash\{1\}$ (see e.g. [P-Sc 91]). In this case $N = 4$, $Z = \{0\} \times \mathbb{C}^2 \cup \mathbb{C}^2 \times \{0\}$ and the action of $G \simeq (\mathbb{C}^*)^2$ is given by

$$(\lambda, \mu) \in (\mathbb{C}^*)^2\colon\ (z_0, z_1, w_0, w_1) \longmapsto (\lambda z_0, \lambda z_1, \lambda^{-a}\mu w_0, \mu w_1).$$

Thus $\mathbb{F}_a = \mathbb{C}^4 \backslash Z/G$ has the form described in 3.1 (we actually allow a negative integer $-a$ following the standard notations: we could equally well write $(\lambda z_0, \lambda z_1, \mu w_0, \lambda^a \mu w_1)$). We are going to give some more information about the cone $\mathcal{T}(\mathbb{F}_a)$.

• $\mathcal{C} = (z_0 = 0)$ and $\mathcal{C}' = (w_0 = 0)$ define two irreducible curves of $\mathbb{F}_a$ whose associated line bundles generate $Pic(\mathbb{F}_a) \simeq \mathbb{Z}^2$. They satisfy

$$\mathcal{C}^2 = 0, \quad \mathcal{C} \cdot \mathcal{C}' = 1, \quad \mathcal{C}'^2 = -a.$$

Moreover $\mathcal{C}'$ is the only irreducible curve with negative self intersection.

• Any curve $H_{\alpha,\alpha'}$ of $\mathbb{F}_a$ is defined as $\{P=0\}$, where $P$ is a bihomogeneous polynomial of bidegree $(\alpha,\alpha')$ in the sense that

$$P(\lambda z_0, \lambda z_1, \lambda^{-a}\mu w_0, \mu w_1) = \lambda^{\alpha}\mu^{\alpha'}P(z_0, z_1, w_0, w_1).$$

Observe that $\alpha$ might be negative, e.g. $P = w_0$ is a bihomogeneous polynomial of bidegree $(-a, 1)$ s.t. $(P=0) = \mathcal{C}'$. More precisely it is always true that $\alpha' \geq 0$ and $\alpha + a\alpha' \geq 0$, since any $H_{\alpha,\alpha'}$ is linearly equivalent to $(\alpha + a\alpha')\cdot\mathcal{C} + \alpha'\mathcal{C}'$.

Since $\mathcal{C}, \mathcal{C}'$ generate $H^{1,1}(\mathbb{F}_a, \mathbb{R}) \simeq \mathbb{R}^2$, we decompose $\mathcal{T}(\mathbb{F}_a) = \cup\mathcal{T}_{\alpha,\alpha'}(\mathbb{F}_a)$, where

$$\mathcal{T}_{\alpha,\alpha'}(\mathbb{F}_a) = \{T \in \mathcal{T}(\mathbb{F}_a)/T \sim (\alpha + a\alpha')[\mathcal{C}] + \alpha'[\mathcal{C}']\}.$$

With these notations, one checks easily that the isomorphism $\mathcal{L}$ described in Theorem 3.1 satisfies $\mathcal{L}(\mathcal{P}_{\alpha,\alpha'}) = \mathcal{T}_{\alpha,\alpha'}(\mathbb{F}_a)$, where $\Psi \in \mathcal{P}_{\alpha,\alpha'}$ satisfies

$$\Psi(\lambda z_0, \lambda z_1, \lambda^{-a}\mu w_0, \mu w_1) = \alpha \log|\lambda| + \alpha' \log|\mu| + \Psi(z_0, z_1, w_0, w_1).$$

As for divisors $\alpha' \geq 0$ and $\alpha$ might be negative. However the latter happens only in exceptional cases described by the following:

PROPOSITION 3.1. *Set* $\omega_1 = \mathcal{L}(\frac{1}{2}\log[|z_0^2| + |z_1^2|])$ *and*

$$\omega_2 = \mathcal{L}\left(\frac{1}{2}\log[(|z_0|^2 + |z_1|^2)^a|w_0|^2 + |w_1|^2]\right).$$

*Let* $T \in \mathcal{T}_{\alpha,\alpha'}(\mathbb{F}_a)$. *Then the following hold:*

• *$T$ is cohomologous to a Kähler form iff $\alpha > 0$ and $\alpha' > 0$.*

• *$T$ is cohomologous to a smooth semi-positive form iff $\alpha > 0$.*

• *There exists $\gamma \geq 0$ and $S \in \mathcal{T}_{\beta,\beta'}(\mathbb{F}_a)$ with $\beta = \alpha + a\gamma \geq 0$ and $\beta' = \alpha' - \gamma \geq 0$ such that $T = S + \gamma[\mathcal{C}']$. In particular if $\alpha < 0$ then $T$ charges the curve $\mathcal{C}' = (w_0 = 0)$.*

*Proof.* Observe that $\omega_1, \omega_2$ are smooth semi-positive forms on $\mathbb{F}_a$ such that

$$\omega_1 \sim [\mathcal{C}] \quad \text{and} \quad \omega_2 \sim a[\mathcal{C}] + [\mathcal{C}'].$$

Therefore $T \in \mathcal{T}_{\alpha,\alpha'}(\mathbb{F}_a)$ is cohomologous to $\alpha\omega_1 + \alpha'\omega_2$. Assume $T$ is cohomologous to a Kähler form. Then $[T]\cdot\mathcal{C} = \alpha' > 0$ and $[T]\cdot\mathcal{C}' = \alpha > 0$. Conversely if $\alpha, \alpha' > 0$, one can compute the Levi forms of $\Psi_1 = \mathcal{L}^{-1}(\omega_1)$ and $\Psi_2 = \mathcal{L}^{-1}(\omega_2)$ to check that $\alpha\omega_1 + \alpha'\omega_2$ is a Kähler form.

Similarly if $T$ is cohomologous to a smooth semi-positive form, then $[T]\cdot H \geq 0$ for any curve $H$ of $\mathbb{F}_a$. This yields $\alpha \geq 0$ when $H = \mathcal{C}'$. Conversely if $\alpha \geq 0$, then $T$ is cohomologous to $\alpha\omega_1 + \alpha'\omega_2$ which is smooth, semi-positive.

It remains to analyze the case $\alpha < 0$: By a theorem of Siu [Siu 74], we can decompose $T = \gamma[\mathcal{C}'] + S$, where $\gamma \geq 0$ and $S \in \mathcal{T}_{\beta,\beta'}(\mathbb{F}_a)$ has no mass on $\mathcal{C}'$.

Clearly $\beta = \alpha + a\gamma$ and $\beta' = \alpha' - \gamma \geq 0$. We claim $\beta \geq 0$, i.e. $[S] \cdot C' \geq 0$. To see this we can approximate $S$ in the weak sense of currents by rational divisors $S_j = \frac{1}{N_j}[P_j]$ which have no $C'$-component (see e.g. [G 99]). It follows that $[S_j] \cdot C' \geq 0$ hence $[S] \cdot C' \geq 0$. □

**3.3. Rational self maps of $\mathbb{F}_a$.** In order to apply Theorem 2.1, we describe the linear map $\Phi_f$ when $X = \mathbb{F}_a$ and give criteria for $f$ to be algebraically stable. In particular we give precise conditions in Section 3.4 so that a polynomial self-map of $\mathbb{C}^2$ admits an algebraically stable extension to some $\mathbb{F}_a$.

Let $f$: $\mathbb{F}_a \to \mathbb{F}_a$ be a dominating rational self-map of $\mathbb{F}_a$. It easily follows from the existence of homogeneous coordinates on $\mathbb{F}_a$ (see [Cox 95] and [Gu 95]) that there exists $F = (P_0, P_1, Q_0, Q_1)$ a polynomial self map of $\mathbb{C}^4$ with the following properties:

(1) The following diagram is commutative

$$\begin{array}{ccc} \mathbb{F}_a & \xrightarrow{f} & \mathbb{F}_a \\ \pi \uparrow & & \uparrow \pi \\ \mathbb{C}^4 \backslash Z & \xrightarrow{F} & \mathbb{C}^4 \backslash Z. \end{array}$$

(2) $P_0$ and $P_1$ are relatively prime. So are $Q_0$ and $Q_1$.

(3) $P_0, P_1$ are bihomogeneous of bidegree $(\alpha, \beta)$, $Q_1$ is bihomogeneous of bidegree $(\gamma, \delta)$ and $Q_0$ is bihomogeneous of bidegree $(\gamma - a\alpha, \delta - a\beta)$.

Moreover any polynomial self-map $H$ of $\mathbb{C}^4$ which satisfies (1) and (2) has the form $H = (\lambda P_0, \lambda P_1, \lambda^{-a}\mu Q_0, \mu Q_1)$ for some constants $(\lambda, \mu) \in (\mathbb{C}^*)^2$. Since the $P_i$'s and the $Q_j$'s define complex curves in $\mathbb{F}_a$, it follows from the previous section that $\beta \geq 0$ and $\delta \geq a\beta$. Moreover $\alpha \geq 0$ since otherwise $w_0$ would divide both $P_0$ and $P_1$ (Proposition 3.1). The induced linear map $\Phi_f$ is given, in the basis $([\omega_1], [\omega_2])$ by the "degrees of $f$":

$$A_f = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \in \mathcal{M}_2(\mathbb{N}).$$

In other words, $f^*\omega_1 \sim \alpha\omega_1 + \beta\omega_2$ and $f^*\omega_2 \sim \gamma\omega_1 + \delta\omega_2$.

