Math. Ann. 313, 437–474 (1999)

Mathematische Annalen
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Approximation of currents on complex manifolds

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Received: 25 March 1998

Mathematics Subject Classification: 32E20, 32F20, 32C20

Introduction

We are interested in the approximation of positive closed currents of bidegree $(1,1)$ on a complex manifold X by rational divisors, i.e. currents of the type $\frac{1}{N_j}[H_j]$, where N_j is an integer and $[H_j]$ denotes the current of integration along a complex hypersurface H_j of X.

When X is a pseudoconvex open set in $\mathbb{C}^m$ s.t. $H^2(X,\mathbb{R})=0$, Lelong proved [Le 72] that one can always find such an approximation in the weak sense of currents. Demailly [De 82] generalized this result to the case where X is a Stein or a projective algebraic manifold, modulo some cohomological assumptions: for example one can weakly approximate a positive closed current T of bidegree $(1,1)$ if it has integer class (i.e. $[T]\in H^2(X,\mathbb{Z})$).

Using rational convexity properties of the complement of the support of a positive closed current T of bidegree $(1,1)$ in $\mathbb{C}^m$, Duval and Sibony [D-S 95] showed that one can approximate T by rational divisors whose support converges to $Supp\, T$ in the Hausdorff metric.

The purpose of this work is an attempt to generalize this result to the case of projective algebraic and Stein manifolds.

We first consider the case of homogeneous manifolds (X is homogeneous if its group of biholomorphisms Aut(X) acts transitively on X). There is in this situation a useful regularization process for positive metrics of holomorphic line bundles which we recall in an Appendix. Our main theorem 1.6 shows that:

Theorem 0.1 *Every positive closed current T of bidegree $(1,1)$ on the projective space $\mathbb{P}^m(\mathbb{C})$ (resp. the Grassmann manifold $G_{k,m}(\mathbb{C})$ of $k-$planes of $\mathbb{C}^m$, resp. the hyperquadric $\mathbb{Q}_m(\mathbb{C})$ for $m \geq 4$) can be weakly approximated by rational divisors whose support converges to $Supp\, T$.*

We give an example (1.3) of a current on an abelian torus for which such an approximation does not hold.

In paragraph 2 we define and study the notion of (strong)rational convexity on a complex manifold X: a compact set K is said to be (strongly)rationally convex if $X \setminus K$ is a union of positive divisors. Our main theorem is a generalization of a result of [D-S 95]:

Theorem 0.2 *Let S be a smooth compact totally real submanifold of a projective algebraic manifold X. Then S is rationally convex iff it is isotropic for some Hodge form, i.e. $\omega_{|S} = 0$ for some Kähler form ω on X s.t. $[\omega] \in H^2(X, \mathbb{Z})$.*

There is no intrinsic definition of polynomial convexity on complex manifolds generalizing the usual notion in $\mathbb{C}^m$. However we define a notion of polynomial convexity relative to a positive closed current T of bidegree $(1,1)$:

Definition 0.3 *The $T-$polynomial hull of a compact subset K of X is*

$$\widehat{K}^T := \left\{ x \in X \;/\; f(x) \leq \sup_K f, \, \forall f \in \mathcal{C}_T(X) \text{ s.t. } dd^c f \geq -T \right\},$$

where $\mathcal{C}_T(X)$ denotes the set of functions $f \in L^1(X)$ s.t. $\exp(f + \varphi)$ is continous whenever φ is a local potential of T. The compact K is said to be $T-$polynomially convex when $\widehat{K}^T = K$.

In many cases, $X \setminus Supp\, T$ satisfies a convexity property (the “condition (C)”: $\forall K \subset\subset X \setminus Supp\, T$, $\widehat{K}^T \subset\subset X \setminus Supp\, T$) which turns out to be intermediate between being “rationally convex” and being Runge. An interesting observation on the Levi problem (theorem 3.7) yields the following:

Theorem 0.4 *Let T be a positive closed current of bidegree $(1,1)$ on a compact Kähler manifold X. If T is cohomologous to a Kähler form and satisfies condition (C), then $X \setminus Supp\, T$ is Stein.*

This generalizes the standard situation where s is a holomorphic section of some positive holomorphic line bundle on X and $T = [\{s = 0\}]$ is the current of integration along the positive divisor $\{s = 0\}$.

Our main approximation result gives an approximation of certain positive closed currents by rational divisors with a control of the supports and the Lelong numbers of the approximants:

Theorem 0.5 *Let T be a positive closed current of bidegree $(1,1)$ on a projective algebraic manifold X. Assume there is $\lambda > 0$ s.t. $[\lambda T] = c_1(L)$ for some holomorphic line bundle L which we assume is positive. Assume $T = [H] + R$, where $H = \sum_{j=1}^{p} \lambda_j [Z_j]$ ($\forall j$, λ_j is a positive constant and Z_j is an irreducible algebraic hypersurface of X) and R is a positive closed current of bidegree $(1,1)$ on X s.t. the level sets of Lelong numbers of R, $E_c(R) = \{x \in X \,/\, \nu(R,x) \geq c\}$, are of codimension greater or equal than 2. Assume moreover that T satisfies condition (C).*
Then there exists $N_j \in \mathbb{N}$ and $s_j \in \Gamma(X, L^{N_j})$ s.t.
i) $T_j = \frac{1}{N_j}[\{s_j = 0\}] \longrightarrow T$ in the weak sense of currents;
ii) $\{s_j = 0\} \longrightarrow Supp\, T$ in the Hausdorff metric;
iii) $\forall x \in X$, $\nu(T_j, x) \longrightarrow \nu(T, x)$.

It can be seen as a combination of a result of Demailly [De 93] and the approximation result of Duval and Sibony [D-S 95].

Finally we take up our main results in paragraph 5 considering the case of Stein manifolds.

We now set some notations and recall a few definitions from complex analytic geometry for the reader's convenience.

Let L be a holomorphic line bundle on a complex manifold X. We always implicitly fix a locally finite open covering $\{\mathcal{U}_\alpha\}$ of X s.t. $L_{|\mathcal{U}_\alpha}$ is trivial and both the $\mathcal{U}_\alpha$'s and the $\mathcal{U}_{\alpha\beta} := \mathcal{U}_\alpha \cap \mathcal{U}_\beta$ are connected and simply connected. The line bundle is then uniquely determined by its transition functions $g_{\alpha\beta} \in \mathcal{O}^*(\mathcal{U}_{\alpha\beta})$. We denote by $Pic(X)$ the Picard group of holomorphic line bundles of X and we use a multiplicative notation for the group law; $\Gamma(X,L)$ denotes the set of holomorphic sections of L on X, i.e. $s \in \Gamma(X,L)$ is a set $\{s_\alpha\}$ of functions $s_\alpha \in \mathcal{O}(\mathcal{U}_\alpha)$ satisfying the compatibility condition $s_\alpha = g_{\alpha\beta} s_\beta$ in $\mathcal{U}_{\alpha\beta}$.

A positive (singular-)metric of L is a set $\varphi = \{\varphi_\alpha\}$ of plurisubharmonic functions (psh for short), $\varphi_\alpha \in PSH(\mathcal{U}_\alpha)$, s.t. $\varphi_\alpha = \varphi_\beta + \log |g_{\alpha\beta}|$ in $\mathcal{U}_{\alpha\beta}$. Note that the curvature current of the metric defined as $dd^c\varphi := dd^c\varphi_\alpha$ in $\mathcal{U}_\alpha$ (where $d = \partial + \overline{\partial}$ and $d^c = \frac{1}{2i\pi}(\overline{\partial} - \partial)$) is globally well defined on X, since $\log |g_{\alpha\beta}|$ is pluriharmonic in $\mathcal{U}_{\alpha\beta}$; it is a positive closed current of bidegree $(1,1)$ on X but not necessarily a smooth form since we allow singularities. Observe also that the difference of two metrics of L is a globally well defined function $f \in L^1(X)$. If $h = \{h_\alpha\} \in \Gamma(X,L)$, then $\log|h| := \{\log|h_\alpha|\}$ defines a positive singular metric of L on X. We denote by $\mathcal{P}(X,L)$ the set of positive metrics of L on X; if $\varphi = \{\varphi_\alpha\} \in \mathcal{P}(X,L)$ and $h = \{h_\alpha\} \in \Gamma(X,L)$, then the norm of h in the metric φ is defined in each $\mathcal{U}_\alpha$ by $|h|_\varphi := |h_\alpha| e^{-\varphi_\alpha}$.

When X is compact Kähler, it follows from the Hodge decomposition theorem that if a positive closed current T of bidegree $(1,1)$ on X has integer class (i.e. $[T] \in H^2(X,\mathbb{Z})$), then $[T]$ is equal to the first Chern class of some holomorphic line bundle L on X. Therefore there exists a positive metric φ of L on X which is a potential for T (i.e. $T = dd^c\varphi_\alpha$ in $\mathcal{U}_\alpha$ and we write then $T = dd^c\varphi$).

A line bundle $L \in Pic(X)$ is said to be pseudoeffective if there exists a singular positive metric φ of L on X and L is positive (resp. semi-positive) if it admits a smooth positive metric φ on X s.t. the curvature form $dd^c\varphi$ is a Kähler form on X (resp. a semi-positive $(1,1)-$form).

We refer to [De 90] for further details and we finally recall a particular version of the solution to the $\overline{\partial}-$problem with L^2-estimates on projective algebraic and Stein manifolds that we use intensively in the present work:

Theorem 0.6 *Let X be a projective algebraic or a Stein manifold of dimension m and fix ω a Kähler form on X. Let L be a holomorphic line bundle on X and suppose there exists a singular metric θ of L s.t. $dd^c\theta \geq \varepsilon\omega$ for some positive constant ε. Then for every smooth $\overline{\partial}-$closed $(m,1)-$form v with values in L, s.t. $\int_X |v|^2 e^{-2\theta} dV_\omega < +\infty$, there exists a smooth $(m,0)-$form u with values in L s.t. $\overline{\partial}u = v$ and*

$$\int_X |u|^2 e^{-2\theta} dV_\omega \leq \frac{1}{\varepsilon} \int_X |v|^2 e^{-2\theta} dV_\omega,$$

where $dV_\omega = \frac{1}{m!}\omega^m$ denotes the Kähler volume element.

Acknowledgements. I would like to express my heartfelt thanks to N.Sibony for suggesting this work and helping me through it. Part of it was done while i was visiting Chalmers University of Technology under the support of the Swedish Royal Academy of Sciences. I am particularly grateful to B.Berndtsson for very stimulating discussions.

1 Approximation of currents on homogeneous manifolds

1.1 A modification procedure

In this section we set up a general construction of holomorphic sections with prescribed bounded norm on the set where a metric of a line bundle is pluriharmonic (proposition 1.1). This is an important technical step in the approximation theorem of the next section and we use it as well to establish rational convexity properties of the complement of the support of positive closed currents on homogeneous manifolds (theorem 2.5).

Proposition 1.1 *Let X be a projective algebraic homogeneous manifold. Let T be a positive closed current of bidegree $(1,1)$ on X s.t. $[T] = c_1(L)$ for some holomorphic line bundle L that we assume is positive.*
Assume T admits a continuous potential, i.e. there exists a continuous positive metric φ of L s.t. $dd^c\varphi = T$.

For $\varepsilon > 0$ *we set* $K_\varepsilon = \{m \in X \,/\, d(m, Supp\, T) \geq \varepsilon\} \subset\subset X \setminus Supp\, T$, *where* d *is a pseudo-distance function defined in the Appendix.*
Let V *be an open subset of* X *s.t.* $K_\varepsilon \subset V \subset\subset X \setminus Supp\, T$ *and fix* $\delta > 0$.

Then we can find an integer M *and construct a continuous positive metric* ψ *of* L^M *and a holomorphic section* h *of* L^M *in* V *s.t.*

i) $K_\varepsilon \subset \{m \in V \,/\, |h|_\psi(m) \geq 1\} = \{m \in V \,/\, |h|_\psi(m) = 1\} \subset\subset V$
ii) $\| \frac{\psi}{M} - \varphi \|_{L^\infty(X)} \leq \delta$
iii) ψ *is* $\mathcal{C}^\infty-$*smooth and* $dd^c\psi > 0$ *in a neighborhood of* $Supp\, T$.

We first prove three lemmas.

Lemma 1.2 *Under the above assumptions* $X \setminus Supp\, T$ *is Stein and we can find large relatively compact Stein open subsets* W *of* $X \setminus Supp\, T$ *and positive integers* k *s.t.* ${L^k}_{|W}$ *is trivial.*

Proof. It is an easy consequence of the Kontinuitätssatz that $X \setminus Supp\, T$ is locally pseudoconvex in X (see [Ce 78]). Since X is homogeneous it is infinitesimally homogeneous (i.e. the global holomorphic vector fields generate the tangent space of X at every point of X, see [Hi 75]). It follows then from a result of Hirschowitz [Hi 75] that $X \setminus Supp\, T$ is Stein iff it admits no "interior integral curve" (a holomorphic map $\gamma : \mathbb{C} \to X \setminus Supp\, T$ with relatively compact image whose tangent vectors belong to some holomorphic vector field on X). If such a curve exists, we can construct by a standard argument (see lemma 1.3 below) a non trivial positive closed current S of bidimension $(1,1)$ with compact support in $X \setminus Supp\, T$. Now T is cohomologous to a Kähler form ω (L is positive), hence $T = \omega - dd^c f$ for some $f \in L^1(X)$ which is smooth outside $Supp\, T$ (see [G-H 78] p149); thus Stokes theorem gives

$$\begin{aligned} \|S\| := \int_X \omega \wedge S &= \int_{X \setminus Supp\, T} dd^c f \wedge S \\ &= \int_{X \setminus Supp\, T} d^c f \wedge dS \\ &= 0. \end{aligned}$$

Therefore $X \setminus Supp\, T$ contains no interior integral curve hence it is Stein (see also theorem 3.8).