*Definition* 3.1. Let $f$: $\mathbb{F}_a \to \mathbb{F}_a$ be a dominating rational self-map. The matrix $A_f = \left[\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right] \in \mathcal{M}_2(\mathbb{N})$ denotes the algebraic degrees of $f$. It is the matrix of the induced linear map $\Phi_f$: $H^{1,1}(\mathbb{F}_a, \mathbb{R}) \to H^{1,1}(\mathbb{F}_a, \mathbb{R})$ in the basis $([\omega_1], [\omega_2])$.

PROPOSITION 3.2. *Let $f$ be a dominating rational self-map of $\mathbb{F}_a$ and denote by $F = (P_0, P_1, Q_0, Q_1)$ a bihomogeneous representative of $f$. The indeterminacy set*

*$I_f$ of $f$ is the discrete set $I_f = I_P \cup I_Q$, where*

$$I_P = \{[z_0 : z_1 : w_0 : w_1] \in \mathbb{F}_a / P_i(z,w) = 0, 0 \leq i \leq 1\}$$
$$I_Q = \{[z_0 : z_1 : w_0 : w_1] \in \mathbb{F}_a / Q_j(z,w) = 0, 0 \leq j \leq 1\}.$$

*Proof.* Obvious. □

The following lemma gives useful criteria to decide whether a map is algebraically stable.

LEMMA 3.1. *Let $f, F$ be as above. The following are equivalent:*
*(1) $f$ is algebraically stable, i.e., there is no curve $C$ of $\mathbb{F}_a$ s.t. $f^n(C \backslash I_{f^n} \subset I_f$.*
*(2) $\forall n \in \mathbb{N}, F^n$ is a bihomogeneous representative of $f^n$.*
*(3) $\forall n \in \mathbb{N}, \Phi_{f^{n+1}} = \Phi_f \circ \Phi_{f^n}$.*

The proof is identical to the case $a = 0$ (see Proposition 1.8 in [Fa-G 99]). To illustrate the usefulness of bihomogeneous representatives, we now characterize the holomorphic self-maps of $\mathbb{F}_a$. The case $a = 0$ is well known (see e.g. Proposition 1.5 in [Fa-G 99]); we therefore assume $a \geq 1$.

PROPOSITION 3.3. *Let $f: \mathbb{F}_a \to \mathbb{F}_a$ be a dominating holomorphic map, $a \geq 1$. Then $A_f = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$ and $f$ admits a unique representative $F = (P_0, P_1, Q_0, Q_1)$ such that $Q_0 = w_0^\alpha$ and $Q_1 = w_1^\alpha + w_0\widetilde{Q_1}$, where $\widetilde{Q_1}$ is a bihomogeneous polynomial of bidegree $(a, \alpha - 1)$. Conversely, any $F$ of this form uniquely defines a holomorphic self-map of $\mathbb{F}_a$.*

*Proof.* Since $I_P = \emptyset$, the wedge product $[P_0 = 0] \wedge [P_1 = 0]$ is well defined and identically 0. This yields $\beta = 0$. Similarly $I_Q = \emptyset$ yields $\delta[\delta a + 2\gamma - a\alpha] = 0$. Now $\delta > 0$ since $\delta \geq a\beta$ and $f$ is dominating, hence $\delta a + 2\gamma = a\alpha$.

It follows that $\gamma - a\alpha = -\gamma - a\delta < 0$. As $Q_0$ is bihomogeneous of bidegree $(\gamma - a\alpha, \delta - a\beta)$, Proposition 3.1 insures $w_0$ divides $Q_0$. Thus $I_Q = \emptyset$ implies $Q_1(z_0, z_1, 0, 1) \neq 0$ for all $[z_0, z_1] \in \mathbb{P}^1$. Therefore $\gamma = 0$, $\alpha = \delta$ and $Q_1 = cw_1^\alpha + w_0\widetilde{Q_1}$, where $\widetilde{Q_1}$ is a bihomogeneous polynomial of bidegree $(a, \alpha - 1)$.

By Proposition 3.1 again, $Q_0$ which is bihomogeneous of bidegree $(-a\alpha, \alpha)$ has necessarily the form $Q_0 = c'w_0^\alpha$. We can normalize $F$ uniquely so that $c = c' = 1$. □

**3.4. Meromorphic extensions of polynomial mappings.** Consider now $f: (z,w) \in \mathbb{C}^2 \mapsto (P(z,w), Q(z,w)) \in \mathbb{C}^2$ a polynomial mapping. Set $\beta = \deg_w P$, $\delta = \deg_w Q$ and write $P(z,w) = \sum_{i=0}^{\beta} A_i(z)w^{\beta-i}$, $Q(z,w) = \sum_{j=0}^{\delta} B_j(z)w^{\delta-j}$. Set $\alpha = \deg A_0$, $\gamma = \deg B_0$ and consider

$$\widetilde{A}_i(z_0, z_1) = z_0^{\alpha+ia}A_i(z_1/z_0), \quad \widetilde{B}_j(z_0, z_1) = z_0^{\gamma+ja}B_j(z_1/z_0).$$

These are homogeneous polynomials in $(z_0, z_1)$ if $a$ is large enough. Now

$$P_1 = \sum_{i=0}^{\beta} \widetilde{A}_i(z_0, z_1) w_0^i w_1^{\beta-i} \quad \text{and} \quad Q_1 = \sum_{j=0}^{\delta} \widetilde{B}_j(z_0, z_1) w_0^j w_1^{\delta-j}$$

are bihomogeneous polynomials of bidegree $(\alpha, \beta)$, $(\gamma, \delta)$ is $a$ is chosen large enough so that the following condition is satisfied:

$$(*) \qquad \forall(i,j), \quad \alpha + ia \geq \deg A_i \quad \text{and} \quad \gamma + ja \geq \deg B_j.$$

In order to get an algebraically stable extension of $f$, we need to make another assumption on $a$. Set $t = \gamma - a\alpha$ and $s = \delta - a\beta$. The map $F = (z_0^{\alpha+a\beta} w_0^{\beta}, P_1, z_0^{t+as} w_0^s, Q_1)$ is a bihomogeneous representative of the extension $\tilde{f}$ of $f$ to $\mathbb{F}_a$, as soon as the following condition is satisfied:

$$(**) \qquad \delta \geq a\beta \quad \text{and} \quad \gamma + a(\delta - \alpha) - a^2\beta \geq 0.$$

In other words $t \geq 0$ and $t + as \geq 0$. Thus $a$ should not be chosen too large if $\beta \neq 0$. The two conditions $(*)$ and $(**)$ might be incompatible, however we have the following:

LEMMA 3.2. *If there exists $a \in \mathbb{N}$ satisfying $(*)$ and $(**)$ then the meromorphic extension $\tilde{f}$ of $f$ to $\mathbb{F}_a$ is algebraically stable.*

*Proof.* The only curves that can be contracted to a point of indeterminacy are the curves $\mathcal{C} = (z_0 = 0)$ and $\mathcal{C}' = (w_0 = 0)$ at infinity. They are either fixed or sent to the point $q_\infty = [0 : 1 : 0 : 1]$. Now $P_1(0, 1, 0, 1) = A_0(0, 1) \neq 0$ since $\deg A_0 = \alpha$, hence $q_\infty \neq I_P$. Similarly $q_\infty \neq I_Q$, therefore $f$ is algebraically stable. □

*Example* 3.1. Consider $f = (z^2 + zw, z^4 + z^3 w^\delta)$, $\delta \geq 2$. Then $\alpha = \beta = 1$, $\gamma = 3$ and

$$\widetilde{A_0} = z_1, \ \widetilde{A_1} = z_0^{a-1} z_1^2, \ \widetilde{B_0} = z_1^3, \ \widetilde{B_\delta} = z_0^{a\delta-1} z_1^4.$$

It follows that $f$ admits an algebraically stable extension to $\mathbb{F}_a$ for $a$ such that $1 \leq a \leq \delta - 1$. However the meromorphic extension of $f$ to $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ are not algebraically stable. Note that the first dynamical degree of $f$ is

$$\lambda_1(f) = \frac{1 + \delta + \sqrt{(\delta - 1)^2 + 12}}{2}$$

hence it is not an integer if $\delta \neq 3$

*Example* 3.2. (Polynomial skew-products) Consider $f = (P(z), Q(z,w))$, where $\deg P = \alpha$, $\deg_w Q = \delta$ and $Q = \sum_{j=0}^{\delta} B_j(z) w^{\delta - j}$ with $\deg B_0 = \gamma$. Condition $(**)$ becomes $\gamma + a(\delta - \alpha) \geq 0$ since $\beta = 0$. Therefore if $\delta \geq \alpha$, $f$ admits an algebraically stable extension in $\mathbb{F}_a$ for $a$ large enough.

*Example* 3.3. Consider $f(z,w) = (w^p, z^q + w^d)$.

(1) If $d^2 > pq$ and $d > p$, then $f$ admits an algebraically stable extension to $\mathbb{F}_a$ for $a$ such that $q/d \leq a < d/p$. Indeed the bihomogeneization process yields

$$P_0 = z_0^{ap} w_0,\ P_1 = w_1^p,\ Q_0 = z_0^{a(d-ap)} w_0^{d-a},\ Q_1 = w_1^d + z_0^{ad-q} w_0^d z_1^q.$$

The indeterminacy set is $I_f = \{[0:1:1:0], [1:0:0:1]\}$ and the curves at infinity are contracted to the point $q_\infty = [0:1:0:1]$. The degrees of the extension are $A_f = \begin{bmatrix} 0 & 0 \\ p & d \end{bmatrix}$ therefore the first dynamical degree of $f$ equals $d$.

(2) If $d^2 \leq pq$ and $q \geq d$, then $f^2 = ([z^q + w^d]^p, w^{pq} + [z^q + w^d]^d)$ admits an holomorphic extension to $\mathbb{P}^2$ and has algebraic degree $pq$. Therefore the first dynamical degree of $f$ is $\lambda_1(f) = \sqrt{pq}$.

(3) There remains to consider the case $q < d < p$. One can check by induction that for all $j$, $f^j$ does not admit an algebraically stable extension to $\mathbb{P}^2$ nor to any $\mathbb{F}_a$. One needs here to consider nonminimal compactifications of $\mathbb{C}^2$. For example when $q = 1$, $d = 2$, $p = 3$, then $f = (w^3, z + w^2)$ becomes algebraically stable in $\mathbb{P}^2$ blown up at two points: blow up first the point $[z:w:t] = [1:0:0]$, then blow up the intersection between the exceptional divisor and the strict transform of $(t = 0)$.

PROPOSITION 3.4. *Let $f(z,w) = (P(z,w), Q(z,w))$ be a dominating polynomial self mapping of $\mathbb{C}^2$ of algebraic degree $d_a(f) := \max(\deg P, \deg Q) = 2$.*

*Then $f$ or $f^2$ admits an algebraically stable extension either to $\mathbb{P}^2$, or $\mathbb{F}_1$ or $\mathbb{F}_2$.*

*Proof.* Consider first the extension of $f$ to $\mathbb{P}^2$. The hyperplane $(t = 0)$ at infinity is either fixed or sent to a point, say $[z:w:t] = [0:1:0]$. Thus $f$ is algebraically stable in $\mathbb{P}^2$ except in the latter case when $[0:1:0]$ is a point of indeterminacy. This means $f$ has the following form

$$f(z,w) = (az + bw + c, z[dw + ez] + L(z,w)),$$

where $L$ is linear.

If $b = 0$ then $f$ is a skew-product with $1 = \delta \geq \alpha = 1$, so $f$ admits an algebraically stable extension to any $\mathbb{F}_a$, $a \geq a_0$ (see Example 3.2, here $a_0 = 2$ would work).

If $d = 0$, $b \neq 0$, then $e \neq 0$ since $d_a(f) = 2$. Thus we get $f^2(z,w) = (bez^2 +$ linear terms, $e(az + bw)^2 +$ linear terms$)$, so $f^2$ admits an holomorphic extension (hence algebraically stable) to $\mathbb{P}^2$.

Finally assume $bd \neq 0$. Using our previous notations, we get $\alpha = 0$, $\beta = \gamma = \delta = 1$ and $P(z,w) = A_0(z)w + A_1(z)$, $Q(z,w) = B_0(z)w + B_1(z)$ with

$$\deg A_0 = 0, \deg A_1 \leq 1, \deg B_0 = 1, \deg B_1 \leq 2.$$

Looking at the extension in $\mathbb{F}_a$ in bihomogeneous coordinates, the condition $(**)$ becomes $1 \geq a$ and $1 + a - a^2 \geq 0$ hence $a \in \{0, 1\}$. On the other hand condition $(*)$ yields $a \geq \deg A_1$, $1 + a \geq \deg B_1$. Thus $f$ admits an algebraically stable extension to $\mathbb{F}_1$. □

*Remark* 3.1. Similar (but much longer) computations show that any polynomial self mapping of $\mathbb{C}^2$ of algebraic degree 3 admits an algebraically stable extension to $\mathbb{P}^2$ or $\mathbb{F}_a$ or $\mathbb{P}^2$ blown up at 2,3 or 4 points. We conjecture that any polynomial self mapping of $\mathbb{C}^2$ admits an algebraically stable extension to some (nonminimal) compactification of $\mathbb{C}^2$.

## 4. Mappings with large topological degree.

**4.1. Growth properties and dynamical degrees.** In this section $f\colon \mathbb{C}^2 \to \mathbb{C}^2$ is a polynomial dominating mapping, $X = \mathbb{C}^2 \cup Y_\infty$ is either $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_a$ and we still denote by $f\colon X \to X$ its meromorphic extension. It follows from Section 3 that there is a smooth semi-positive $(1,1)$-form $\omega$ on $X$ s.t. $f^*\omega \sim \lambda\omega$—here $\lambda$ denotes the spectral radius of the induced linear map $\Phi_f$ on $H^{1,1}(X, \mathbb{R})$. Indeed one can take the Fubini-Study Kähler form $\omega = \omega_{FS}$ if $X = \mathbb{P}^2$ and $\omega = t_1\omega_1 + t_2\omega_2$ if $X = \mathbb{F}_a$, where $t_1, t_2 \geq 0$ satisfy $A_f \cdot (t_1, t_2) = \lambda(t_1, t_2)$. The form $\omega$ is unique up to normalization: if $f$ is algebraically stable with $\lambda > 1$, we normalize it so that $\int_{\mathbb{C}^2} T \wedge \omega_{FS} = 1$, where $T$ is the Green current constructed in Theorem 2.1. Actually there is one exceptional case where $\omega$ is not uniquely determined. This is when $\beta = \gamma = 0$ and $\alpha = \delta$ on $X = \mathbb{F}_a$. However this corresponds to a polynomial skew-product of $\mathbb{C}^2$ whose simple dynamics was completely settled in [Fa-G 99].

Define $\psi \in PSH(\mathbb{C}^2)$ by $\psi(z,w) = \frac{1}{2}\log[1 + |z|^2 + |w|^2]$ if $X = \mathbb{P}^2$ or

$$\psi(z,w) = \frac{t_1}{2}\log[1 + |z|^2] + \frac{t_2}{2}\log[(1 + |z|^2)^a + |w|^2]$$

if $X = \mathbb{F}_a$ so that $dd^c\psi = \omega$ in $\mathbb{C}^2$.

THEOREM 4.1. *Assume $f(Y_\infty \backslash I_f) = q_\infty \notin I_f$. Then the following hold:*

• *the map $f$ is algebraically stable on $X$, $q_\infty$ is a superattractive fixed point and the first dynamical degree of $f$ satisfies $\lambda = \lambda_1(f) > 1$.*

• *the sequence $g_j(p) = \lambda^{-j}\psi \circ f^j(p)$ converges pointwise towards a function $g \in PSH(\mathbb{C}^2)$ which satisfies $g \circ f = \lambda g$.*

• *the set* $\Omega_\infty = \{p \in \mathbb{C}^2 / g(p) > 0\}$ *is the basin of attraction of* $q_\infty$, *the function* $g$ *is pluriharmonic in* $\Omega_\infty$ *and continuous in* $\mathbb{C}^2$.

• *set* $\mathcal{K}^+ := \mathbb{C}^2 \backslash \Omega_\infty$, *then* $\overline{\mathcal{K}^+} \cap Y_\infty = Supp\, T \cap Y_\infty = I_f \neq \emptyset$.

*Proof.* Since $\mathbb{C}^2$ is $f$-invariant, the only curves that could be contracted to points of indeterminacy are contained in $Y_\infty$. The latter is sent to $q_\infty$ which is fixed and does not belong to $I_f$, therefore $f$ is algebraically stable (Lemma 3.4). Since $f$ is polynomial, $Df(q_\infty)$ has a 0-eigenvalue in directions transverse to $Y_\infty$. The other eigenvalue is also 0 since $Y_\infty$ is contracted to $q_\infty$, hence $q_\infty$ is a superattractive fixed point. The first dynamical degree of $f$ equals the spectral radius of the induced linear map $\Phi_f$ (since $f$ is algebraically stable), hence $\lambda_1(f) = \lambda$. Clearly $\lambda > 1$ otherwise $f$ would act linearly at infinity contradicting $f(Y_\infty \backslash I_f) = q_\infty$.

Since $f$ is algebraically stable, we can apply Theorem 2.1: the sequence $\lambda^{-n}(f^n)^*\omega$ converges towards a Green current $T \in \mathcal{T}(X)$ satisfying $f^*T = \lambda T$. The choice of potential is unique up to the addition of a constant, therefore the convergence of $(g_j)$ is a consequence of Theorem 2.1 if we normalize the potential $g$ of $T$ in $\mathbb{C}^2$ so that $\inf_{\mathbb{C}^2} g = 0$.

To show that the basin of attraction of $q_\infty$ is precisely the set where $g > 0$, one needs to estimate the growth of $f$ outside this basin. This was done in [Fa-G 99] in the case $a = 0$ and the proof for every $a$ is quite similar: one shows the existence of $C > 0$ and $\gamma < \lambda$ such that for all points $p$ outside the basin of $q_\infty$

$$1 + \|f^j(p)\| \leq C[1 + \|p\|]^{\gamma^j}, \quad \forall_j \in \mathbb{N}.$$

The case $X = \mathbb{P}^2$ is similar though the estimate is easier to establish ($\gamma = \lambda - 1$ is easily shown to be convenient in this case). Since $\Omega_\infty$ is a Fatou component, the Green current vanishes on $\Omega_\infty$ (Theorem 2.2), i.e., $g$ is pluriharmonic on $\Omega_\infty$. The upper-semi-continuity of $g \geq 0$ guarantees that $g$ is continuous at every point of $\partial\mathcal{K}^+ \subset (g = 0)$, hence $g$ is continuous in $\mathbb{C}^2$.