Let ϱ be a smooth strictly p.s.h. exhaustion function of $X \setminus Supp\, T$. By Sard's lemma we can find $R \in \mathbb{R}$ as big as we like so that $W = \{x \in X \setminus Supp\, T \,/\, \varrho(x) < R\}$ is a smooth relatively compact Stein open subset of $X \setminus Supp\, T$. Therefore the cohomology of W is finite dimensional and $H^1(W, \mathbb{R}) = H^1(W, \mathbb{Z}) \otimes \mathbb{R}$.

Since W is Stein, it follows from Cartan's theorem B that the Picard group $Pic(W) = H^1(W, \mathcal{O}^*)$ is isomorphic to $H^2(W, \mathbb{Z})$.

Now L admits a flat metric in W since $dd^c\varphi = 0$ in W hence the image of the first Chern class of L via $H^2(W, \mathbb{Z}) \to H^2(W, \mathbb{R}) \simeq H^2_{dR}(W, \mathbb{R})$ is 0.

In terms of Čech cohomology this means that a finite number of equations in a finite number of unknowns with coefficients in $\mathbb{Z}$ admits a solution in $\mathbb{R}$. Therefore it must have some solution in $\mathbb{Q}$, and multiplying the equations by some large integer k gives a solution in $\mathbb{Z}$ to the corresponding system, i.e. $c_1(L^k) = 0$ in $H^2(W, \mathbb{Z})$. Since c_1 is an isomorphism between $Pic(W)$ and $H^2(W, \mathbb{Z})$, this shows that $L^k{}_{|W}$ is trivial. Q.E.D.

Lemma 1.3 *Let Ω be an open subset of a complex Kähler manifold X and let $\gamma : \mathbb{C} \to \Omega$ be a non constant holomorphic map with relatively compact image in Ω. Then there exists a non trivial positive closed current of bidimension $(1, 1)$ with support in $\overline{\gamma(\mathbb{C})}$.*

Proof. Fix ω a Kähler form on X and let S_R be the current of integration over the analytic disc $\gamma(\Delta(R))$, where $\Delta(R)$ denotes the disc of radius R centered at 0 in $\mathbb{C}$. We normalize S_R in the following way:

$$< S_R, \theta >:= \frac{1}{\int_{\gamma(\Delta(R))} \omega} \int_{\gamma(\Delta(R))} \theta,$$

where θ is any test form of bidegree $(1, 1)$ on X. Therefore S_R are positive currents of bidimension $(1, 1)$ and of mass 1 on X. We want to extract a weak limit that is closed; it will have compact support in $\overline{\gamma(\mathbb{C})}$.

We claim there exists a sequence of radii $R_j \to +\infty$ s.t.

$$\frac{R_j^{1/2} \left[\int_{\partial\Delta(R_j)} |\gamma'|^2_\omega \right]^{1/2}}{\int_{\Delta(R_j)} |\gamma'|^2_\omega} \to 0.$$

Assume the contrary. Then there exists $c > 0$ s.t. $\forall R > 0$,

$$\frac{R^{1/2} \left[\int_{\partial\Delta(R)} |\gamma'|^2_\omega \right]^{1/2}}{\int_{\Delta(R)} |\gamma'|^2_\omega} \geq c > 0.$$

Set $f(t) = \int_{\Delta(e^t)} |\gamma'|^2_\omega$. This is a well defined function of $t \in \mathbb{R}$ which is smooth, positive, and s.t. $f'(t) = e^t \int_{\partial\Delta(e^t)} |\gamma'|^2_\omega \geq 0$. We thus have $\frac{f'(t)}{f^2(t)} \geq c^2 > 0$. This implies

$$\frac{1}{f(0)} \geq \frac{-1}{f(t)} + \frac{1}{f(0)} \geq c^2 t, \ \forall t \geq 0,$$

a contradiction.

Fix such a sequence R_j. Since $\|S_{R_j}\| = 1$, there exists a subsequence $(S_{R_{j_k}})$ which converges in the weak sense of currents towards a positive current S of bidimension $(1, 1)$. Again $\|S\| = 1$ and S has relatively compact support in Ω. We claim that S is closed. Indeed let θ be a test form of degree 1. Then

$$< dS, \theta >= - < S, d\theta >= - \lim_{k \to +\infty} \left[\frac{\int_{\partial \Delta(R_{j_k})} \gamma^* \theta}{\int_{\Delta(R_{j_k})} \gamma^* \omega} \right].$$

Now there exists $C > 0$ s.t. $|\gamma^* \theta| \leq C |\gamma'|_\omega$ and Cauchy-Schwarz inequality gives

$$\left| \int_{\partial \Delta(R_{j_k})} \gamma^* \theta \right| \leq C \int_{\partial \Delta(R_{j_k})} |\gamma'|_\omega \leq C R_{j_k}^{1/2} \left[\int_{\partial \Delta(R_{j_k})} |\gamma'|_\omega^2 \right]^{1/2}.$$

On the other hand $\gamma^* \omega = |\gamma'|_\omega^2$ hence by definition of R_j,

$$\frac{| \int_{\partial \Delta(R_{j_k})} \gamma^* \theta |}{\int_{\Delta(R_{j_k})} |\gamma'|_\omega^2} \to 0,$$

hence S is closed. Q.E.D.

Let $PH(W)$ be the real vector space of pluriharmonic functions in W. Let $\{\mathcal{U}_\alpha\}$ be an open covering of W s.t. both the $\mathcal{U}_\alpha$'s and the $\mathcal{U}_{\alpha\beta} = \mathcal{U}_\alpha \cap \mathcal{U}_\beta$ are connected and simply connected.

Let $\varphi \in PH(W)$. We can write $\varphi = \Re(h_\alpha)$ in $\mathcal{U}_\alpha$ where $h_\alpha \in \mathcal{O}(\mathcal{U}_\alpha)$ and we set $v_{\alpha\beta} = \frac{1}{2i\pi} [h_\alpha - h_\beta]$. This is a holomorphic function in $\mathcal{U}_{\alpha\beta}$ which has imaginary part equal to zero hence $v_{\alpha\beta}$ is constant. Moreover $v_{\alpha\beta} + v_{\beta\gamma} + v_{\gamma\alpha} \equiv 0$ in $\mathcal{U}_{\alpha\beta\gamma}$, thus $\{v_{\alpha\beta}\}$ is a real Čech $1-$cocycle and defines a class $[v_{\alpha\beta}(\varphi)] \in H^1(W, \mathbb{R})$.

Lemma 1.4 *The map*

$$\begin{aligned} \Phi : PH(W) &\to \quad H^1(W, \mathbb{R}) \\ \varphi \quad &\to [v_{\alpha\beta}(\varphi)] = \Phi(\varphi) \end{aligned}$$

is a morphism of real vector spaces s.t. $\ker \Phi = \Re(\mathcal{O}(W))$.

Given $\varphi \in PH(W)$*, there exists* $H \in \mathcal{O}^*(W)$ *s.t.* $\varphi = \log |H|$ *in* W *iff* $\Phi(\varphi) \in H^1(W, \mathbb{Z}) \subset H^1(W, \mathbb{R})$.

If moreover W *is Stein, then* Φ *is surjective.*

Proof. The first assertion is clear.

Let $\varphi \in PH(W)$ and assume $\Phi(\varphi) = [v_{\alpha\beta}] \in H^1(W, \mathbb{Z}) \subset H^1(W, \mathbb{R})$. Then there exists $c_{\alpha\beta} \in \mathbb{Z}$ and $w_\alpha \in \mathbb{R}$, s.t. $v_{\alpha\beta} = c_{\alpha\beta} + w_\alpha - w_\beta$. Thus we have

$$h_\alpha + 2i\pi w_\alpha = h_\beta + 2i\pi w_\beta + 2i\pi c_{\alpha\beta}$$

and we can define a global holomorphic function by setting $H = e^{h_\alpha + 2i\pi w_a}$ in $\mathcal{U}_\alpha$. Clearly $\log |H| = \varphi$ in W.

Conversely assume there exists $H \in \mathcal{O}^*(W)$ s.t. $\log |H| = \varphi$ in W. We fix a determination $h_\alpha = \log H$ of the complex logarithm of H in $\mathcal{U}_\alpha$. Two such determinations only differ by an integer multiple of $2i\pi$ hence

$$v_{\alpha\beta} = \frac{1}{2i\pi}\left[h_\alpha - h_\beta\right] = c_{\alpha\beta} \in \mathbb{Z}.$$

This shows that $[v_{\alpha\beta}] \in H^1(W, \mathbb{Z}) \subset H^1(W, \mathbb{R})$.

Assume now that W is Stein and let $[v_{\alpha\beta}] \in H^1(W, \mathbb{R})$. It also defines a class in $H^1(W, \mathcal{O}) = \{0\}$. Thus there exists $h_\alpha \in \mathcal{O}(\mathcal{U}_\alpha)$ s.t. $v_{\alpha\beta} = \frac{1}{2i\pi}\left[h_\alpha - h_\beta\right]$. Hence $f = \Re h_\alpha$ is a globally well defined pluriharmonic function on W s.t. $\Phi(f) = [v_{\alpha\beta}]$. Q.E.D.

Remark 1.5 *This can be seen as a reformulation in terms of Čech cohomology of lemma 1.3 in [D-S 95].*

Proof of proposition 1.1. Let φ^ε be the regularized metric of φ defined in the Appendix. It is a smooth metric for the same line bundle L that decreases uniformly towards φ when ε decreases towards 0 since we assumed φ is continuous. We can therefore assume that $\|\varphi^\varepsilon - \varphi\|_{L^\infty(X)} \leq 5.\delta/8$ (otherwise, replace ε by some smaller constant)
Moreover, $\varphi^\varepsilon = \varphi$ on K_ε, and $\varphi^\varepsilon > \varphi$ in $X \setminus K_\varepsilon$, hence we can also assume that $\varphi^\varepsilon \geq \varphi + 4\delta'/8$ on ∂V (with $0 < \delta' \leq \delta$).

Let W be a smooth Stein open subset of $X \setminus Supp\, T$ s.t. $V \subset\subset W \subset\subset X \setminus Supp\, T$ and $H^1(W, \mathbb{R})$ is equal to $H^1(W, \mathbb{Z}) \otimes \mathbb{R}$ and of finite dimension, and ${L^k}_{|W}$ is trivial for some positive integer k (see lemma 1.2).

Therefore $k\varphi$ is a continuous metric of L^k on X which defines a pluriharmonic function in W. Since $\Phi : PH(W) \to H^1(W, \mathbb{R})$ is surjective, we can find $f_1, \dots, f_p$ in $PH(W)$ s.t. $(\Phi(f_j))$ is a $\mathbb{Z}-$basis of $H^1(W, \mathbb{R})$. We can thus choose $\lambda = (\lambda_1, \dots, \lambda_p) \in \mathbb{R}^p$ so small that $\theta_\lambda = k\varphi + \sum_{j=1}^p \lambda_j.f_j$ (which is a continuous metric of L^k in W) satisfies $\Phi(\theta_\lambda) \in H^1(W, \mathbb{Q}) = H^1(W, \mathbb{Z}) \otimes \mathbb{Q}$ and $\|\frac{1}{k}\theta_\lambda - \varphi\|_{L^\infty(\overline{V})} < \delta'/8$.
Fix $M_1 \in \mathbb{N}$ s.t. $\Phi(M_1.\theta_\lambda) \in H^1(W, \mathbb{Z})$. By lemma 1.4 we can find a holomorphic section h of L^M in W ($M = kM_1$) which has constant norm equal to 1 in the metric $M_1\theta_\lambda$ (i.e. $|h|e^{-M_1\theta_\lambda} \equiv 1$).

Since L is positive, we can find a smooth metric G of L s.t. $dd^cG > 0$ in X. Consider $f_\eta = \eta.G + (1-\eta)\left[\varphi^\varepsilon - 2\delta'/8\right]$ $(0 \leq \eta < 1)$. It is a smooth metric of L s.t. $dd^c f_\eta > 0$ in X if $\eta > 0$ and $\|f_\eta - (\varphi^\varepsilon - 2\delta'/8)\|_{L^\infty(X)} = \eta.\|G - (\varphi^\varepsilon - 2\delta'/8)\|_{L^\infty(X)} \leq \delta'/8$ if we choose η small enough.

We define

$$\psi = \begin{cases} M_1 \sup(kf_\eta, \theta_\lambda) & \text{in V} \\ M.f_\eta & \text{in } X \setminus V \end{cases}$$

It is a well defined continuous positive metric of L^M in X since the maximum of two continuous positive metrics of a holomorphic line bundle is a continuous positive metric of the same line bundle and

$$f_\eta \leq (\varphi^\varepsilon - 2\delta'/8) + \delta'/8 = \varphi - \delta'/8 < \frac{1}{k}\theta_\lambda \text{ on } K_\varepsilon$$

whereas

$$f_\eta \geq (\varphi^\varepsilon - 2\delta'/8) - \delta'/8 \geq \varphi + \delta'/8 > \frac{1}{k}\theta_\lambda \text{ on } \partial V.$$

We have $dd^c\psi = M.dd^c f_\eta \geq M\eta dd^c G > 0$ in $X \setminus V$ and

$$\|\psi/M - \varphi\| \leq \max\left\{\|f_\eta - \varphi\|_{L^\infty(X)}; \|\theta_\lambda/k - \varphi\|_{L^\infty(\overline{V})}\right\} \leq \delta$$

since

$$\begin{aligned}\|f_\eta - \varphi\|_{L^\infty(X)} &\leq \|f_\eta - (\varphi^\varepsilon - 2\delta'/8)\|_{L^\infty(X)} + 2\delta'/8 + \|\varphi_\varepsilon - \varphi\|_{L^\infty(X)} \\ &\leq 3\delta'/8 + 5\delta/8 \leq \delta.\end{aligned}$$

Finally, $\psi = M_1.\theta_\lambda = \log|h|$ in a neighborhood of K_ε hence

$$K_\varepsilon \subset \left\{m \in V \,/\, |h|e^{-\psi} \geq 1\right\} = \left\{m \in V \,/\, |h|e^{-\psi} = 1\right\} \subset\subset V,$$

and the proof is complete. Q.E.D.