The current $T$ is supported on $\overline{\mathcal{K}^+}$. If we show that every point of indeterminacy belongs to *Supp* $T$, then $I_f \subset Supp\, T \cap Y_\infty \subset \overline{\mathcal{K}^+} \cap Y_\infty \subset I_f$, where the latter inclusion comes from the fact that every point of $Y_\infty \backslash I_f$ belongs to the basin $\Omega_\infty$. Recall that $T = \omega + dd^c\psi_\infty$, where

$$\psi_\infty = \sum_{j \geq 0} \frac{1}{\lambda^j} \psi \circ f^j \quad \text{and} \quad \frac{1}{\lambda} f^*\omega = \omega + dd^c\psi.$$

Thus $\psi$ has positive Lelong number at every point of $I_f$ since $\omega$ is Kähler (this follows from our assumption $f(Y_\infty \backslash I_f) = q_\infty$: if $\omega$ is not Kähler, then $X = \mathbb{F}_a$ and $f$ is a "skew-product," i.e. $\beta\gamma = 0$, but in this case one of the two lines at infinity is not contacted by $f$). Therefore $T$ has positive Lelong number at every point of $I_f$, in particular $I_f \subset Supp\, T$.

Finally note that $I_f$ is nonempty otherwise $f$ would be holomorphic and $Y_\infty$ could not be contracted to the point $q_\infty$ since $f$ is finite-to-1 (this follows from Proposition 3.3). □

In order to construct interesting invariant measures starting from the Green current $T$, we need to relate the growth of the mapping $f_{|Supp\, T}$ to the dynamical degrees of $f$. The following lemma is a basic observation that we are going to use several times.

LEMMA 4.1. *Let $f, \lambda, T$ be as above and let $d_t$ denote the topological degree of $f$. Then*

$$\int_{\mathbb{C}^2} f^*\omega_{FS} \wedge T = \delta := d_t/\lambda.$$

*Proof.* It is well known that the set $Z = \{p \in \mathbb{C}^2/\#f^{-1}(p) \neq d_t\}$ is a proper algebraic subset of $\mathbb{C}^2$. Since $\omega_{FS} \wedge T$ is a probability measure in $\mathbb{C}^2$ which does not charge hypersurfaces, we infer

$$\int_{\mathbb{C}^2} f^*(\omega_{FS} \wedge T) = \int_{\mathbb{C}^2\setminus Z} f^*(\omega_{FS} \wedge T) = \langle \omega_{FS} \wedge T, f_* 1\rangle = d_t.$$

Therefore $\int_{\mathbb{C}^2} f^*\omega_{FS} \wedge T = \lambda^{-1} \int_{\mathbb{C}^2} f^*(\omega_{FS} \wedge T) = \delta$. □

PROPOSITION 4.1. *Let $f, \mathcal{K}^+$ be as in Theorem 4.1. Let $d_t$ denote the topological degree of $f$ and set $\delta = d_t/\lambda$.*

(1) *If there exist constants $C, \gamma$ such that*

$$(*) \qquad 1 + \|f(p)\| \geq C[1 + \|p\|]^\gamma, \quad \forall p \in \mathcal{K}^+,$$

*then $\gamma \leq \delta$. The set $I_f$ is an attracting set for $f_{|\mathcal{K}^+}$ if $\gamma > 1$.*

(2) *If $I_f$ is an attracting set for $f_{|\mathcal{K}^+}$, then $\delta \geq 1$, $f$ is not normal and $K^+ := \{p \in \mathbb{C}^2/(f^n(p))_{n\geq 0}$ is bounded$\}$ is a compact polynomially convex subset of $\mathbb{C}^2$.*

(3) *If there exist constants $C, \gamma$ such that*

$$(**) \qquad 1 + \|f(p)\| \leq C[1 + \|p\|]^\gamma, \quad \forall p \in \mathcal{K}^+,$$

*then $\gamma \leq \delta$. The set $I_f$ is a repelling set for $f_{|\mathcal{K}^+}$ if $\gamma < 1$.*

(4) *If $I_f$ is a repelling set for $f_{|\mathcal{K}^+}$, then $\delta \leq 1$, $f$ is normal hence $K^+ = \mathcal{K}^+$ is closed and $K := \{p \in \mathbb{C}^2/(f^n(p))_{n\in\mathbb{Z}}$ is bounded$\}$ is a compact polynomially convex subset of $\mathbb{C}^2$.*

*Proof.* (1) Set $u(p) = \log^+\|f(p)\|$ and $u_\varepsilon = \max([1+\varepsilon]u - C_\varepsilon, \gamma\log^+\|p\|)$. Then $u$, $u_\varepsilon$ are plurisubharmonic functions on $\mathbb{C}^2$. If $R > 0$ is fixed, we can choose $\varepsilon_0 > 0$ and $C_\varepsilon \gg 1$ so that $u_\varepsilon \equiv \gamma\log^+\|p\|$ in a neighborhood of $B(R) = \{p \in \mathbb{C}^2/\|p\| < R\}$ for any $0 < \varepsilon < \varepsilon_0$. Moreover it follows from $(*)$

that $u_\varepsilon \equiv [1+\varepsilon]u - C_\varepsilon$ on $Supp\, T \backslash B(R_\varepsilon)$ for $R_\varepsilon$ large enough. Let $\chi \geq 0$ be a smooth test function in $\mathbb{C}^2$ s.t. $\chi \equiv 1$ in a neighborhood of $B(R)$. Then

$$\gamma \int_{B(R)} dd^c \log^+ \|p\| \wedge T = \int_{B(R)} dd^c u_\varepsilon \wedge T \leq \int_{\mathbb{C}^2} \chi dd^c u_\varepsilon \wedge T$$
$$= [1+\varepsilon] \int_{\mathbb{C}^2} \chi dd^c u \wedge T \leq [1+\varepsilon] \int_{\mathbb{C}^2} f^* \omega_{FS} \wedge T,$$

where the last equality follows from Stokes theorem. Letting $\varepsilon \to 0$ and $R \to +\infty$, this yields

$$\gamma = \gamma \int_{\mathbb{C}^2} \omega_{FS} \wedge T \leq \int_{\mathbb{C}^2} f^* \omega_{FS} \wedge T$$

hence $\gamma \leq \delta$ by Lemma 4.1.

(2) If $\gamma > 1$, it follows from $(*)$ that $I_f = \overline{\mathcal{K}^+} \cap Y_\infty$ is an attracting set for $f_{|\mathcal{K}^+}$. Conversely if $I_f$ is an attracting set for $f$, then there is an inequality $(*)$ with either $\gamma > 1$ or $\gamma = 1$ and $C \geq 1$. It follows from (1) that $\delta \geq \gamma \geq 1$.

Assume $I_f$ is an attracting set for $f_{|\mathcal{K}^+}$. Let $\mathcal{B}^+(I_f)$ denote the set of points which are attracted by $I_f$ under iteration. This is an open subset of $\mathcal{K}^+$ which contains a neighborhood of infinity in $\mathcal{K}^+$ and is nonempty since $\overline{\mathcal{K}^+} \cap Y_\infty = I_f \neq \emptyset$. Therefore $f$ is not normal and $K^+$ is a compact subset of $\mathbb{C}^2$. Set $u_n(p) = \log^+ \|f^n(p)\| \in PSH(\mathbb{C}^2)$. If $p \in \mathbb{C}^2 \backslash K^+$, then $f^n(p) \to Y_\infty$ therefore $u_n(p) \to +\infty$ whereas $\sup_{K^+} \sup_n u_n = \sup_{K^+} \log^+ \|\cdot\| < +\infty$, hence $K^+$ is polynomially convex.

Proofs of (3) and (4) are similar to those of (1) and (2). We say that $I_f$ is a repelling set for $f_{|\mathcal{K}^+}$ if it is an attracting set for $f^{-1}_{|\mathcal{K}^+}$ in the following sense: there exists $V$ an open neighborhood of $Y_\infty$ in $\mathbb{C}^2$ s.t. $f^{-1}(V \cap \mathcal{K}^+) \subset\subset V \cap \mathcal{K}^+$ and $f^{-j}(V \cap \mathcal{K}^+) \to I_f$ in the Hausdorff metric. It clearly follows that $f$ is normal and more precisely $K^+ = \mathcal{K}^+$. Let $\mathcal{B}^-(I_f)$ denote the set of points whose backward orbit is attracted by $I_f$. If $I_f$ is a repelling set for $f_{|\mathcal{K}^+}$ then $\mathcal{B}^-(I_f)$ is an open subset of $\mathcal{K}^+$ which contains a neighborhood of infinity, therefore the set $K$ of points of bounded orbit (both forward and backward) is compact in $\mathbb{C}^2$. To see that $K$ is polynomially convex, one can consider the functions $v_n = d_t^{-n}(f^n)_* \log^+ \|\cdot\|$. □

*Remark* 4.1. The maximal $\gamma$ such that $1 + \|f(p)\| \geq C[1+\|p\|]^\gamma$, $\forall p \in \mathbb{C}^2$ is called the Lojasiewicz exponent of $f$ at infinity and is usually denoted by $L_\infty(f)$. This is a rational number which can be computed explicitly by means of a simple algebraic formula (see [C-K 92]). Note that $Y_\infty$ is an attracting set for $f$ if $L_\infty(f) > 1$.

**4.2. $f^*$-invariant measures.**

THEOREM 4.2. *Let $f$ be as in Theorem 4.1. Assume $I_f$ is an attracting set for $f_{|\mathcal{K}^+}$ and $\delta = d_t/\lambda > 1$. Set*

$$\mu_N = \frac{1}{N}\sum_{j=1}^{N}\frac{1}{\delta^j}(f^j)^*\omega \wedge T.$$

*Then $(\mu_N)$ is a sequence of probability measures in $\mathbb{C}^2$. Any cluster point $\mu$ has support in the compact set $K^+ = \{p \in \mathbb{C}^2/(f^n(p))_{n\geq 0}$ is bounded$\}$ and satisfies $f^*\mu = d_t\mu$.*

*If $\mu$ does not charge pluripolar sets, then it is an invariant measure ($f_*\mu = \mu$) which is mixing and of maximal entropy*

$$h_\mu(f) = h_{top}(f) = \log d_t(f).$$

*Proof.* It follows from Lemma 4.1 that $\mu_N$ is a probability measure. Since $I_f$ is an attracting set for $f_{|\mathcal{K}^+}$, the set $K^+$ is compact (Proposition 4.1) and $f^j(p) \to I_f$ for every point in $Supp\ T\backslash K^+$. Assume $X = \mathbb{P}^2$ and $q_\infty = [1:0:0]$. Then if $f^j = (f_1^j, f_2^j)$ we have

$$\frac{1}{2}\log[1 + \|f^j\|^2] = \log|f_2^j| + u_j, \text{ with } u_j \text{ bounded on } Supp\ T\backslash K^+.$$

It follows that $\mu_N \to 0$ outside $K^+$. A similar proof applies for the other compactifications of $\mathbb{C}^2$. The invariance of $T$ yields

$$\frac{1}{d_t}f^*\mu_N = \frac{N+1}{N}\mu_{N+1} - \frac{1}{N}\omega \wedge T,$$

hence $f^*\mu = d_t\mu$ follows from $\mu_{N+1} - \mu_N \to 0$.

Let $\chi$ be a test function. Then $f_*\chi$ is well defined outside some analytic subset and $f_*f^*\chi = d_t\chi$. Therefore

$$\langle f_*\mu, \chi\rangle = \left\langle \frac{1}{d_t}f_*f^*\mu, \chi\right\rangle = \left\langle \mu, \frac{1}{d_t}f_*f^*\chi\right\rangle = \langle \mu, \chi\rangle$$

if $\mu$ does not charge pluripolar sets. Moreover since $d_t > \lambda = \lambda_1(f)$, a result of Russakovskii and Shiffman [R-Sh 97] asserts that $\mu$ satisfies the following equidistribution property: there exists a pluripolar set $\mathcal{E}_f$ such that

$$\forall_p \in \mathbb{C}^2\backslash\mathcal{E}_f, \frac{1}{d_t}(f^j)^*\varepsilon_p \longrightarrow \mu,$$

where $\varepsilon_p$ denotes the Dirac mass at point $p$. As was observed in [Fa-G 99], this implies that $\mu$ is mixing whenever $\mu$ does not charge pluripolar sets.

Finally the functional equation $f^*\mu = d_t\mu$ insures that $f$ has constant Jacobian $d_t$ with respect to $\mu$. The Rohlin-Parry formula (see [Pa 69]) yields $h_\mu(f) \geq \log d_t$. On the other hand $h_{top}(f) \leq \log d_t$ by a result of Friedland [Fr 91], it follows therefore from the variational principle (see e.g. [Wa 82]) that these are equalities, hence $\mu$ has maximal entropy. □

What remains is to make sure that $\mu$ does not charge pluripolar sets. A natural idea is to construct a partial Green function $v$ which measures the (slower) growth of orbits on *Supp T*. A similar construction appears in [Fa-G 99] in the case of polynomial skew-products of $\mathbb{C}^2$ and in [G-S 00] in the study of polynomial automorphisms of $\mathbb{C}^k$. We have the following:

Proposition 4.2. *Let $f$ be as above. Assume there exists $C > 0$ s.t.*

$$\forall p \in Supp\ T, \qquad 1 + \|f(p)\| \leq C[1 + \|p\|]^\delta.$$

*Then $v_j = \delta^{-j} \log^+ \|f^j(p)\|$ (almost) decreases on Supp T towards a function $v \in L^\infty_{loc}(Supp\ T)$ which satisfies $v \circ f = \delta v$. Therefore $(\mu_N)$ converges towards the probability measure $\mu = dd^c(vT)$ which does not charge pluripolar sets.*

*Proof.* The growth control on $f$ on *Supp T* implies $v_{j+1} \leq v_j + C'\delta^{-j}$. Therefore $(v_j)$ is almost decreasing and $v = \lim v_j$ is well defined at every point of *Supp T*. Since $v$ is upper-semi-continuous and nonnegative, it is locally bounded hence $v \cdot T$ is a well-defined "pluripositive current" in the sense of Sibony [S 85]. There are Chern-Levine-Nirenberg inequalities for $dd^c(v \cdot T)$ similar to the classical ones (see [Fa-G 99]). They insure that $\mu = \lim dd^c(v_j \cdot T) = \lim \mu_j$ does not charge pluripolar sets. □

*Example* 4.1. Let $f$: $(z, w) \in \mathbb{C}^2 \mapsto (P(w), Q(z) + R(w))$, where $P, Q, R$ are polynomials of degree $p, q, d$ with $d > \max(p, q)$. Then $f$ admits an algebraically stable extension to $\mathbb{P}^2$ with $I_f = [1 : 0 : 0]$ and $f(Y_\infty \backslash I_f) = q_\infty = [0 : 1 : 0]$. Note that the topological degree of $f$ is $d_t = pq$ and the first dynamical degree equals $d$.

(a) If $\delta = pq/d > 1$, then the hyperplane $Y_\infty$ at infinity is an attracting set for $f$. This can be checked directly or by computing the Lojasiewicz exponent of $f$ at infinity which is $L_\infty(f) = \delta = pq/d > 1$. More precisely we have the following growth control: there exists $C > 1$ such that

$$\text{(a)} \qquad \frac{1}{C}[1 + \|p\|]^\delta \leq 1 + \|f(p)\| \leq C[1 + \|p\|]^\delta, \quad \forall p \in \mathcal{K}^+ = \mathbb{C}^2 \backslash \Omega_\infty.$$

Thus $f$ satisfies the assumptions of Theorem 4.2 and Proposition 4.2.

(b) If $\delta = pq/d < 1$, then $I_f$ is a repelling set for $f_{|\mathcal{K}^+}$ and moreover we have the following growth control for $f^{-1}$: there exists $C > 1$ such that

$$\text{(b)} \quad \frac{1}{C}[1 + \|f(p)\|]^{1/\delta} \leq 1 + \|p\| \leq C[1 + \|f(p)\|]^{1/\delta}, \quad \forall p \in \mathcal{K}^+ = \mathbb{C}^2 \backslash \Omega_\infty.$$

*Proof.* (a) We set $V_\varepsilon = \{(z, w) \in \mathbb{C}^2 / \max(|z|, |w|) > 1/\varepsilon\}$. We leave it to the reader to check that there exists $\varepsilon_0 > 0$ such that $0 < \varepsilon < \varepsilon_0 \Rightarrow f(V_\varepsilon) \subset V_{\varepsilon/2}$. In particular $f(V_\varepsilon \cap \mathcal{K}^+) \subset V_{\varepsilon/2} \cap \mathcal{K}^+$ since $\mathcal{K}^+$ is $f$-invariant. Now $\overline{\mathcal{K}^+} \cap Y_\infty = I_f = [1 : 0 : 0]$, therefore

$$\mathcal{K}^+ \cap V_\varepsilon = \{(z, w) \in \mathcal{K}^+ / |z| > 1/\varepsilon \text{ and } |w| < c(\varepsilon)|z|\},$$

where $c(\varepsilon) \to 0$ as $\varepsilon \to 0$. We claim that there exists $C_1 > 1$, $\varepsilon_1 > 0$ such that if $0 < \varepsilon < \varepsilon_1$ and $(z, w) \in V_\varepsilon \cap \mathcal{K}^+$ then

$$(*) \qquad \frac{1}{C_1}|z|^q \leq |w|^d \leq C_1|z|^d.$$

Assume on the contrary that $|w|^d > C_1|z|^q$ where $C_1 \gg 1$, then if $(z', w') = f(z, w)$, we get $|w'| = |Q(z) + R(w)| \geq C'|w|^d \gg |z'| = |P(w)|$ contradicting $|w'| < c(\varepsilon/2)|z'|$. Similarly if $|w|^d \leq |z|^q/C_1$ then $|w'| \geq C''|z|^q \geq C''C_1|w|^d \gg |z'|$, a contradiction.

Therefore $(*)$ is satisfied and this yields $|z|^\delta/C_2 \leq |z'| \leq C_2|z|^\delta$ for any $(z, w) \in V_\varepsilon \cap \mathcal{K}^+$. The desired growth control follows from compactness of $\mathcal{K}^+ \backslash V_\varepsilon \cap \mathcal{K}^+$.

(b) Straightforward adaptation of the previous case. □

*Remark* 4.2. Similar growth control could easily be obtained for mappings of the form $f(z, w) = (P(z) + A(z, w), Q(z) + R(w) + B(z, w))$ where the polynomials $A$ and $B$ have small degrees compared to those of $P$, $Q$, $R$.