1.2 Approximation of $(1,1)-$positive closed currents

Theorem 1.6 *Let X be a projective algebraic homogeneous manifold and T a positive closed current of bidegree $(1,1)$ on X. Assume that $\exists\lambda \in \mathbb{R}^{*+}$ s.t. $[\lambda T] \in H^2(X,\mathbb{Z})$, hence $[\lambda T] = c_1(L)$ for some holomorphic line bundle which, we assume, is positive.*

Then there exists (H_j) algebraic hypersurfaces of X and (N_j) integers s.t.

$$\begin{cases} \frac{1}{\lambda N_j}[H_j] \to T \text{ in the weak sense of currents} \\ \text{and} \\ H_j \to Supp\, T \text{ in the Hausdorff metric.}\end{cases}$$

Remark 1.7 *The cohomological and the positivity assumptions on the cohomology class of T are always satisfied if $H^{1,1}(X) = \mathbb{C}$. This the case if X is the complex projective space $\mathbb{P}^m(\mathbb{C})$, the Grassmann manifold $G_{k,m}(\mathbb{C})$ of complex k-planes of $\mathbb{C}^m$ or the hyperquadric $\mathbb{Q}_m(\mathbb{C})$ ($m \geq 4$).*

Proof of the theorem. We can assume that T is smooth since, on a homogeneous manifold, we can regularize T in such a way that $[T^\varepsilon] = [T]$ and T^ε tends to T in the sense of the theorem (see Appendix).

We can also assume $\lambda = 1$ and we denote by $\varphi = \{\varphi_\alpha \in PSH(\mathcal{U}_\alpha)\}$ a smooth positive metric of L s.t. $dd^c\varphi = T$ (two such metrics only differ by a constant).

Let $K_n = \left\{m \in X \,/\, d(m, Supp\,T) \geq \frac{1}{n}\right\}$, and $\delta_n > 0$ a sequence converging towards 0. We fix neighborhoods $V_n \subset\subset X \setminus Supp\,T$ of K_n. By proposition 1.1, we can find integers M_n, continuous positive metrics ψ_n of L^{M_n} and holomorphic sections h_n of L^{M_n} in V_n with the prescribed properties.

Fix $(a_j)_{j\in\mathbb{N}}$ a sequence of points dense in $Supp\,T$.We are going to construct for each n, an integer N_n and a global holomorphic section S_n of $L^{N_n.M_n}$ s.t.

$$|S_n|e^{-N_n\psi_n} \leq 1 \text{ on } X$$
$$|S_n|e^{-N_n\psi_n} \geq \frac{1}{2} \text{ on } \{a_1, \ldots, a_n\} \cup K_n$$

Thus we get by ii) of proposition 1.1:

$$\frac{1}{N_n.M_n}\log|S_n| \leq \varphi + \delta_n \text{ on } X$$
$$\frac{1}{N_n.M_n}\log|S_n| \geq \varphi - \delta_n - \frac{\log 2}{M_n} \text{ on } \{a_1, \ldots, a_n\}$$
$$|S_n| > 0 \text{ on } K_n.$$

The first two inequalities show the convergence of $\frac{1}{N_n.M_n}\log|S_n|$ in L^1_{loc} towards φ (c.f. lemma 15.1.7. in [Hö 85]), and the Lelong-Poincare equation then gives the convergence of $\frac{1}{N_n.M_n}[\{S_n = 0\}]$ towards T in the weak sense of currents, while the last inequality shows that $\{S_n = 0\} \subset X \setminus K_n$, and since K_n exhausts $X \setminus Supp\,T$, this gives the convergence of $\{S_n = 0\}$ towards $Supp\,T$ in the Hausdorff metric.

We construct now the sections S_n. From now on, n is fixed and we might not mention the subscript. We fix an open covering $\{U_\alpha\}$ of X which is fine enough s.t. $\forall 1 \leq i \leq n$, $\exists! a_i \in U_{\alpha_i}$ and $L_{|U_{\alpha_i}}$ is trivial.

Since ψ is smooth in a neighborhood of $Supp\,T$, and $dd^c\psi > 0$ on $Supp\,T$, there are holomorphic polynomials P_i s.t. $\psi_{\alpha_i}(x) - \Re(P_i)(x) \geq c_i d^2_{eucl}(a_i, x)$, in a neighborhood W_i of a_i, $W_i \subset U_{\alpha_i}$, for some strictly positive constants c_i. We choose the W_i's small enough so that $W_i \cap U_\beta = \emptyset$, $\forall \beta \neq \alpha_i$.

Let $\chi_i \in \mathcal{C}^\infty_0(W_i)$ with $0 \leq \chi_i \leq 1$, and $\chi_i \equiv 1$ in a neighborhood of a_i. We define smooth sections f_i of L^{NM} by $f_i^\alpha = 0$ if $\alpha \neq \alpha_i$ and $f_i^{\alpha_i} = \chi_i e^{NP_i}$ (N is an integer to be chosen later).

Let χ be a test function in a neighborhood of $K' = \left\{|h|e^{-\psi} \geq 1\right\}$ ($0 \leq \chi \leq 1$ and $\chi \equiv 1$ in a neighborhood of K'), s.t $Supp\,\chi$ is disjoint from the supports of the χ_i's.

Set $u = \chi.h^N + \sum_{i=1}^n f_i$. This is a smooth global section of L^{NM} on X, hence $\overline{\partial} u$ is a smooth $\overline{\partial}-$closed $(0,1)-$form with values in L^{NM}, i.e. a smooth $\overline{\partial}-$closed $(m,1)-$form with values in $L^{NM} \otimes K_X^*$ (m is the dimension of X).

We set $N = N_1 + N_2$ where we fix $N_2 \in \mathbb{N}$ s.t $L^{N_2.M} \otimes K_X^*$ is positive and N_1 will be chosen later. Fix ω a Kähler metric on X, $\varepsilon > 0$ and G a smooth metric of $L^{N_2.M} \otimes K_X^*$ s.t $dd^cG \geq \varepsilon.\omega$. We solve $\overline{\partial} v = \overline{\partial} u$ on X with L^2 estimates associated to the metric $\theta = N_1.\psi + G$ and get

$$\int_X |v|^2 e^{-2\theta} dV_\omega \leq \frac{1}{\varepsilon} \int_X |\overline{\partial} u|^2 e^{-2\theta} dV_\omega.$$

Since $Supp\, \overline{\partial}\chi \subset \left\{|h|e^{-\psi} < 1\right\}$, and $Supp\, \overline{\partial}\chi_i \subset \left\{|e^{MP_i}|e^{-\psi} < 1\right\}$, we can fix $a < 1$ s.t. $|\overline{\partial} u|^2 e^{-2N\psi} \leq C_1 a^{2N_1}$ with C_1 independent of N_1. Thus

$$\int_X |v|^2 e^{-2\theta} dV_\omega \leq C_2 a^{2N_1}.$$

We estimate now v on X. It is standard (see lemma 15.1.8 in [Hö 85]) that

$$\begin{aligned} |v(x)|^2 &\leq C_3 \left(r^2 \sup_{B(x,r)} |\overline{\partial} v|^2 + r^{-2m} \|v\|^2_{L^2(B(x,r))} \right) \\ &\leq C_4 e^{2N\psi(x)} e^{2N_1\eta} \left[(\sup |\overline{\partial} u|^2 e^{-2N}) + \|v\|^2_\theta \right] \\ &\leq C_5 (e^\eta a)^{2N_1} e^{2N\psi(x)}, \end{aligned}$$

with C_5 independent of N_1 and where η is the uniform oscillation of ψ on the balls $B(x,r)$. Here $B(x,r)$ implicitly stands for the pull back of an euclidean ball via a coordinate chart. We choose r so that $e^\eta a < 1$ and N_1 so that $C_5(e^\eta a)^{N_1} < \frac{1}{9}$ and set $S_n = \frac{3}{4}(u - v)$. Then S_n is a holomorphic $(m,0)-$form with values in $L^{NM} \otimes K_X^*$, i.e. it is a holomorphic section of L^{NM} on X and it satisfies all our requirements. Q.E.D.

Remark 1.8 *It follows from the Borel-Weil theorem (see e.g. [Ak 95]) that a semi-positive holomorphic line bundle L on a projective algebraic homogeneous manifold X is either positive, or the pull-back p^*L' under a morphism $p : X \to Y$, where Y is a projective algebraic homogeneous manifold of lower dimension, of a positive holomorphic line bundle L' on Y. A positive closed current T of bidegree $(1,1)$ on X s.t. $[T] = c_1(L)$ thus satisfies $T = p^*T'$ for some positive closed current T' of bidegree $(1,1)$ on Y with $[T'] = c_1(L')$, and the approximation of T reduces to that of T' on Y.*

1.3 On a counterexample of Grauert

We explain here an example due to Grauert ([Na 63]) of a pseudoconvex domain in a complex torus which is not holomorphically convex hence not Stein. We show that there is in fact no complex hypersurface of the torus contained in this pseudoconvex domain, whereas it contains the support of a $(1,1)-$positive closed current. Finally we give an explicit example in the 2-dimensional case of this situation, where the parameters are chosen so that the torus is algebraic. This provides a counterexample to the approximation theorem of the previous section if we omit the cohomological and the positivity assumption.
Let Λ be the lattice of $\mathbb{C}^m$ generated by

$$\lambda_1 = (1, 0, \ldots, 0) \text{ and } \lambda_j = (ia_j, a_{j2}, \ldots, a_{jm}) = (ia_j, \lambda'_j),\ 2 \leq j \leq 2m$$

where $(\lambda_j)_{1\leq j\leq 2m}$ is a $\mathbb{R}-$free family in $\mathbb{C}^m \simeq \mathbb{R}^{2m}$, and $(a_j)_{2\leq j\leq 2m}$ are real constants s.t. a_2 and a_3 are $\mathbb{Z}-$independent. We denote by X the corresponding complex torus and $\pi : \mathbb{C}^m \to X = \mathbb{C}^m/\Lambda$ the canonical projection.

Consider $U_\alpha = \left\{z \in \mathbb{C}^m \,/\, 0 < \Re(z_1) < \frac{1}{\alpha}\right\}$, and $D_\alpha = \pi(U_\alpha)$, where $\alpha > 1$. Set $\varphi(z) = \frac{1}{1-\alpha\Re(z_1)} + \frac{1}{\Re(z_1)}$. An easy computation shows that φ is plurisubharmonic in U_α, and it is moreover a $\mathcal{C}^\infty-$smooth exhaustion function for U_α. Now since φ is invariant by any element $\lambda \in \Lambda$ s.t. $\overline{\lambda + U_\alpha} \cap \overline{U_\alpha} \neq \emptyset$, φ also defines a smooth-psh exhaustion function for D_α, hence D_α is pseudoconvex.

Proposition 1.9 *There is no compact analytic subset of X of dimension $m-1$ contained in the domain D_α.*

Proof. Assume the contrary and let A be such a set which, we can assume, is connected. $f(z) = \Re(z_1)$ is a well defined pluriharmonic function on D_α. If A is compact, f attains its maximum on A thus is constant on A and $A \subset \pi(\{\Re(z_1) = t\})$ for some real constant t. This necessarily means that $A \subset \pi(\{z_1 = c\})$, with equality if A is of dimension $m-1$. But since a_2 and a_3 are $\mathbb{Z}-$independent, $\pi(\{z_1 = c\})$ is dense in $\pi(\{\Re z_1 = \Re c\})$, hence it cannot be closed. A contradiction. Q.E.D.

On the other hand, $T = dd^c(max(\Re(z_1) - \frac{1}{2\alpha}, 0))$ is a well defined positive closed current of bidegree $(1,1)$ on X s.t. $Supp\, T = \pi(\{\Re(z_1) = \frac{1}{2\alpha}\}) \subset\subset D_\alpha$. This current is not approximable in the sense of our theorem, since D_α is a neighborhood of $Supp\, T$ which does not contain any hypersurface of X.

Note that there is no $\lambda > 0$ s.t. $[\lambda T] \in H^2(X, \mathbb{Z})$ since a_2 and a_3 are $\mathbb{Z}-$independent; T is moreover not cohomologous to a Kähler form (it is cohomologous to $c.\frac{i}{2} dz_1 \wedge d\overline{z}_1$) and in particular $X \setminus Supp\, T$ is not Stein

since it admits non trivial $(1,1)-$positive closed currents with compact support (any $dd^c(\max(\Re(z_1)-t,0))$, for $t \neq \frac{1}{2\alpha}$).
We exhibit now an explicit algebraic example:
Recall that a torus $\mathbb{C}^m/\Lambda$ is algebraic iff there exists $H \in GL_m(\mathbb{C})$ a positive definite hermitian matrix s.t. $\Im H(\Lambda,\Lambda) \subset \mathbb{Z}$ (see [G-H 78] p303). Let Λ be the lattice in $\mathbb{C}^2$ generated by

$$\lambda_1 = (1,0)\ ; \quad \lambda_2 = (i,0)\ ;$$
$$\lambda_3 = (i\sqrt{2},1)\ ; \quad \lambda_4 = (0,i\sqrt{2});$$

and define

$$H = \begin{pmatrix} 1 & -i\sqrt{2} \\ i\sqrt{2} & 2+\sqrt{2} \end{pmatrix}$$

H is a hermitian matrix which satisfies

$$tr\, H = 3+\sqrt{2} > 0 \text{ and } det\, H = \sqrt{2} > 0,$$

hence it is positive definite. Of course, $\Im H(\lambda,\lambda) = 0\,, \forall\lambda \in \Lambda$, and we easily check that:

$$H(\lambda_1,\lambda_2) = -i \text{ and } H(\lambda_1,\lambda_3) = 0 \text{ and } H(\lambda_1,\lambda_4) = 2$$
$$H(\lambda_2,\lambda_3) = 0 \text{ and } H(\lambda_2,\lambda_4) = 2i \text{ and } H(\lambda_3,\lambda_4) = -2i$$

hence $\Im H(\Lambda,\Lambda) \subset \mathbb{Z}$ and the complex torus $X = \mathbb{C}^2/\Lambda$ is algebraic.

2 Rational convexity on compact complex manifolds

Recall that the rational hull of a compact set K of $\mathbb{C}^m$ is defined as the complement of the union of hypersurfaces of $\mathbb{C}^m$ that do not intersect K. Duval and Sibony show in [D-S 95] that one can replace the hypersurfaces in the definition by positive closed currents of bidegree $(1,1)$ in $\mathbb{C}^m$ whose support does not intersect K.

Therefore there are several natural generalizations of this notion to complex manifolds whether one considers the hull with respect to effective divisors (resp. positive divisors) or positive closed currents of bidegree $(1,1)$ (with or without cohomological restrictions). Although these notions might coincide (e.g. on $\mathbb{P}^m(\mathbb{C})$), most of the time they differ considerably (e.g. on abelian tori). We are going to consider the strongest notion of rational convexity (see definition 2.1 below) because it allows the use of L^2-techniques and it is the proper notion to consider for the generalization of the main theorem in [D-S 95] (see theorem 2.9 below), which was our main motivation for the study of rational convexity.

Definition 2.1 *Let K be a compact subset of a projective algebraic manifold X. We define the rational hull of K by*

$$r(K) := \{m \in X \,/\, \forall H \text{ positive divisor of } X,\ m \in H \Rightarrow H \cap K \neq \emptyset\},$$

and K is said to be rationally convex when $r(K) = K$.

Lemma 2.2

$$\begin{aligned} r(K) = \{ & m \in X \,/\, \left|\frac{f}{g}(m)\right| \leq \sup_K \left|\frac{f}{g}\right|,\ \forall L \in Pic(X) \text{ positive and} \\ & \forall f, g \in \Gamma(X, L) \text{ s.t. } \{g = 0\} \cap K = \emptyset \\ & \text{and } m \notin \{f = 0\} \cap \{g = 0\}\} \end{aligned}$$

therefore $r(K)$ is compact and $r(r(K)) = r(K)$.

Proof. Let $m \notin r(K)$; since positive divisors coincide (modulo linear equivalence) with positive line bundles on a projective algebraic manifold, we can find a holomorphic section s of a positive line bundle L s.t. $s(m) = 0$ and $\{s = 0\} \cap K = \emptyset$. Since L is positive, we can find $k \geq 1$ and a holomorphic section f of L^k s.t. $f(m) \neq 0$. Set $g = s^k$; we thus have $|\frac{f}{g}(m)| = +\infty$ whereas $\sup_K |\frac{f}{g}| < +\infty$, hence $m \notin r'(K)$, where $r'(K)$ denotes the right hand side in the lemma.

Fix now $m \notin r'(K)$ and f, g holomorphic sections of a positive holomorphic line bundle L s.t. $|\frac{f}{g}(m)| > \sup_K |\frac{f}{g}|$. Either $g(m) = 0$ and we are done, or $g(m) \neq 0$ and we may consider $s = f - \frac{f}{g}(m).g$: it is a holomorphic section of L s.t. $s(m) = 0$ and $\{s = 0\} \cap K = \emptyset$. Q.E.D.

Example 2.3 $i)$ *When $X = \mathbb{P}^m(\mathbb{C})$ and $K \subset\subset \mathbb{C}^m \subset \mathbb{P}^m(\mathbb{C})$, this coincides with the usual notion of rational convexity.*

$ii)$ *$\mathbb{P}^m(\mathbb{R}) = \{[z_0, \ldots, z_m] \in \mathbb{P}^m(\mathbb{C}) \,/\, \frac{z_i}{z_j} \in \mathbb{R}$ whenever $z_j \neq 0\}$ is a smooth compact totally real submanifold of $\mathbb{P}^m(\mathbb{C})$ which is rationally convex and intersects every hyperplane of $\mathbb{P}^m(\mathbb{C})$. Indeed let $[x] \in \mathbb{P}^m(\mathbb{C}) \setminus \mathbb{P}^m(\mathbb{R})$; we can assume $x_0 = 1$ and $x_1 \notin \mathbb{R}$ (otherwise rotate coordinates). Consider the homogeneous polynomial of degree 2, $P_\varepsilon(z) = z_1^2 - x_1^2 z_0^2 + \varepsilon[z_2^2 + \ldots + z_m^2 - (x_2^2 + \ldots + x_m^2) z_0^2]$; clearly $P_\varepsilon(x) = 0$, but for $\varepsilon > 0$ small enough, $\mathbb{P}^m(\mathbb{R}) \cap \{P_\varepsilon = 0\} = \emptyset$ since $x_1^2 \notin \mathbb{R}^+$.*

$iii)$ *Let T be a non trivial positive closed current of bidegree $(1,1)$ on X, then its support intersects every positive divisor, hence $r(Supp\, T) = X$. Indeed assume θ is the current of integration along a positive divisor which does not intersect the support of T; since θ is cohomologous to a Kähler form ω, we have $\|T\| = \int_X \omega \wedge T = \int_X \theta \wedge T = 0$, a contradiction.*

2.1 A fundamental lemma

Lemma 2.4 *Let X be a projective algebraic manifold of dimension m. Let L be a positive holomorphic line bundle on X and let φ be a positive continuous metric of L on X. Let s be a holomorphic section of L defined on an open subset V of X and assume $K = \{a \in V \,/\, \|s(a)\|_\varphi = |s(a)|e^{-\varphi(a)} \geq 1\}$ is compact.*

Then K is rationally convex.

Proof. More precisely, $a \in X \setminus K$ being fixed, we are going to construct a global holomorphic section S of L^M (M a large integer to be chosen later) s.t. $S(a) = 0$ and $\{S = 0\} \cap K = \emptyset$.

Let $\chi \in \mathcal{C}_0^\infty(V)$ be s.t. $0 \leq \chi \leq 1$ and $\chi \equiv 1$ in a neighborhood of K, and $a \notin Supp\,\chi$. We consider $v = \overline{\partial}(\chi s^M) = \overline{\partial}\chi.s^M$. This is a smooth $\overline{\partial}-$closed $(0,1)-$form with values in L^M or else a smooth $\overline{\partial}-$closed $(m,1)-$form with values in $L^M \otimes K_X^*$.

Since L is positive, there exists $M_1 \in \mathbb{N}$ and global holomorphic sections $h_1, \ldots, h_m$ of L^{M_1} which form a local coordinate system at a. More precisely, we can find the h_j's s.t. $h_j(a) = 0$ and $\bigcap_{j=1}^m \{h_j = 0\} = \{a\}$. Thus $G_1 = \frac{m}{2}\log[\sum_{j=1}^m |h_j|^2]$ is a singular metric of L^{mM_1} which is smooth in $X \setminus \{a\}$ and admits a logarithmic singularity of coefficient m at the point a.

Fix ω a Kähler metric on X. Since L is positive, there exists $M_2 \in \mathbb{N}$ s.t. $L^{M_2} \otimes K_X^*$ is positive. Taking M_2 large enough, we can even assume the existence of a smooth metric G_2 of $L^{M_2} \otimes K_X^*$ s.t $\Theta_{G_2}(L^{M_2} \otimes K_X^*) := dd^c G_2 \geq \omega$.

We now solve $\overline{\partial}u = v$ on X with L^2-estimates associated to the metric $\psi = G_1 + G_2 + M_3\varphi$ of $L^M \otimes K_X^*$ ($M = mM_1 + M_2 + M_3$), which satisfies $dd^c\psi \geq \omega$. We obtain therefore a smooth section u of L^M st:

$$\int_X |u|^2 e^{-2\psi} dV_\omega \leq \int_X |v|^2 e^{-2\psi} dV_\omega,$$

where dV_ω denotes the Kähler volume element $\frac{1}{m!}\omega^m$.

Since $\chi \equiv 0$ in a neighborhood of a which is the only singularity of the metric ψ, the integral on the right hand-side is obviously convergent. Moreover, $\chi \equiv 1$ hence $\overline{\partial}\chi \equiv 0$ in a neighborhood of K, thus we can fix $\alpha \in]0,1[$ s.t. $|s|e^{-\varphi} \leq \alpha < 1$ on $Supp\,\overline{\partial}\chi$. Hence

$$\int_X |u|^2 e^{-2\psi} dV_\omega \leq C_1 \alpha^{2M_3},$$

where C_1 is a constant independent of M_3.

Since ψ has a logarithmic singularity of coefficient $m = dim_{\mathbb{C}} X$ at the point a, we necessarily have $u(a) = 0$.

Fix $\eta > 0$ s.t $\alpha e^{\eta} < 1$. Fix $r > 0$ s.t. $\chi \equiv 1$ on the pseudo-balls $B(y, 2r)$ (which are the pull-back of euclidean balls via a coordinate chart), $y \in K$, and the oscillation of ψ (which is uniformly continuous on any compact neighborhood of K which avoids a) is smaller than η. Observe that u is holomorphic hence $|u|^2$ is subharmonic on the pseudo-balls $B(y, r)$ $y \in K$, so:

$$\begin{aligned} |u(y)|^2 &\leq C_2 \int_{B(y,r)} |u|^2 dV_\omega \\ &\leq C_2 e^{2M_3(\varphi(y)+\eta)} \int_{B(y,r)} |u|^2 e^{-2M_3\varphi} dV_\omega \\ &\leq C_3 e^{2\psi(y)} e^{2M_3\eta} \int_{B(y,r)} |u|^2 e^{-2\psi} dV_\omega \\ &\leq C_4 e^{2\psi(y)} (\alpha e^{\eta})^{2M_3}, \end{aligned}$$

where $C_4 = C_1.C_3$ is a constant independent of M_3.

Fix $\delta > 0$ s.t. $|s|^{M_1+M_2} e^{-(G_1+G_2)} \geq \delta > 0$ on K and fix M_3 large enough so that $|u|e^{-\psi} \leq \frac{\delta}{2}$.

Now we set $S = \chi.s^M - u$. This is a global holomorphic section of L^M s.t $S(a) = 0$ and $|S|e^{-\psi} \geq \frac{\delta}{2} > 0$ on K, hence $\{S = 0\} \cap K = \emptyset$ (ψ is smooth on K) and we are done. Q.E.D.

Theorem 2.5 *Let T be a positive closed current of bidegree $(1,1)$ on a projective algebraic homogeneous manifold X s.t $[T] = c_1(L)$ for some positive holomorphic line bundle L. Then for every $\varepsilon > 0$, the compact set $K_\varepsilon = \{m \in X \,/\, d(m, Supp\, T) \geq \varepsilon\}$ is rationally convex.*

Proof. Fix $\varepsilon > 0$ and $a \in X \setminus K_\varepsilon$. Using proposition 1.1, we can fix a neighborhood V of K_ε which does not contain a, a sufficiently big integer M, a global continuous metric ψ of L^M and a local holomorphic section h of L^M defined on V s.t $K_\varepsilon \subset F = \{m \in V \,/\, |h(m)|e^{-\psi(m)} \geq 1\} \subset\subset V$. But F is rationally convex by the previous lemma and $a \notin V$ hence we can construct a global holomorphic section S of some power of L s.t. $S(a) = 0$ and $\{S = 0\} \cap K_\varepsilon \subset \{S = 0\} \cap F = \emptyset$. Q.E.D.

Remark 2.6 *If $T = dd^c \max(\Re z_1, c)$ in the example 1.3, $X \setminus Supp\, T$ cannot be exhausted by rationally convex compact sets, since it contains non trivial positive closed currents with compact support (see example 2.3.iii).*

2.2 Rational convexity of totally real submanifolds

Proposition 2.7 *Let X be a projective algebraic manifold equipped with a Kähler metric ω. Let K be a compact subset of X.*

i) For every $a \notin r(K)$, there exists a positive closed current T of bidegree $(1,1)$ on X which admits a continuous potential and s.t. T is smooth and strictly positive at a, T vanishes in a neighborhood of $r(K)$ and moreover $[T] \in H^2(X,\mathbb{Z})$.

ii) For every $\varepsilon > 0$ and every fixed neighborhood V of $r(K)$, we can find a smooth $(1,1)-$form ω_ε which satisfies the following properties:

$a)$ $\omega_\varepsilon \geq \omega$ *in* $X \setminus V$
$b)$ $\omega_\varepsilon \equiv 0$ *in a neighborhood of* $r(K)$
$c)$ $\omega_\varepsilon \geq -\varepsilon.\omega$ *in V*
$d)$ $[\omega_\varepsilon] \in H^2(X,\mathbb{Z})$.