Note also that these estimates are stable under composition. Thus (a) (or (b)) applies for mappings $f = f_1 \circ \cdots \circ f_s$, where each $f_i$ has the form described in Example 4.1.

## 5. Mappings with small topological degree.

**5.1. Construction of $f_*$-invariant currents.** Let $f\colon X \to X$ be a dominating meromorphic self-map of compact Kähler manifold $X$. Given $R \in \mathcal{T}(X)$, we would like to define the push-forward $f_*R$ of $R$ by $f$. When $f$ is holomorphic, this can be done by duality setting $\langle f_*R, \theta\rangle := \langle R, f^*\theta\rangle$ for every test form $\theta$. When $f$ is merely meromorphic, we can consider $\tilde{G}$ a desingularization of the graph

$G_f \subset X \times X$ of $f$. We have a commutative diagramm

$$\begin{array}{ccc} & \tilde{G} & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ X & \longrightarrow & X, \end{array}$$

where $\pi_1, \pi_2$ are holomorphic proper maps. The current $\pi_1^* R$ is a well-defined element of $\mathcal{T}(\tilde{G})$ (see the introduction of Section 2.1) hence we can consider $f_* R := (\pi_2)_*(\pi_1^* R)$. One checks that this definition is independent of the choice of desingularization of $G_f$. It preserves cohomology classes hence induces a linear map on $H^{1,1}(X, \mathbb{R})$ which is dual to the map $\Phi_f$ defined in Section 2.1 in case $X$ is a compact complex surface. There is a useful alternative construction. Denote by $d_t$ the topological degree of $f$ and set

$$Z_f = \{p \in X / \# f^{-1}(p) \neq d_t\}.$$

The latter is well known to be a proper analytic subset of $X$. If $\varphi$ is a local potential of $R$, we can consider $dd^c(f_* \varphi)$, where $f_* \varphi(p) = \sum_{f(q)=p} \varphi(q)$ is well defined on $X \backslash Z_f$. This definition clearly does not depend on the choice of local potentials and yields a positive closed current of bidegree $(1,1)$ in $X \backslash Z_f$ which coincides there with $(\pi_2)_*(\pi_1^* R)$. Thus $dd^c(f_* \varphi)$ has bounded mass near $Z_f$ and we can consider its trivial extension through $Z_f$. When $R$ is smooth, these two currents coincide everywhere since they do not charge complex hypersurfaces.

It is easy to check that $(f^{j+1})_* R = (f^j)_*(f_* R)$ as soon as $f$ is algebraically stable. Moreover we have the basic identity

$$f_* f^* R = d_t R \text{ in } X \backslash Z_f.$$

*Remark* 5.1. The dynamical study of push-forward of currents appears in [F-S 98] in the context of endomorphisms of $\mathbb{P}^2$. Although our interest is rather in mappings with "small" topological degree, some arguments of Fornaess and Sibony can easily be adapted to our situation and we refer to [F-S 98] for further details on push-forward of currents.

THEOREM 5.1. *Let $X$ be a compact Kähler manifold and $f: X \to X$ a dominating meromorphic self-map which is algebraically stable. Let $\omega \in \mathcal{T}(X)$ with continuous potential and assume $f_* \omega \sim \lambda \omega$, where $\lambda > d_t(f)$.*

*Then there exists $T^- \in \mathcal{T}(X)$ such that:*

(1) *$\lambda^{-n}(f^n)_* \omega \to T^-$ in the weak sense of currents. When $f$ is holomorphic, there is uniform convergence of potentials therefore $T^-$ admits a continuous potential.*

(2) *The current $T^-$ satisfies $f_* T^- = \lambda T^-$ and $T^- \sim \omega$.*

(3) *If* $\omega' \in \mathcal{T}(X)$ *is cohomologous to* $\omega$ *and admits a locally bounded potential, the* $\lambda^{-n}(f^n)_*\omega' \to T^-$.

(4) *The current* $T^-$ *is extremal in the cone*

$$\mathcal{K}_{f_*}^{[\omega]} := \{R \in \mathcal{T}(X)/f_*R \sim \lambda R \text{ and } R \sim \omega\}.$$

*Proof.* The proof is very similar to those of Theorem 2.1 and Proposition 2.1. We therefore only sketch the construction of the potential of $T^-$. Let $\varphi \in L^1(X)$ be such that $\lambda^{-1}f_*\omega = \omega + dd^c\varphi$. Since $\varphi$ is quasiplurisubharmonic, we can assume $\varphi \leq 0$. Since $f$ is algebraically stable, we get $(f^{j+1})_*\omega = (f^j)_*(f_*\omega)$ for all integers $j$. Thus we can iterate the previous equation to get $\lambda^{-j}(f^j)_*\omega = \omega + dd^c\varphi_j$, where

$$\varphi_j = \sum_{l=0}^{j-1} \frac{1}{\lambda^l}(f^l)_*\varphi$$

is a decreasing sequence of quasiplurisubharmonic functions. If $\varphi_\infty \not\equiv -\infty$, the current $T^- = \omega + dd^c\varphi_\infty$ satisfies all our requirements. Thus it remains for us to show that the limit $\varphi_\infty$ is not identically $-\infty$. Since $\lambda^{-j}(f^j)_*\omega$ is bounded in $\mathcal{T}(X)$, we can construct $\sigma \in \mathcal{T}(X)$ such that $f_*\sigma = \lambda\sigma$ and $\sigma \sim \omega$. Let $v \in L^1(X)$ be a potential for $\sigma$; we can assume

$$v - \frac{1}{\lambda}f_*v = \varphi.$$

Then it follows that $v - \lambda^{-j}(f^j)_*v = \varphi_j$. Now $v$ is bounded from above on $X$, hence there exists $C > 0$ such that

$$\varphi_j \geq v - \frac{1}{\lambda^j}(f^j)_*C = v - C\left(\frac{d_t}{\lambda}\right)^j.$$

Since $d_t < \lambda$ we infer $\varphi_\infty \geq v$ hence $\varphi_\infty \not\equiv -\infty$. □

*Remark* 5.2. When $d_t = 1$, i.e. when $f$ is bimeromorphic, then $f_*\omega = (f^{-1})^*\omega$ hence $T^-$ is the Green current associated to $f^{-1}$.

Assume now $X = \mathbb{P}^2$ or $\mathbb{F}_a$. Then the linear action induced by $f_*$ on $H^{1,1}(X, \mathbb{R})$ is dual to the action induced by $f^*$. We let $\omega$ denote a normalized Kähler form such that $f_*\omega \sim \lambda\omega$, where $\lambda$ denotes the spectral radius of the linear actions $f_*, f^*$. Observe that the eigenspace associated to $\lambda$ is one-dimensional since we assume $\lambda > d_t(f)$.

THEOREM 5.2. *Let* $f$ *be as in Theorem 5.1 with* $X = \mathbb{P}^2$ *or* $\mathbb{F}_a$. *Assume there exists a finite set* $S$ *which is* $f^{-1}$*-attracting. Then* $T^-$ *is an extremal point of the cone* $\mathcal{T}(X)$.

*Proof.* The proof goes along the same lines as that of Theorem 2.4. Given $S \in \mathcal{T}(X)$ such that $0 \leq S \leq T^-$ we need to show $S = xT^-$ for some $x \in [0,1]$. Observe first that one can adapt the proof of Theorem 2.3 to show that $T^-$ does not charge complex hypersurface of $X$. In particular $T^-$ does not charge the analytic subsets $Z_{f^j}$ for all $j \geq 1$. Consider

$$T_j := \left(\frac{\lambda}{d_t}\right)^j \overline{(f^j)^* T^-} \text{ and } S_j := \left(\frac{\lambda}{d_t}\right)^j \overline{(f^j)^* S}$$

where $\overline{\cdots}$ means that we take the trivial extension through $Z_{f^j}$ of these currents. We have $\lambda^{-j}(f^j)_*(T_j) = T^-$ in $X \backslash Z_{f^j}$. However $T^-$ does not charge $Z_{f^j}$. We claim neither does $\lambda^{-j}(f^j)_*(T_j)$ so that they coincide everywhere on $X$. Indeed from the invariance $(f^j)_* T^- = \lambda^j T^-$ we get $T_j = d_t^{-j}\overline{(f^j)^*(f^j)_* T^-}$. Thus if $X = \mathbb{P}^2$ we have $T_j \sim \alpha_j \omega_{FS}$ with $\alpha_j \leq 1$. It follows that

$$\omega_{FS} \sim T^- \leq \lambda^{-j}(f^j)_*(T_j) \sim \alpha_j \omega_{FS},$$

hence $\alpha_j = 1$ and $\lambda^{-j}(f^j)_*(T_j)$ actually equals $T^-$. When $X = \mathbb{F}_a$ we have $T^- \sim \omega$ where $\mathbb{R}[\omega]$ is the eigenspace associated to the spectral radius $\lambda$ of the linear action induced by $f_*$ on $H^{1,1}(X,\mathbb{R}) \simeq \mathbb{R}^2$. Therefore $T_j \sim \alpha_j \omega + \theta_j$ with $\alpha_j \leq 1$ and $\lambda^{-j}(f^j)_* \theta_j \longrightarrow 0$. We infer similarly $\theta_j \sim 0$ and $\alpha_j = 1$. This shows $T^- = \lambda^{-j}(f^j)_*(T_j)$ and $T_j \sim \omega$. Since $\lambda^{-j}(f^j)_*(S_j) \leq T^- = \lambda^{-j}(f^j)_*(T_j)$ we also have $S = \lambda^{-j}(f^j)_*(S_j)$ on $X$ and $S_j \sim S \sim x\omega$ for some $x \in [0,1]$.