Proof. i) Let $a \in X \setminus r(K)$. There exists a global holomorphic section s of a positive holomorphic line bundle L on X s.t. $s(a) = 0$ and $\{s = 0\} \cap K = \emptyset$, hence $\{s = 0\} \cap r(K) = \emptyset$ since $r(r(K)) = r(K)$. Let G be a smooth metric of L on X s.t. $dd^cG > 0$. Changing s in $\lambda.s$ for some large real positive constant λ if necessary, we can assume $|s|e^{-G} > 1$ on $r(K)$. Consider $\psi = \max(\log|s|, G)$. This is a well defined continuous positive metric of L on X s.t. $\psi \equiv G$ in a neighborhood of a and $\psi \equiv \log|s|$ in a neighborhood of $r(K)$. Therefore $T = dd^c\psi$ satisfies all our requirements since moreover $[T] = [dd^c\psi] = c_1(L) \in H^2(X,\mathbb{Z})$.

ii) Since $X \setminus V$ is compact, we can find a finite number of points a_j s.t. $T = \sum T_{a_j}$ is a $(1,1)-$positive closed current which is strictly positive in $X \setminus V$, vanishes in a neighborhood of $r(K)$ and has integer class. Since moreover T admits a continuous potential, we can use a regularization theorem due to Richberg [Ri 68] to approximate T by smooth forms with small negative part to obtain what we need. Q.E.D.

Recall that a submanifold S of X is totally real if $\forall x \in S$, the real tangent space $T_x^{\mathbb{R}}(S)$ of S at x contains no complex line. We show now that a compact totally real submanifold of X is rationally convex iff it is isotropic for some Hodge form (i.e. a Kähler form whose cohomology class belongs to $H^2(X,\mathbb{Z})$). More precisely, we have the following

Theorem 2.8 *Let S be a smooth compact totally real submanifold of a projective algebraic manifold X. The following are equivalent:*

$i)$ *S is rationally convex.*
$ii)$ *There exists a smooth Hodge form θ for X s.t. $j^*\theta = 0$,*

where $j : S \to X$ denotes the inclusion map.

Proof. $i) \Rightarrow ii)$: Since S is smooth and totally real, there exists a positive function ρ which is smooth and strictly plurisubharmonic in a neighborhood of S and s.t. $S = \rho^{-1}(0)$ and $\nabla\rho = 0$ on S. Indeed S can be defined as the zero set of a finite number of smooth globally defined funtions g_i, then $\rho := \sum g_i^2$ will be strictly psh in a neighborhood of S since S is totally real. Set $S_\delta = \{m \in X \,/\, \rho(m) < \delta\}$ and fix $\delta > 0$ small enough.

Fix $\chi \in \mathcal{C}_0^\infty(S_{2\delta})$ with $0 \leq \chi \leq 1$ and $\chi \equiv 1$ in a neighborhood of $\overline{S_\delta}$ and define $\omega_1 = dd^c(\chi.\rho)$. Fix a Kähler metric ω on X and a positive integer A s.t. $\omega_1 \geq -A.\omega$ on X and $\omega_1 \geq \frac{1}{A}\omega$ on $\overline{S_\delta}$.

We can use proposition 2.7 with $V = S_\delta$, $\varepsilon = \frac{1}{4A^2}$ and the fact that $r(S) = S$ to construct a smooth $(1,1)-$form ω_2 which satisfies:

$$
\begin{aligned}
&a)\, \omega_2 \geq \omega \text{ in } X \setminus S_\delta \\
&b)\, \omega_2 \equiv 0 \text{ in a neighborhood of S} \\
&c)\, \omega_2 \geq -\frac{1}{4A^2}\omega \text{ in } S_\delta \\
&d)\, [\omega_2] \in H^2(X, \mathbb{Z}).
\end{aligned}
$$

Consider now $\theta = \omega_1 + 2A\omega_2$. This is a smooth strictly positive $(1,1)-$form on X s.t. $j^*\theta = j^*\omega_1 = j^*(dd^c\rho) = d(j^*d^c\rho) = 0$ since the gradient of ρ vanishes on S. Furthermore $[\theta] \in H^2(X, \mathbb{Z})$ since $[\omega_1] = 0$, hence θ is the desired Hodge form.

$ii) \Rightarrow i)$:
There exists a positive holomorphic line bundle L on X s.t. $c_1(L) = [\theta]$. We need the following

Lemma 2.9 *There exists a Stein neighborhood V of S and an integer k s.t. $L^k{}_{|V}$ is trivial.*

We show how the lemma implies the theorem.
We can assume $k = 1$, therefore positive metrics of L define psh functions on V. We only need to follow the corresponding proof in [D-S 95], where the psh functions are replaced by positive metrics of the line bundles L^M, which can be viewed as functions on V.

Starting with a smooth strictly positive metric φ of L with $j^*(dd^c\varphi) = 0$, we define a small perturbation $\varphi_\varepsilon = \varphi + \sum_{j=1}^p \varepsilon_j\psi_j$ which is again a strictly positive metric of L with ψ_j smooth functions with compact support in V and such that $M\varphi_\varepsilon$ has periods in $2\pi\mathbb{Z}$ on S. The latter allows us to construct a smooth function h on S with values in $\mathbb{R}$ that we extend locally in V in a function h_s (which is equivalently a smooth section of L^M above V) satisfying

$$
\begin{aligned}
&a)\, \overline{\partial}h_s = 0 \text{ to order s on } S \\
&b)\, M\varphi_\varepsilon - \log|h_s| \text{ vanishes to order 2 on} S.
\end{aligned}
$$

Using lemma 3.3 in [D-S 95] we can modify $M\varphi_\varepsilon$ locally in V and obtain a new strictly positive smooth metric $\tilde{\varphi}$ of L^M which, together with h_s, fulfils the hypotheses of lemma 3.2 in [D-S 95]. As the construction of the holomorphic section h of L^M on V only requires the solution of the $\overline{\partial}-$equation on a Stein neighborhood S_δ of S, and as L is trivial there, we can again use the same construction as in [D-S 95] and then apply our lemma 2.4 to conclude that S is rationally convex.

There remains to prove lemma 2.9.

Proof of lemma 2.9. Let $V = S_\delta$ be a tubular neighborhood of S; then $H^2(V,\mathbb{Z}) \simeq H^2(S,\mathbb{Z})$. Since V is Stein, $Pic(V) \simeq H^2(V,\mathbb{Z})$. But $c_1(L_{|S}) = [j^* dd^c\varphi] = [0]$, hence $c_1(L_{|V}) = [0]$, i.e. the image of the first Chern class of L in $H^2_{dR}(V,\mathbb{R})$ via the morphism induced by the canonical inclusion $\mathbb{Z} \to \mathbb{R}$ is trivial. As we have already explained in the proof of lemma 1.2, this implies that $L^k_{|V}$ is trivial for some integer k. Q.E.D.

3 T-polynomial convexity

There is no intrinsic definition of polynomial convexity on complex manifolds extending the usual notion in $\mathbb{C}^m$. Indeed, $K = \{[1, e^{i\theta}, e^{-i\theta}] \in \mathbb{P}(\mathbb{C}^2)\,/\,0 \le \theta \le 2\pi\}$ is polynomially convex when viewed as a subset of $\mathbb{C}^2 = \mathbb{P}^2(\mathbb{C}) \setminus \{z_0 = 0\}$, but it is not polynomially convex as a subset of the other chart $\mathbb{P}^2(\mathbb{C}) \setminus \{z_1 = 0\} \simeq \mathbb{C}^2$.

However we define a notion of polynomial convexity relative to a fixed positive closed current T of bidegree $(1,1)$ on a complex manifold X. It is an interesting tool to describe the convexity properties of $X \setminus Supp\, T$ (see 3.1) and there is an analogue of the classical Oka principle when T is the current of integration along a positive divisor of a projective algebraic manifold X (see 3.3). The case of Stein manifold will be considered in 5.1.

Definition 3.1 *Let T be a positive closed current of bidegree $(1,1)$ on a complex manifold X, and let K be a compact subset of X.*
We define the $T-$polynomial hull of K by

$$\widehat{K}^T := \left\{ x \in X \;/\; f(x) \le \sup_K f,\ \forall f \in \mathcal{C}_T(X) \text{ s.t. } dd^c f \ge -T \right\},$$

where $\mathcal{C}_T(X)$ denotes the set of functions $f \in L^1(X)$ s.t. $\exp(f+\varphi)$ is continous whenever φ is a local potential of T. Note in particular that any f in $\mathcal{C}_T(X)$ is lower semi-continuous.

The compact K is said to be $T-$polynomially convex when $\widehat{K}^T = K$.

We list a few elementary properties of these hulls:

i) $\widehat{K}^T$ is closed and $\widehat{\widehat{K}^T}^T = \widehat{K}^T$.
ii) $\forall \lambda > 0$, $\widehat{K}^{\lambda T} = \widehat{K}^T$.

iii) If $T = T_1 + T_2$ is a sum of two positive closed current of bidegree $(1,1)$ then $\widehat{K}^T \subset \widehat{K}^{T_1} \cap \widehat{K}^{T_2}$.

iv) When $X = \mathbb{P}^m(\mathbb{C})$, $T = [\{z_0 = 0\}]$ and K is a compact subset of $\mathbb{C}^m = \mathbb{P}^m(\mathbb{C}) \setminus \{z_0 = 0\}$, then $\widehat{K}^T$ is the usual polynomial hull of K in $\mathbb{C}^m$. Indeed, a function $f \in \mathcal{C}_T(X)$ defines a psh log-homogeneous function in $\mathbb{C}^{m+1}$ via $\varphi(z) = f([z]) + \log|z_0|$ hence $\varphi_{|\{z_0=1\}} = f_{|\mathbb{C}^m} \in PSH(\mathbb{C}^m)$ and is s.t. $f(\zeta) \le \log^+|\zeta| + C$; conversely any function $\psi \in PSH(\mathbb{C}^m)$ with log-growth defines a log-homogeneous psh function in $\mathbb{C}^{m+1}$ setting $\varphi(z) = \psi(z_1/z_0, \dots, z_m/z_0) + \log|z_0|$ if $z_0 \neq 0$ and $\varphi(0,\zeta) = \limsup_{z \to (0,\zeta),\, z_0 \neq 0} \varphi(z)$. The function φ corresponds to a function $f \in \mathcal{C}_T(X)$ via $f([z]) = \varphi(z) - \log|z_0|$. Thus $\widehat{K}^T$ equals the hull of K with respect to the psh functions of log-growth in $\mathbb{C}^m$, and it is standard that this hull is exactly the polynomial hull of K (see also the second assertion of proposition 3.2 below).

Proposition 3.2 *Let X be a complex manifold.*

When $[T]$ is equal to the first Chern class $c_1(L)$ of a holomorphic line bundle L on X, then

$$\widehat{K}^T = \left\{ x \in X \ / \ (\psi - \varphi)(x) \le \sup_K (\psi - \varphi),\ \forall \psi \in \mathcal{P}_c(X,L) \right\},$$

where $\mathcal{P}_c(X,L)$ denotes the set of positive metrics ψ of L on X s.t. e^ψ is continuous, and φ is a positive metric of L on X s.t. $dd^c\varphi = T$.

Moreover if we define

$$p_T(K) = \left\{ x \in X \ / \ |h|_{k\varphi}(x) \le \sup_K |h|_{k\varphi},\ \forall k \in \mathbb{N},\ \forall h \in \Gamma(X, L^k) \right\},$$

then $\widehat{K}^T \subset p_T(K)$, with equality if L is positive and X is a projective algebraic homogeneous manifold (resp. a Stein manifold).

Proof. Fix $\mathcal{U}_\alpha$ an open covering of X trivializing L. If $[T] = c_1(L)$, then there exists a positive metric $\varphi = \{\varphi_\alpha\}$ of L on X s.t. $dd^c\varphi = T$, and two such metrics (with respect to this covering) only differ by a pluriharmonic function which is globally well defined on X, thus the definition of the right hand side is independent of the choice of the potential φ of T. Let $\psi = \{\psi_\alpha\}$ be a positive metric of L s.t. e^ψ is continuous, then $f = \psi - \varphi$ is a globally well defined function on X which lies in $\mathcal{C}_T(X)$ and s.t. $dd^c f \ge -dd^c\varphi = -T$.

Conversely, if $f \in \mathcal{C}_T(X)$ is s.t. $dd^c f \ge -T$, then $\psi = \{f + \varphi_\alpha\}$ is a positive metric of L on X s.t. e^ψ is continuous; the first assertion follows.

Let $k \in \mathbb{N}$ and $h \in \Gamma(X, L^k)$, then $\psi = \frac{1}{k}\log|h|$ defines a positive metric of L on X s.t. e^ψ is continuous, hence $\widehat{K}^T \subset p_T(K)$.

Conversely let $a \in X \setminus \widehat{K}^T$; there exists $\psi \in \mathcal{P}_c(X, L)$ s.t. $(\psi - \varphi)(a) > \sup_K(\psi - \varphi)$. If X is homogeneous we can assume ψ is smooth (otherwise replace ψ by its regularized metric ψ^ε for $\varepsilon > 0$ small enough). If moreover L is positive, we can assume ψ has strictly positive curvature, replacing if necessary ψ by $(1 - \eta)\psi + \eta G$, where G is a smooth metric of L s.t. $dd^c G > 0$ and $\eta > 0$ is small enough.

Since $dd^c\psi(a) > 0$, there exists a holomorphic polynomial P and a positive constant c s.t. $\psi_\alpha(x) - \Re(P)(x) \geq cd(x, a)^2$ in a neighborhood of $a \in \mathcal{U}_\alpha$. Let χ be a positive test function defined in this neighborhood, s.t. $\chi \equiv 1$ in a smaller neighborhood of a and $0 \leq \chi \leq 1$. If X is projective algebraic we can solve $\overline{\partial} v = \overline{\partial}(\chi e^{NP})$ with L^2-estimates associated to the weight $N\psi$ and construct, in the same vein as what has been done in the proof of theorem 1.6, a holomorphic section h of L^N on X s.t. $|h|e^{-N\psi} \leq 1$ on X and $|h(a)|e^{-N\psi(a)} \geq 1/2$. Thus for a choice of N large enough, we get

$$(\frac{1}{N} \log |h| - \varphi)(a) > \sup_K(\frac{1}{N} \log |h| - \varphi),$$

hence $a \in X \setminus p_T(K)$. The Stein case will be considered in proposition 5.4. Q.E.D.

3.1 Steinness of $X \setminus Supp\, T$

Definition 3.3 *T is said to satisfy condition (C) if*

$$\forall K \subset\subset X \setminus Supp\, T, \quad \widehat{K}^T \subset\subset X \setminus Supp\, T.$$

Example 3.4 *i) When X is homogeneous, the regularization process insures that every positive closed current of bidegree $(1, 1)$ s.t. $[T] = c_1(L)$ satisfies condition (C). Indeed $T = dd^c\varphi$ for some positive metric of L, and the regularized metrics φ^ε of φ (see Appendix) satisfy $\varphi^\varepsilon - \varphi \equiv 0$ in a neighborhood of K if $\varepsilon > 0$ is small enough whereas $\varphi^\varepsilon - \varphi > 0$ in a neighborhood of $Supp\, T$.*

ii) When $T = [\{s = 0\}]$ where $s \in \Gamma(X, L)$ and L is semi-positive, then T satisfies condition (C). Indeed L admits a positive continuous metric ψ on X, hence ψ is locally bounded on $Supp\, T$ whereas $\log |s| \equiv -\infty$ on $Supp\, T$.

iii) If $\pi : \tilde{X} \to X$ is the blow-up at a point p of a compact complex manifold X (dim$_{\mathbb{C}}(X) \geq 2$), and if T is the current of integration along the exceptional divisor $E = \pi^{-1}(p)$, then $\forall K \subset\subset \tilde{X} \setminus Supp\, T$, $\widehat{K}^T = \tilde{X}$, since every function $f \in \mathcal{C}_T(\tilde{X})$ s.t. $dd^c f \geq -T$ defines a plurisubharmonic function in $\tilde{X} \setminus E \simeq X \setminus \{p\}$, hence is constant; thus $T = [E]$ does not satisfy condition (C).

Lemma 3.5 *If $K = \widehat{K}^T$, then for any open neighborhood V of K, there exists a non negative function $f \in L^1(X)$ s.t. $f + \varphi$ is upper semi-continuous whenever φ is a local potential of T and moreover $dd^c f \geq -T$ on X with $f \equiv 0$ on K and $f > 0$ in $X \setminus V$.*

Proof. Let $a \in X \setminus K = X \setminus \widehat{K}^T$, then there exists $f_a \in \mathcal{C}_T(X)$ s.t. $dd^c f_a \geq -T$ and $f_a(a) > 0 \geq \sup_K f_a$. Since f is lower semi-continuous at a, $f > 0$ in a small ball $B(a, \varepsilon_a)$.

We can consider $f_a^+ = \max(f_a, 0)$. Then $f_a^+ \equiv 0$ on K and $f_a^+ + \varphi$ is upper semi-continuous (u.s.c.) as a maximum of two u.s.c. functions. Moreover $f_a^+ = \max(f_a + \varphi, \varphi) - \varphi$ hence $dd^c f_a^+ \geq -T$ on X.

Now we can cover $X \setminus V$ by a finite number of balls $B(a_i, \varepsilon_{a_i})$ and consider $f = \frac{1}{p} \sum_{i=1}^p f_{a_i}^+$ to conclude. Q.E.D.

Corollary 3.6 *If T satisfies condition (C), then $X \setminus \mathit{Supp}\, T$ admits a psh exhaustion function.*

Proof. By hypothesis, we can exhaust $X \setminus \mathit{Supp}\, T$ by an increasing sequence of compact sets K_j that satisfy $\widehat{K_j}^T = K_j$ and $K_j \subset (K_{j+1})^\circ$.

By the previous lemma, we can find for each j, a non negative function f_j which is psh in $X \setminus \mathit{Supp}\, T$, identically 0 on K_j and positive outside $(K_{j+1})^\circ$. Multiplying by some large constant, we can even assume $f_j \geq 2^j$ on $K_{j+2} \setminus (K_{j+1})^\circ$. Therefore $f = \sum_{j \geq 0} f_j$ is a psh exhaustion function for $X \setminus \mathit{Supp}\, T$ (the sum is finite on each compact set). Q.E.D.

Theorem 3.7 *Let Ω be a complex manifold which admits a psh exhaustion function. Then Ω is Stein iff there is no non trivial positive dd^c-closed current of bidimension $(1,1)$ with compact support in Ω.*

Proof. If Ω is Stein, it admits a smooth strictly psh exhaustion function φ. Let S be a positive current of bidimension $(1,1,)$ with compact support in Ω and s.t. $dd^c S = 0$. Then $dd^c \varphi \wedge S$ is a well defined positive measure which measures the mass of S. Stokes theorem gives

$$\int_\Omega dd^c \varphi \wedge S = \int_\Omega \varphi dd^c S = 0,$$

hence $S \equiv 0$.

Conversely, let f be a psh exhaustion function of Ω, and define $\Omega_j = \{x \in \Omega \,/\, f(x) < j\}$. We set

$$\mathcal{H} = \{S \text{ current of bidimension } (1,1) \text{ on } \Omega \text{ s.t. } dd^c S = 0\},$$

and

$$\mathcal{K}_j = \{S \quad \text{positive current of bidim. } (1,1) \text{ on } \Omega \text{ s.t. } \|S\| = 1 \\ \text{and } \mathit{Supp}\, S \subset \overline{\Omega_j}\}.$$

Then $\mathcal{H}$ is a hyperplane of the set $\mathcal{T}_{(1,1)}(\Omega)$ of currents of bidimension $(1,1)$ on Ω and $\mathcal{K}_j$ is a convex compact subset of $\mathcal{T}_{(1,1)}(\Omega)$.

We assume that $\forall j \in \mathbb{N}$, $\mathcal{H} \cap \mathcal{K}_j = \emptyset$; the theorem of Hahn-Banach insures the existence of a linear functional Φ_j on $\mathcal{T}_{(1,1)}(\Omega)$ s.t. $\Phi_j(\mathcal{H}) = 0$ and $\Phi_j(\mathcal{K}_j) \geq c_j > 0$. This functional is defined by a smooth $(1,1)$-form ω_j, s.t. $\Phi_j(S) = \int_X S \wedge \omega_j$. Since it belongs to $\mathcal{H}^\perp = \overline{\{dd^c g\}}$, we can write $\omega_j = \lim dd^c g_k$; thus for k_j large enough, $\varphi_j = g_{k_j}$ is a smooth function on X s.t. $\int_\Omega S \wedge dd^c \varphi_j > 0$ for every $S \in \mathcal{K}_j$, since $\mathcal{K}_j$ is compact, hence φ_j is strictly psh in a neighborhood of $\overline{\Omega_j}$.

Without loss of generality we can assume $-\frac{3}{4} \leq \varphi_j \leq -\frac{1}{2}$ on $\overline{\Omega_j}$ (otherwise replace φ_j by $A_j \varphi_j + B_j$ for some properly chosen constants A_j and B_j). Consider now

$$\psi_j = \begin{cases} \varphi_j & \text{in } \Omega_{j-1} \\ \max(\varphi_j, f - j) & \text{in } \Omega_j \setminus \Omega_{j-1} \\ f - j & \text{in } \Omega \setminus \Omega_j \end{cases}$$

This is clearly a psh function in Ω which is strictly psh in Ω_{j-1} and moreover $\psi_j \geq -\frac{3}{4}$ in Ω and $\psi_j \leq 0$ in Ω_j.

We set finally $\psi = f + \sum_{j \geq 1} 2^{-j} \psi_j$. Then ψ is an exhaustion function for Ω which is strictly psh; it follows from a result of Fornaess-Narasimhan ([F-N 80]) that Ω is Stein. Q.E.D.

Theorem 3.8 *Let T be a positive closed current of bidegree $(1,1)$ on a compact Kähler manifold X. Assume T is cohomologous to a Kähler form and satisfies condition (C), then $X \setminus Supp\, T$ is Stein.*

Proof. Let ω be a Kähler form cohomologous to T. Since T is real and X is Kähler, $T - \omega$ is in fact dd^c-exact; there exists a distribution f on X s.t. $T = \omega - dd^c f$. Note that f is a smooth strictly psh function in $X \setminus Supp\, T$.

Let S be a positive dd^c-closed current of bidimension $(1,1)$ with compact support in $X \setminus Supp\, T$. Stokes theorem gives

$$\int_X \omega \wedge S = \int_{X \setminus Supp\, T} \omega \wedge S = \int_{X \setminus Supp\, T} dd^c f \wedge S = \int_{X \setminus Supp\, T} f dd^c S = 0,$$

hence $S \equiv 0$.

Now T satisfies condition (C) thus $X \setminus Supp\, T$ admits a psh exhaustion function (corollary 3.6); therefore $X \setminus Supp\, T$ is Stein by the previous theorem. Q.E.D.

Remark 3.9 *This can be seen as a generalization of the standard result: $X \setminus \{s = 0\}$ is Stein, when s is a holomorphic section of some positive holomorphic line bundle on X.*

3.2 Oka principle

In this section we want to investigate the case where T is the current of integration along a positive divisor of a projective algebraic manifold X.

Let L be a positive holomorphic line bundle on X, $s \in \Gamma(X, L)$ and $T = [\{s = 0\}]$. By the Kodaira embedding theorem, there exists $k \in \mathbb{N}$ and a basis $(s_0 = s^k, s_1, \ldots, s_N)$ of $\Gamma(X, L^k)$ s.t. the map

$$\begin{aligned} \Phi : X &\to \mathbb{P}^N(\mathbb{C}) \\ x &\mapsto [s_0(x), \ldots, s_N(x)] \end{aligned}$$

defines a holomorphic embedding of X onto a subvariety V of $\mathbb{P}^N(\mathbb{C})$ with $L = \Phi^*(\mathcal{O}(1)_{|V})$. If K is a compact subset of $X \setminus Supp\, T$, $\Phi(K)$ thus is a compact subset of $V \setminus \{z_0 = 0\} \subset \mathbb{C}^N$ and one easily checks that

$$\Phi^{-1}(\widehat{\Phi(K)}) = p_T(K) \subset\subset X \setminus Supp\, T,$$

where $\widehat{\Phi(K)}$ denotes the usual polynomial hull of $\Phi(K)$ in $\mathbb{C}^N$. Indeed every holomorphic section of L^p defines a holomorphic section of $\mathcal{O}(p)_{|V}$ that extends to a holomorphic section of $\mathcal{O}(p)$ on $\mathbb{P}^N(\mathbb{C})$ and conversely.

If f is a function holomorphic in a neighborhood of $p_T(K)$ then $F = f \circ \Phi^{-1}$ is a function holomorphic in a neighborhood of $\widehat{\Phi(K)}$ in V that extends to a function holomorphic in a neighborhood of $\widehat{\Phi(K)}$ in $\mathbb{C}^N$. The classical theorem of Oka-Weil asserts that F is a uniform limit on $\widehat{\Phi(K)}$ of polynomials in z_i/z_0, $1 \leq i \leq N$. Therefore we have the following

Theorem 3.10 (Oka-Weil) *Let $T = [\{s = 0\}]$ with $s \in \Gamma(X, L)$, L positive. Let K be a compact subset of $X \setminus Supp\, T$. Then every function holomorphic in a neighborhood of $p_T(K)$ is a uniform limit on $p_T(K)$ of polynomials in s_i/s^k where $s_i \in \Gamma(X, L)$ (and k is an integer s.t. L^k is very ample), i.e. of meromorphic functions of the type h/s^p where $h \in \Gamma(X, L^p)$.*

Definition 3.11 *Let $L \in Pic(X)$. We say that $(H_t)_{t \geq t_0}$ is a continuous $L-$family of algebraic hypersurfaces if the following holds:*

i) $\forall t \geq t_0$, there exists $d_t \in \mathbb{N}$ and $s_t \in \Gamma(X, L^{d_t})$ s.t $H_t = \{s_t = 0\}$;

ii) $t \mapsto d_t$ is bounded on each compact set;

iii) $(t, x) \mapsto s_t(x)$ is continuous on $[t_0, +\infty[\times X$.

Moreover the family is said to join x to H_∞ avoiding a compact set K if

i) $x \in H_{t_0}$ and $\forall t \geq t_0$, $H_t \cap K = \emptyset$;

ii) $\sup_{x \in H_t} d(x, H_\infty) \to 0$ as $t \to +\infty$.

Definition 3.12 *Let K be a compact subset of $X \setminus \{s = 0\}$, where $s \in \Gamma(X, L)$, L positive, and set $T = [\{s = 0\}]$. The Oka-hull $O_T(K)$ of K relative to T is defined by saying that a point $x \in X$ lies in $X \setminus O_T(K)$ iff there exists a continuous $L-$family of algebraic hypersurfaces (H_t) joining x to $H_\infty = \{s = 0\}$ avoiding K.*

Theorem 3.13 (Oka principle) *Let* $s \in \Gamma(X, L^d)$, *whith* $L \in Pic(X)$ *positive and set* $T = [\{s = 0\}]$. *Then*

$$\forall K \subset\subset X \setminus Supp\, T, \;\; p_T(K) = O_T(K).$$

Proof. Let $x \in X \setminus p_T(K)$; there exists $k \in \mathbb{N}$ and $h \in \Gamma(X, L^k)$ s.t. $h/s^k(x) = 1 > \sup_K |h/s^k|$. Therefore $(H_t := \{h - ts^k = 0\})_{t \geq 1}$ is a continuous $L-$family of algebraic hypersurfaces which joins x to H_∞ avoiding K, hence $x \notin O_T(K)$.

Conversely, let $x \in p_T(K)$ and assume there exists a continuous L-family of algebraic hypersurfaces $(H_t = \{s_t = 0\})_{t \geq 1}$ that joins x to $H_\infty = \{s = 0\}$ avoiding K.

Since $p_T(K)$ is compact, there exists $r \geq 1$ s.t. $H_r \cap p_T(K) \neq \emptyset$ and $\forall t > r, \; H_t \cap p_T(K) = \emptyset$.