Define $R_j = T_j - S_j \geq 0$ and fix potentials $u_j, v_j, w_j \in L^1(X)$ so that $T_j = \omega + dd^c u_j$, $S_j = x\omega + dd^c v_j$, $R_j = (1-x)\omega + dd^c w_j$. We normalize these potentials so that $u_j = v_j + w_j$ and $\sup_X v_j = \sup_X w_j = 0$. This insures that they do not converge uniformly towards $-\infty$. We claim $\lambda^{-j}(f^j)_*(u_j) \rightarrow 0$ in $L^1(X)$. Indeed,

$$dd^c(\lambda^{-j}(f^j)_*(u_j)) = \lambda^{-j}(f^j)_*(T_j) - \lambda^{-j}(f^j)_*(\omega) = T^- - \lambda^{-j}(f^j)_*(\omega) \rightarrow 0.$$

Therefore $\lambda^{-j}(f^j)_*(u_j) \rightarrow C \leq 0$ (possibly $C = -\infty$).

We now use the fact that there exists a finite $f^{-1}$-attracting set $S$ to show $C = 0$ ($S = I_f$ in Theorem 5.3 below). Fix $V$ a small neighborhood of $S$ such that $f^{-1}(V) \subset\subset V$ and $\cap \overline{f^{-j}(V)} \subset S$. Since $S$ is finite, we get $T^- = 0$ in $V$. Since $T_j, S_j, R_j$ are all supported on *Supp* $T^-$, it follows from Harnack inequalities that there exists a constant $C_V$ independent of $j$ such that $-C_V \leq u_j \leq 0$ in $V$. This yields

$$-C_V \left(\frac{d_t}{\lambda}\right)^j \leq \frac{1}{\lambda^j}(f^j)_* u_j \leq 0 \text{ in } V,$$

since $f^{-1}(V) \subset V$. Therefore $\lambda^{-j}(f^j)_* u_j \rightarrow 0$ in $V$, hence $C = 0$.

Now $0 \geq v_j = u_j - w_j \geq u_j$ therefore $\lambda^{-j}(f^j)_*(v_j) \to 0$. This shows

$$S = \lambda^{-j}(f^j)_*(S_j) = x\lambda^{-j}(f^j)_*(\omega) + dd^c(\lambda^{-j}(f^j)_*(v_j)) \to xT^-. \qquad \square$$

**5.2. Invariant measures.** We now come back to the situation described in Section 4.1: $X$ is either $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_a$ and $f$: $X \to X$ is the algebraically stable meromorphic extension of a polynomial self-map of $\mathbb{C}^2$. Let $\omega, \omega' \in \mathcal{T}(X)$ with continuous potential such that $f^*\omega \sim \lambda\omega$ and $f_*\omega' \sim \lambda\omega'$. We assume $\lambda > d_t$-here $d_t$ stands, as usual, for the topological degree of $f$. We normalize $\omega, \omega'$ by imposing $\int_X \omega \wedge \omega' = 1$. By Theorems 2.1 and 5.1 we can define

$$T^+ = \lim_{n\to+\infty} \frac{1}{\lambda^n}(f^n)^*\omega \text{ and } T^- = \lim_{n\to+\infty} \frac{1}{\lambda^n}(f^n)_*\omega'.$$

THEOREM 5.3. *Let $f$ be as in Theorem 4.1. Assume $I_f$ is a repelling set for $f$ and* $\delta := d_t/\lambda < 1$.

*Then $\mu = T^+ \wedge T^-$ is an invariant probability measure with support in the compact set $K = \{p \in \mathbb{C}^2/(f^n(p))_{n\in\mathbb{Z}}$ is bounded$\}$. The measure $\mu$ is mixing. It does not charge pluripolar sets and has maximal entropy*

$$h_\mu(f) = h_{top}(f) = \log\lambda.$$

The proof is divided into three steps. To simplify notations we only treat the case $X = \mathbb{P}^2$. In this case the first dynamical degree $\lambda$ equals the algebraic degree $d$ of $f$.

*Step* 1 (*Invariance of* $\mu$). It follows from the work of Bedford and Taylor (see e.g. [K 91]) that $\mu$ is a well-defined positive Radon measure. This is clear in $\mathbb{C}^2$ where $T^+$ has locally bounded potential. Near every point of indeterminacy $p \in I_f$, $T^+$ admits a potential that is continuous outside $p$ (Theorem 4.1). So $\mu$ is globally well defined (see e.g. [F-S 95b]) since $Supp\ T \cap Y_\infty = I_f$.

We now show that $\mu$ has compact support in $\mathbb{C}^2$. Let $V$ be a neighborhood of $I_f$ such that $f^{-1}(V) \subset V$ and $\cap_{j\geq 0} f^{-j}(V) = I_f$. Denote by $\mathcal{B}^-(I_f) = \cup_{j\geq 0} f^j(V)$ the basin of attraction of $I_f$ for $f^{-1}$. We claim that $\mathbb{C}^2 = K^- \cup \mathcal{B}^-(I_f)$, where $K^- = \{p \in \mathbb{C}^2/(f^{-n}(p))_{n\geq 0}$ is bounded$\}$. Indeed if $p_j \to Y_\infty \backslash I_f$ with $p_j \in f^{-n_j}(p)$ for some point $p \in \mathbb{C}^2$, then $p = f^{n_j}(p_j) \to q_\infty$ since $q_\infty$ is a (super)attractive fixed point for $f$, a contradiction. Since $I_f$ is a finite number of points, we can choose coordinates so that $I_f \cap (w = 0) = \emptyset$. It follows that $(f^n)_*(\log^+\|(z,w)\|) = (f^n)_*(\log|w|) + O(d_t^n)$ in the basin $\mathcal{B}^-(I_f)$ so $T^-$ has support in $K^-$. On the other hand $T^+$ has support in $K^+$ which clusters only on $I_f$ in $Y_\infty$ (see Theorem 4.1 and Proposition 4.1). This shows $\mu$ has support in the compact set $K$ (Proposition 4.1).

Observe that $\mu$ does not charge proper analytic subsets as follows from Chern-Levine-Nirenberg inequalities (see [K 91]). In particular $\mu$ does not charge the set $Z_f = \{p \in \mathbb{C}^2/\#f^{-1}(p) \neq d_t\}$. The invariance of $\mu$ will follow from the following:

LEMMA 5.1. *Let $R, S$ be two positive closed currents of bidegree* $(1,1)$. *Assume $f_*R \wedge S$ does not charge the set $Z_f$ and $S$ has locally bounded potential. Then*

$$f_*(R \wedge f^*S) = f_*R \wedge S.$$

*Proof.* Let $\chi$ be a test function and assume first $S$ is smooth. We have

$$\langle f_*(R \wedge f^*S), \chi \rangle := \langle R \wedge f^*S), f^*\chi \rangle = \langle R, f^*(\chi S) \rangle = \langle f_*R, \chi S \rangle = \langle f_*R \wedge S, \chi \rangle.$$

For the general case, we can regularize $S$ and use the monotone convergence theorem in the style of Bedford-Taylor (see [K 91]). □

Since $\mu = T^- \wedge T^+ = d^{-1}T^- \wedge f^*T^+$, we get $f_*\mu = d^{-1}f_*T^- \wedge T^+ = \mu$. Thus $\mu$ is an invariant measure with compact support in $\mathbb{C}^2$.

*Step* 2 (*Mixing*). We now show that $\mu$ is mixing. Given $\chi, \theta$ two test functions, we need to prove (see [Wa 82]) that

$$\int \theta \chi \circ f^j \, d\mu \longrightarrow \int \theta \, d\mu \int \chi \, d\mu.$$

We can assume without loss of generality that $0 \leq \theta, \chi, \leq 1$. Observe that

$$\int \theta \chi \circ f^j \, d\mu = \left\langle \theta T^-, \frac{1}{\lambda^j}(f^j)^*(\chi T^+) \right\rangle = \left\langle \frac{1}{\lambda_j}(f^j)_*(\theta T^-), \chi T^+ \right\rangle.$$

Set $R_j = \lambda^{-j}(f^j)_*(\theta T^-)$. The invariance of $T^-$ guarantees $0 \leq R_j \leq T^-$. Moreover any cluster point $R$ of $(R_j)$ is closed by Proposition 5.1 below. Since $T^-$ is extremal in $\mathcal{T}(X)$, we infer $R = cT^-$ where

$$c = \langle cT^-, \omega \rangle = \lim \langle R_j, \omega \rangle = \lim \langle \theta T^-, \lambda^{-j}(f^j)^*\omega \rangle = \int \theta \, d\mu.$$

Thus $c = c_\theta$ is independent of $R$ and this shows that $(R_j)$ actually converges towards $c_\theta T^-$. Denote by $g^+$ the continuous potential of $T^+$. Then

$$\langle R_j \wedge T, \chi \rangle = \langle dd^c\chi \wedge R_j, g^+ \rangle + 2\langle dR_j \wedge d^c\chi, g^+ \rangle + \langle dd^c R_j, \chi g^+ \rangle.$$

The first term converges towards $c_\theta \langle dd^c \chi \wedge T^-, g^+ \rangle = c_\theta c_\chi$ since $g^+$ is continuous and $dd^c\chi \wedge R_j \to c_\theta dd^c \chi \wedge T^-$ in the sense of Radon measures. The last two terms converge to 0 since $\|dR_j\|, \|dd^c R_j\| \to 0$ (Proposition 5.1 below). This shows $\mu$ is mixing.