Since (H_t) avoids K, the function $(t, x) \mapsto f_t(x) = \frac{s^{d_t}}{s_t^d}(x)$ is bounded on $[r, r+1] \times K$.

Now let $y \in p_T(K) \cap H_r$; $|f_r(y)| = +\infty$ hence by continuity of H_t, $|f_t(y)| \to +\infty$ as $t \to r^+$. Thus there exists $t > r$ s.t. $|f_t(y)| > \sup_K |f_t|$. Since $H_t \cap p_T(K) = \emptyset$, f_t is holomorphic in a neighborhood of $p_T(K)$ hence we can approximate it uniformly on $p_T(K)$ by functions of the type h/s^p with $h \in \Gamma(X, L^p)$, thus y cannot lie in $p_T(K)$, a contradiction. Q.E.D.

4 Approximation of currents on projective algebraic manifolds

In this section we wish to extend theorem 1.6 to the case of non homogeneous projective algebraic manifolds. We need to make an extra assumption (T satisfies condition (C)) which is always satisfied in the homogeneous case; on the other hand we obtain a control on the Lelong numbers of the approximants (such a control was obtained by Demailly in [De 93]) and this gives a refinement of theorem 1.6 in the homogeneous case (see corollary 4.3). Recall that by a theorem of Siu [Siu 74], the level sets $E_c(T) = \{x \in X \,/\, \nu(T, x) \geq c\}$ of Lelong numbers of a positive closed current T on X are proper closed analytic subsets of X for each $c > 0$.

Theorem 4.1 *Let* T *be a positive closed current of bidegree* $(1,1)$ *on a projective algebraic manifold* X. *Assume* $[\lambda T] = c_1(L)$ *for some holomorphic line bundle* L *which we assume is positive. Assume* $T = [H] + R$, *where* $H = \sum_{j=1}^p \lambda_j [Z_j]$ *(*$\forall j$, λ_j *is a positive constant and* Z_j *is an irreducible algebraic hypersurface of* X*) and* R *is a positive closed current of bidegree* $(1,1)$ *on* X *s.t. the level sets of Lelong numbers of* R, $E_c(R) = \{x \in X \,/\, \nu(R, x) \geq c\}$, *are of codimension greater or equal than* 2. *Assume moreover that* T *satisfies condition* (C).
Then there exists $N_j \in \mathbb{N}$ *and* $s_j \in \Gamma(X, L^{N_j})$ *s.t.*

i) $T_j = \frac{1}{N_j}[\{s_j = 0\}] \longrightarrow T$ *in the weak sense of currents;*
ii) $\{s_j = 0\} \longrightarrow Supp\, T$ *in the Hausdorff metric;*
iii) $\forall x \in X$, $\nu(T_j, x) \longrightarrow \nu(T, x)$.

We first need a proposition:

Proposition 4.2 *Under the hypotheses of the theorem (with* $\lambda = 1$*), let* φ *be a positive metric of* L *which is a potential for* T*, let* K *be a compact subset of* $X \setminus Supp\, T$ *s.t.* $K = \widehat{K}^T$ *and fix* ω *a Kähler form on* X.

Then for every open set V *s.t.* $K \subset V \subset\subset X \setminus Supp\, T$ *and every* $\delta > 0$*, we can find* $M \in \mathbb{N}$ *and construct a positive metric* ψ *of* L^M *on* X *and a section* $h \in \Gamma(V, L^M)$ *s.t.*

i) $K \subset \{m \in V \,/\, |h|_\psi \geq 1\} = \{m \in V \,/\, |h|_\psi \equiv 1\} \subset\subset V$,
ii) $\|\psi/M - \varphi\|_{L^\infty(\overline{V})} \leq \delta$ *and* $\|\psi/m - \varphi\|_{L^1(X)} \leq \delta$,
iii) $\sup_{x\in X} |\nu(\psi/M, x) - \nu(\varphi, x)| \leq \delta$,
iv) $dd^c\psi \geq \varepsilon.\omega$ *in a neighborhood of* $Supp\, T$ *for some constant* $\varepsilon > 0$,
v) ψ *is continuous in* $X \setminus Supp\, T$ *and smooth on a dense subset of* $X \setminus E_{c_0}(T)$ *for some* $c_0 > 0$.

Proof. Following Demailly, we set $\varphi_s = \frac{1}{s}\sup_{1\leq j\leq N}[\log|f_j|]$ where $(f_1, \ldots, f_N)$ is an orthonormal basis of sections of $\Gamma(X, L^s)$ with finite L^2-norm $\int_X \|f\|^2_{s\varphi} dV_\omega$. It is proved in [De 93] (proposition 9.1) that:

a) $\|\varphi_s - \varphi\|_{L^1(X)} \to 0$ as $s \to +\infty$ and the convergence is uniform on compact subsets of $X \setminus \{x \in X \,/\, \varphi$ is not continuous at $x\}$, hence in particular on compact subsets of $X \setminus Supp\, T$;

b) $\nu(T, x) - m/s \leq \nu(T_s, x) \leq \nu(T, x)$, $\forall x \in X$, where $m = \dim_{\mathbb{C}} X$ and $T_s = dd^c\varphi_s$.

Clearly $E^+(T_s) := \{x \in X \,/\, \nu(T_s, x) > 0\}$ is equal to $E_{1/s}(T_s) = \{x \in X \,/\, \nu(T_s, x) \geq 1/s\} \subset E_{1/s}(T)$ and φ_s is smooth on a dense subset of $X \setminus E^+(T_s)$.

Let U be a relatively compact open neighborhood of ∂V in $X \setminus Supp\, T$ s.t. $K = \widehat{K}^T \subset\subset V \setminus \overline{U}$. Let $x \in \overline{U}$. There exists $\psi_x \in \mathcal{P}_c(X, L)$ and $\delta_x > 0$ s.t. $(\psi_x - \varphi)(x) \geq \delta_x > 0 > -\delta_x \geq \sup_K(\psi_x - \varphi)$. Since $(\psi_x - \varphi)$ is lower semi-continuous (l.s.c.) at x, $(\psi_x - \varphi) > \delta_x$ in a small ball $B(x, \varepsilon_x)$. We can cover $\overline{U}$ by a finite number of balls $B(x_i, \varepsilon_i)$ and consider $\psi_1, \ldots, \psi_p$ the corresponding metrics.

We set $\delta' = \min(\delta, \delta_1/8, \ldots, \delta_p/8)$ and we fix s large enough so that $\|\varphi_s - \varphi\|_{L^\infty(\overline{V \cap U})} < \delta'$ and $\|\varphi_s - \varphi\|_{L^1(X)} < \delta'$. Thus $\forall i = 1, \ldots, p$, $(\psi_i - \varphi_s) \geq 4\delta' > 0$ in $B(x_i, \varepsilon_i)$ and $\sup_K(\psi_i - \varphi_s) \leq -4\delta' < 0$. Therefore

$$f_s := \frac{1}{p}\sum_{i-1}^{p} \max(\psi_i - \varphi_s, 0)$$

is a non negative function in $L^1(X)$ s.t. $f_s + \varphi_s$ is u.s.c. and which satisfies $dd^c f_s \geq -T_s$ on X with $f_s \equiv 0$ on $K = \widehat{K}^T$ and $f_s \geq 4\delta' > 0$ in U.

Since T is cohomologous to a Kähler form (L is positive) and satisfies condition (C), $X \setminus \mathit{Supp}\, T$ is Stein by theorem 3.8. We can therefore find, as in the proof of proposition 1.1, a relatively compact Stein open subset W of $X \setminus \mathit{Supp}\, T$ that contains $\overline{V}$, and construct integers k and M_1, a small perturbation θ_λ of $k\varphi$ in W and a holomorphic function h in W s.t.

a) $L^k_{|W}$ is trivial;

b) $\|\frac{1}{k}\theta_\lambda - \varphi\|_{L^\infty(\overline{V})} < \delta'$;

c) $M_1\theta_\lambda = \log|h|$.

Now let G be a smooth metric of L s.t. $dd^c G > 0$ on X and consider

$$\psi = \begin{cases} \max\left(M_1.\theta_\lambda; M[(1-\eta)(f_s+\varphi_s) + \eta G - 2\delta']\right) & \text{in } V \\ M[(1-\eta)(f_s+\varphi_s) + \eta G - 2\delta'] & \text{on } X \setminus V, \end{cases}$$

where $M = k.M_1$. Then for a choice of $\eta > 0$ small enough and s large enough, ψ is a positive metric of L^M which satisfies all our requirements. Q.E.D.

Proof of theorem 4.1. Replacing T by T/λ_j if necessary, where (λ_j) is a sequence of positive rational numbers converging to λ, we can assume $\lambda = 1$. Let K_n be a sequence of compact subsets of $X \setminus \mathit{Supp}\, T$ that exhausts $X \setminus \mathit{Supp}\, T$ and s.t. $\widehat{K_n}^T = K_n$. We fix $\delta_n > 0$ a sequence converging towards 0 and open neighborhoods V_n of K_n that are relatively compact subsets of $X \setminus \mathit{Supp}\, T$. Using proposition 4.2, we construct integers M_n, positive metrics ψ_n of L^{M_n} on X and holomorphic sections h_n of L^{M_n} in V_n with the prescribed properties.

Fix (a_j) a dense sequence of points in $\mathit{Supp}\, T$, s.t. $\forall n \in \mathbb{N}$, $a_1, \dots, a_n$ belong to $\mathit{Supp}\, T \setminus E_{c_n}(T)$ where (c_n) is a sequence of positive numbers converging to 0 s.t. ψ_n is smooth and $dd^c\psi_n > 0$ at the points $a_1, \dots, a_n$ (see iv) and v) of proposition 4.2).

Let (F_n) be an increasing sequence of compact subsets of $X \setminus E^+(T)$ s.t. $\bigcup F_n = X \setminus E^+(T)$ and $K_n \cup \{a_1, \dots, a_n\} \subset F_n \subset\subset X \setminus E_{c_n}(T)$. We are going to construct for each n an integer N_n and a section $S_n \in \Gamma(X, L^{N_n.M_n})$ s.t.

$$\begin{aligned} &|S_n|e^{-N_n\psi_n} \leq 1 && \text{on } F_n \\ &|S_n|e^{-N_n\psi_n} \geq 1/2 && \text{on } K_n \cup \{a_1, \dots, a_n\} \\ &\nu\left(\tfrac{1}{N_n}\log|S_n|, x\right) \\ &\quad \geq (1 - 1/\sqrt{N_n})\nu(dd^c\psi_n, x) - 1/N_n, \ \forall x \in E_{c_n}(dd^c\psi_n) \\ &\textstyle\int_X |S_n|e^{-N_n\psi_n} dV_\omega \leq C, \end{aligned}$$

where C is a positive constant independent of n in the last inequality.

The first two inequalities, together with Lelong-Poincaré equation and ii) of proposition 4.2, imply that $T_n = dd^c(\frac{1}{N_n.M_n}\log|S_n|)$ converges weakly towards T in $X \setminus E^+(T)$. Since $T = [H] + R$ with $\mathrm{codim}_{\mathbb{C}} E_c(R) \geq 2$, $\forall c > 0$, the Hausdorff dimension of $E^+(R) = E^+(T) \setminus \cup_{j=1}^p Z_j$ is less or equal than $2m-4$ hence T_n actually converges towards T on $X \setminus \cup_{j=1}^p Z_j$ (see e.g. [F-S 95]).

One easily checks that $\forall x \in X \setminus \cup Z_j$, $\limsup \nu(T_n, x) \leq \nu(T, x)$ since $T_n \to T$ in the weak sense of currents. Therefore the third inequality together with iii) of proposition 4.2 insures that $\forall x \in E^+(T) \setminus \cup Z_j$, $\nu(T_n, x) \to \nu(T, x)$ hence $\forall x \in X \setminus \cup Z_j$, $\nu(T_n, x) \to \nu(T, x)$. Now $(\|T_n\|)$ is bounded by the last inequality and ii) of proposition 4.2, thus (T_n) admits a subsequence that converges weakly towards a positive closed current T' of bidegree $(1,1)$ on X. Clearly $T' \equiv T$ on $X \setminus \cup Z_j$ and $T' \geq T$ on $\cup Z_j$ by the third inequality, therefore $T' \equiv T$ on X since X is compact Kähler and $[T'] = [T] = c_1(L)$; thus T_n converges weakly towards T on X and $\nu(T_n, x) \to \nu(T, x)$, $\forall x \in X$.

Finally since $|S_n| > 0$ on K_n and $T_n \to T$, $\{S_n = 0\}$ converges towards $Supp\, T$ in the Hausdorff metric.

From now on, n is fixed, and we will not mention the subscript. We proceed as in the proof of theorem 1.6 and construct, using i),iv),v) of proposition 4.2, a smooth $(m,0)-$form $u = \chi.h^N + \sum_{i=1}^n \chi_i e^{N.P_i}$ with values in $L^{N.M} \otimes K_X^*$ s.t. $|u|e^{-N.\psi} = 1$ on $K \cup \{a_1, \ldots, a_n\}$ and $|u|e^{-N.\psi} < 1$ outside a neighborhood of this set. We solve $\overline{\partial} v = \overline{\partial} u$ on X with L^2-estimates associated to a weight $\theta = N_1\psi + G$ and get an estimate

$$\int_X |v|^2 e^{-2\theta} dV_\omega \leq \frac{1}{\varepsilon}\int_X |\overline{\partial} u|^2 e^{-2\theta} dV_\omega \leq C_1 a^{2N_1}.$$

We use now the fact that ψ is continuous on $X \setminus E_{c_n}(T)$ hence uniformly continuous on a compact neighborhood of F which is relatively compact in $X \setminus E_{c_n}(T)$. We obtain in the same vein as what has been done in the proof of theorem 1.6 a uniform estimate for v:

$$|v(x)|^2 \leq C_2 (e^\eta a)^{2N_1} e^{2N\psi(x)},\ \forall x \in F,$$

where $e^\eta a < 1$ and C_2 is a constant independent of N_1. We choose N_1 large enough so that $|v|e^{-N\psi} \leq \frac{1}{3}$ on F and we set $S = \frac{3}{4}(u - v)$. Thus S satisfies the first two inequalities.