The next proposition is the key tool to deduce ergodic properties of invariant measures from extremality properties of invariant currents. It relies on the use of Cauchy-Schwartz inequality in the style of Ahlfors-Beurling. Such a result was initiated by Bedford and Smillie (see Lemma 1.2 in [B-Sm 92]) in the context of Hénon mappings (see also Proposition 6.1 in [Si 99]). In the context of endomorphisms of $\mathbb{P}^2$, Fornaess and Sibony gave a similar result for push-forward of "truncated currents" (Proposition 5.4 in [F-S 98]). We leave the technical adaptation to the reader.

PROPOSITION 5.1. *Let $R$ be a positive closed current of bidegree $(1,1)$ in a ball $B$ of $\mathbb{C}^2$ and $\chi \geq 0$ a test function in $B$. Define*

$$S_n = \frac{1}{\lambda^n}(f^n)^*(\chi R) \text{ and } R_n = \frac{1}{\lambda^n}(f^n)_*(\chi R).$$

*Then $(S_n), (R_n)$ are bounded sequences of positive currents. We have $\|dS_n\|$, $\|dR_n\| = O(\lambda^{-n/2})$ and $\|dd^c R_n\| = O(\lambda^{-n})$, $\|dd^c S_n\| = O((d_t/\lambda)^n)$. In particular any cluster point of these sequences is a closed positive current.*

*Step* 3 (*Entropy of* $\mu$). We now show that $\mu$ has maximal entropy $\log d$, following the lines of the proof of Theorem 4.4 in [B-Sm 92]. Observe first that $h_\mu(f) \leq h_{top}(f) \leq \log d$. The first inequality follows from the variational principle (see e.g. [Wa 82]) and the second is due to Friedland [Fr 91]. We therefore only need to show that $h_\mu(f) \geq \log d$.

Let $U$ be a connected neighborhood of $q_\infty$ such that $f(U) \subset U$ and $\cap_{j\geq o} f^j(U) = \{q_\infty\}$. Let $\omega'$ be a smooth semi-positive closed $(1,1)$-form on $\mathbb{P}^2$ such that $\omega' \sim \omega$ and $\omega' \equiv 0$ near $q_\infty$. Shrinking $U$ if necessary, we can assume $\omega' \equiv 0$ in $U$. Let $L$ be a line in $\mathbb{C}^2$ which intersects the line at infinity in $U$. For a generic choice of $L$, we have $d^{-n}(f^n)_*[L] \to T^-$. This is the dual version of an equidistribution result of Russakovskii and Shiffman which can be proved analogously since $d > d_t$. We set

$$\nu_n := [L] \wedge \frac{1}{d^n}(f^n)^*(\omega') \text{ and } \mu_n := \frac{1}{n}\sum_{j=0}^{n-1}(f^j)_*(\nu_n).$$

We show below (Lemma 5.2) that $\mu_n \to \mu = T^+ \wedge T^-$. Observe that $\nu_n$ is a probability measure with compact support in $\mathbb{C}^2$. Indeed

$$\int_{\mathbb{C}^2} \nu_n = \frac{1}{d^n}\int_{\mathbb{C}^2}[L] \wedge (f^n)^*\omega' = \frac{1}{d^n}\int_{\mathbb{P}^2}[L] \wedge (f^n)^*\omega' = 1,$$

since $(f^n)^*\omega' = 0$ in $U$.

Fix $\varepsilon > 0$ and let $\xi = \{\xi_i\}$ be a measurable partition of $\mathbb{P}^2$ such that $diam(\xi_i) < \varepsilon$ and $\mu(\partial\xi_i) = 0$. By a result of Misiurewicz (see [Mi 76] and [B-Sm 92] for an adaptation to this context), we have

$$h_\mu(f) \geq \limsup_{n\to+\infty} \frac{1}{n} H_{\nu_n}\left(\bigvee_{i=0}^{n-1} f^{-1}(\xi)\right).$$

Now every element of $\vee_{i=0}^{n-1} f^{-i}(\xi)$ is contained in an $\varepsilon$-ball in the metric $d_n(p,q) = \max_{0\leq i\leq n-1} d(f^i(p), f^i(q))$ -here $d$ stands e.g. for the Fubini-Study metric. If $B$ is an $\varepsilon$-ball, we have

$$\nu_n(B) = \frac{1}{d^n}\int_B [D_R]\wedge (f^n)^*(\omega') \leq \frac{C}{d^n} Aera(f^n(B\cap D_R))$$

since $\omega'$ is smooth. We infer

$$\frac{1}{n} H_{\nu_n}\left(\bigvee_{i=0}^{n-1} f^{-i}(\xi)\right) \geq \log d - \frac{\log C}{n} - \frac{1}{n} v_1^0(f,n,\varepsilon),$$

where $v_1^0(f,n,\varepsilon) = \sup_B Aera(f^n(B\cap D_R))$. The main result of Yomdin in [Y 87] asserts that $\lim_{\varepsilon\to 0}\lim_{n\to+\infty} \frac{1}{n} v_1^0(f,n,\varepsilon) = 0$. This yields $h_\mu(f) \geq \log d$.

LEMMA 5.2. $\lim_{n\to+\infty} \mu_n = \mu$.

*Proof.* It follows from Lemma 5.1 that

$$(f^j)_*(\nu_n) = \frac{1}{d^j}(f^j)_*(\psi[D_R]) \wedge \frac{1}{d^{n-j}}(f^{n-j})^*(\omega').$$

Let $(k_n)$ be a sequence of integers such that $k_n \to +\infty$ and $k_n = o(n)$. We can decompose $\mu_n$ as $\mu_n = \mu_n' + \lambda_n$ where $\lambda_n \to O$ and

$$\mu_n' = \frac{1}{n}\sum_{j=k_n}^{n-k_n} (f^j)_*(\nu_n) = R_n \wedge T^+ + dd^c\left(\frac{1}{n}\sum_{j=k_n}^{n-k_n} \frac{(u_{n-j} - G^+)}{d^j}(f^j)_*(\psi[D_R])\right).$$

Here $u_n$ denotes the potential of $\frac{1}{d^n}(f^n)^*(\omega')$ and $R_n = \frac{1}{n}\sum_{j=k_n}^{n-k_n}\frac{1}{d^j}(f^j)_*([L])$. The second term converges towards 0 because $u_n$ uniformly converges towards $G^+$ on compact subsets $\mathbb{C}^2$. Now $(R_n)$ converges towards $T^-$ so we can argue as in the proof of the ergodicity of $\mu$: since $\|dR_n\|$, $\|dd^c R_n\| \to 0$ (Proposition 5.1), we get $\mu_n'' = R_n \wedge T^+ \to T^- \wedge T^+ = \mu$. □

*Remark* 5.3. We assumed $I_f$ is repelling to insure that $\mu$ is compactly supported. Since $f$ is polynomial, $T^+$ has locally bounded potential in $\mathbb{C}^2$ so $\mu = T^+ \wedge T^-$ is well defined in $\mathbb{C}^2$, hence in $X\backslash I_f$, hence in $X$. A careful analysis of the potentials near $I_f$ should show that $\mu$ is of total mass in $\mathbb{C}^2$. This would be a first step towards a generalization of Theorem 5.3: one expects the measure $\mu$ to still mixing and of maximal entropy. This was partially done in [Fa-G 99] in the case of birational polynomial mappings.

*Example* 5.1.

(1) Consider $f$: $(z, w) \in \mathbb{C}^2 \mapsto (P(w), Q(z) + R(w))$, where $P$, $Q$, $R$ are polynomials of degree $p = \deg P$, $q = \deg Q$, $d = \deg R$ with $d > pq$. Then $f$ admits an algebraically stable extension to $\mathbb{P}^2$ with $I_f = [1 : 0 : 0]$ and $f(Y_\infty \backslash I_f) = [0 : 1 : 0] = q_\infty$. Note that $f^*\omega_{FS} \sim d\omega_{FS} \sim f_*\omega_{FS}$ and $\lambda_1(f) = d > pq = d_t(f)$, hence we can consider $T^+$ and $T^-$. Moreover $I_f$ is a repelling set for $f_{|Supp\, T}$ (see Example 4.1). Thus $f$ satisfies the assumptions of Theorem 5.3. One can check here that $\mu = T^+ \wedge T^-$ is precisely the equilibrium measure of the compact $K$ of points with bounded orbits.

(2) Consider $f = (w, w^a z + B(w))$, where $B$ is a polynomial of degree $b < a$. Then $f$ admits an algebraically stable extension to $X = \mathbb{P}^1 \times \mathbb{P}^1$ with $\lambda = [a + \sqrt{a^2 + 4}]/2$. We have $I_f = I_{f^2} = \{(0, \infty); (\infty, 0)\}$ and $f^2(Y_\infty \backslash I_f) = q_\infty := (\infty, \infty)$. The map $f$ is birational, i.e. $d_t = 1$. One easily checks that $I_f$ is an attracting 2-cycle for $f$, so $f$ satisfies the assumptions of Theorem 5.3.

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