To get the third one, observe that $u \equiv 0$ in a neighborhood of any $x \in E_{c_n}(dd^c\psi)$. Thus v is holomorphic there and the convergence of $\int_X |v|^2 e^{-2\theta} dV_\omega$ forces v to vanish at x at an order greater or equal than $N_1\nu(dd^c\psi, x) - 1$. Therefore

$$\nu(\frac{1}{N}\log|S|, x) \geq \frac{N_1}{N}\nu(dd^c\psi, x) - \frac{1}{N} \geq (1 - \frac{1}{\sqrt{N}})\nu(dd^c\psi, x) - \frac{1}{N},$$

for a choice of N_1 large enough, since $N = N_1 + N_2$ where N_2 is a fixed integer.

Finally we observe that $\int_X |v|^2 e^{-2\theta} dV_\omega \le C_1$ hence

$$\int_X |v|^2 e^{-2N\psi} dV_\omega \le C_2 \ ;$$

since $\int_X |u|^2 e^{-2N\psi} dV_\omega \le C_3$ we obtain

$$\int_X |S| e^{-N\psi} dV_\omega \le C_4 \int_X |S|^2 e^{-2N\psi} dV_\omega \le C_5 \ ,$$

where all the constants involved are independent of N_1. Q.E.D.

Let T be a positive closed current of bidegree $(1,1)$ on a projective algebraic manifold X. By a theorem of Siu [Siu 74], we can decompose T as

$$T = \sum_{j\ge 1} \lambda_j [Z_j] + R,$$

where each Z_j is an irreducible analytic subset of X of pure codimension 1, the λ_j's are positive constants and R is a positive closed current of bidegree $(1,1)$ on X s.t. $\forall c > 0$, $E_c(R) = \{x \in X \,/\, \nu(R,x) \ge c\}$ is a closed analytic subset of X of codimension greater or equal than 2.

When X is homogeneous we can approximate T by currents

$$T_n = \sum_{j=1}^{n} \lambda_j [Z_j] + R + \sum_{j=n+1}^{+\infty} \lambda_j \omega_j^{\varepsilon_n}, = [H_n] + R_n$$

where ω_j^{ε} denotes the regularization of $[Z_j]$ and we choose a sequence $\varepsilon_n \to 0$. Since ω_j^{ε} is cohomologous to $[Z_j]$, T_n is cohomologous to T; since $\lambda_j \to 0$ as $j \to +\infty$, $T_n \to T$ in the weak sense of currents with convergence of the Lelong numbers and convergence of the supports in the Hausdorff metric ($Supp\, \omega_j^{\varepsilon_n} \to Z_j$ as $n \to +\infty$). Thus it is sufficient to approximate each T_n in the sense of theorem 4.1 to get a similar approximation for T.

Now since X is homogeneous each T_n satisfies condition (C) and since $\omega_j^{\varepsilon_n}$ is smooth, $\mathrm{codim}_{\mathbb{C}} E_c(R_n) = \mathrm{codim}_{\mathbb{C}} E_c(R) \ge 2$; therefore we have the following refinement of theorem 1.6:

Corollary 4.3 *Let X be a projective algebraic homogeneous manifold and let T be a positive closed current of bidegree $(1,1)$ on X s.t. $[T] = c_1(L)$ for some positive holomorphic line bundle L on X. Then there exists $N_j \in \mathbb{N}$ and $s_j \in \Gamma(X, L^{N_j})$ s.t.*

i) $T_j = \frac{1}{N_j}[\{s_j = 0\}] \longrightarrow T$ *in the weak sense of currents;*

ii) $\{s_j = 0\} \longrightarrow Supp\, T$ *in the Hausdorff metric;*

iii) $\forall x \in X$, $\nu(T_j, x) \longrightarrow \nu(T, x)$.

When does a positive closed current of bidegree $(1,1)$ satisfy condition (C) ? When $\pi : \tilde{X} \to X$ is the blow up at a point p of a compact manifold X, we have seen previously (Example 3.4.iii) that the current of integration along the exceptional divisor does not satisfy condition (C). One might hope that a stronger positivity assumption on the line bundle will imply that T satisfies condition (C). However in [D-P-S 94], the authors give an example of a line bundle L on a ruled surface X that is numerically effective (i.e. $L.\mathcal{C} \geq 0$, for every curve $\mathcal{C}$ of X) and which only admits one positive (singular) metric φ (up to additive constants); thus $T = dd^c\varphi$ does not satisfy condition (C). We briefly recall their construction:

Example 4.4 *Let $\tau \in \mathbb{C}$ s.t. $\Im(\tau) > 0$, and consider the manifold X defined as the quotient of $\mathbb{C} \times \mathbb{P}^1$ by the equivalence relation*

$$(z', [w']) \sim (z, [w]) \text{ iff } \exists (a,b) \in \mathbb{Z}^2 \text{ s.t. } \quad z' = z + a + b\tau \ \& \\ [w'] = [w_0, w_1 + bw_0]$$

We denote by π the canonical projection of $\mathbb{C} \times \mathbb{P}^1$ onto X and by p_1 the canonical projection of $\mathbb{C}$ onto the elliptic curve $E = \mathbb{C}/\mathbb{Z}[\tau]$. The mapping

$$\begin{aligned} p : X &\longrightarrow \mathbb{C}/\mathbb{Z}[\tau] \\ \pi(z, [w]) &\longmapsto \ p_1(z) \end{aligned}$$

expresses X as a ruled surface over E. We denote by E_∞ the elliptic curve at infinity $\pi(\{(z,[0,1]) \in \mathbb{C} \times \mathbb{P}^1 \,/\, z \in \mathbb{C}\}) \simeq E$.

It can be shown (see [D-P-S 94]) that the line bundle L_∞ corresponding to the divisor E_∞ is nef and only admits one positive metric (up to additive constants), hence $T_\infty = [E_\infty]$ does not satisfy condition (C) since $\forall K \subset\subset X \setminus Supp\, T_\infty$, $\widehat{K}^{T_\infty} = X$. Moreover $\Omega_\infty = X \setminus Supp\, T_\infty$ is Stein since the function $f(z,[w]) = (\Im w)^2 + (\Re w - \Im z / \Im \tau)^2$ is smooth, well defined in Ω_∞ and easily seen to be a strictly psh exhaustion function for Ω_∞; therefore condition (C) is not necessary for $X \setminus Supp\, T$ to admits a psh exhaustion function.

Question 4.5 *Let T be a positive closed current of bidegree $(1,1)$ on a projective algebraic manifold X s.t. $[T] = c_1(L)$ for some positive holomorphic line bundle L on X. Does T satisfy condition (C) ?*

5 Stein manifolds

The proofs of our main results show, with very slight modifications, that they also hold on Stein manifolds (essentially the uniform estimates have to be performed on compact subsets of the manifold).

However, as there are some considerable simplifications on these manifolds to technical problems we came across when working on algebraic manifolds, we therefore reformulate in this section some of the main results after recalling a few well-known facts about Stein manifolds.

Fact 5.1 *If X is Stein, then every holomorphic line bundle on X is positive and $Pic(X) \simeq H^2(X, \mathbb{Z}) \simeq Div(X)$ (modulo linear equivalence).*

A theorem of Docquier and Grauert [D-G 60] asserts that every locally pseudoconvex open subset of a Stein manifold is Stein. Thus we have:

Fact 5.2 *Let T be a positive closed current of bidegree $(1,1)$ on a Stein manifold X then $X \setminus Supp\, T$ is Stein.*

5.1 Approximation of currents

The $T-$polynomially convex hull $\widehat{K}^T$ of a compact subset K of a complex manifold X is a closed subset of X defined in 3.1. Notice that it is compact if X is Stein, since X admits a smooth psh exhaustion function.

Proposition 5.3 *Let T be a positive closed current of bidegree $(1,1)$ on a Stein manifold X s.t. $[T] = c_1(L)$ for some $L \in Pic(X)$. Then T satisfies condition (C) and for every compact subset K of X, we have*

$$\widehat{K}^T = p_T(K) = \left\{ x \in X \,/\, \forall h \in \mathcal{O}(X),\ |h|(x)e^{-\varphi(x)} \leq \sup_K |h|e^{-\varphi} \right\},$$

where φ is a metric of L which is a potential for T.

Proof. We can assume X is a closed complex submanifold of $\mathbb{C}^N$ (for N large enough). By a theorem of Docquier-Grauert [D-G 60], there exists a holomorphic retraction $\pi : V \to X$ defined in a neighborhood V of X in $\mathbb{C}^N$.

Let $\{\mathcal{U}_\alpha\}$ be an open covering of X trivializing L. Let $g_{\alpha\beta} \in \mathcal{O}^*(\mathcal{U}_{\alpha\beta})$ be the associated transition functions of L and let $\varphi = \{\varphi_\alpha \in PSH(\mathcal{U}_\alpha)\}$ be a positive metric of L which is a potential for T. Considering a finer open covering, we can assume the φ_α's are defined in some slightly bigger open subset $\mathcal{U}'_\alpha$ so that $B(z,\varepsilon) \cap X \subset \mathcal{U}'_\alpha$ for every $z \in \mathcal{U}_\alpha \cap B(0, 2R)$ and $\varepsilon > 0$ small enough, where R is a positive constant to be chosen later.

Now π^*L is a positive holomorphic line bundle on V and $G_{\alpha\beta} := g_{\alpha\beta} \circ \pi \in \mathcal{O}^*(\tilde{\mathcal{U}}_{\alpha\beta})$ are its transition functions associated to the trivializing open covering $\{\tilde{\mathcal{U}}_\alpha\} = \{\pi^{-1}(\mathcal{U}_\alpha)\}$. Thus $\tilde{\varphi} = \{\tilde{\varphi}_\alpha\} := \{\varphi_\alpha \circ \pi\}$ satisfies $\tilde{\varphi}_\alpha = \tilde{\varphi}_\beta + \log|G_{\alpha\beta}|$ in $\tilde{\mathcal{U}}_{\alpha\beta} = \tilde{\mathcal{U}}_\alpha \cap \tilde{\mathcal{U}}_\beta$, i.e. $\tilde{\varphi}$ is a positive metric of π^*L on V. A straightforward computation shows that $(z \in Supp\, dd^c(\tilde{\varphi})) \iff (\pi(z) \in Supp\, dd^c\varphi)$.

We can regularize this metric in $V \cap B(0,2R)$ in the following way; we set

$$\tilde{\varphi}_\alpha^\varepsilon(z) := \int_{\mathbb{C}^N} \tilde{\varphi}_\alpha(w)\chi_\varepsilon(z-w)d\lambda(w),$$

where χ_ε is a smooth non-negative function in $\mathbb{C}^N$, invariant by rotation, with compact support equal to $B(0,\varepsilon)$ and s.t. $\int_{\mathbb{C}^N} \chi_\varepsilon \equiv 1$. Thus for $\varepsilon > 0$ small enough, $\tilde{\varphi}_\alpha^\varepsilon$ is a psh function in $\tilde{\mathcal{U}}_\alpha \cap B(0,2R)$ and one easily checks that:

i) $\tilde{\varphi}_\alpha^\varepsilon = \tilde{\varphi}_\beta^\varepsilon + \log|G_{\alpha\beta}|$ in $\tilde{\mathcal{U}}_{\alpha\beta} \cap B(0,2R)$;

ii) $\tilde{\varphi}_\alpha^\varepsilon(z) = \tilde{\varphi}_\alpha(z)$ if $d(z, Supp\, dd^c\tilde{\varphi}) > \varepsilon$ and $\tilde{\varphi}_\alpha^\varepsilon(z) > \tilde{\varphi}_\alpha(z)$ otherwise (here d stands for the euclidean distance in $\mathbb{C}^N$).

Let θ_R be a non negative test function in $\mathbb{C}^N$ s.t. $\theta_R \equiv 1$ in a neighborhood of $\overline{B}(0,R) = \{z \in \mathbb{C}^N \,/\, |z| \leq R\}$ and $\theta \equiv 0$ outside $B(0,2R) \cap V$. Let f be a smooth positive metric of L on X and consider

$$\tilde{\psi}^\varepsilon = \theta_R.\tilde{\varphi}^\varepsilon + (1-\theta_R) f \circ \pi + A\max(|z|^2 - R^2, 0).$$

Then for a choice of $A > 0$ large enough, $\tilde{\psi}^\varepsilon$ defines a continuous positive metric of $\pi^* L$ on V s.t. $\tilde{\psi}^\varepsilon \equiv \tilde{\varphi}^\varepsilon$ on $V \cap \overline{B}(0,R)$.

Now let K be a compact subset of $X \setminus Supp\, T$ and fix $R > 0$ large enough so that $\widehat{K}^T \subset\subset X \cap B(0,R)$. Then for $\varepsilon > 0$ small enough, the continuous positive metric $\psi^\varepsilon := \tilde{\psi}^\varepsilon_{|X}$ of L on X satisfies $\inf_{Supp\, T \cap \overline{B}(O,R)}(\psi^\varepsilon - \varphi) > 0 = \sup_K(\psi^\varepsilon - \varphi)$, hence $\widehat{K}^T \subset\subset X \setminus Supp\, T$ and T satisfies condition (C).

The equality $\widehat{K}^T = p_T(K)$ follows from a standard application of the techniques of L^2-estimates on Stein manifold (see [Hö 88] and the proof of proposition 3.2) noticing that we can regularize any positive metric of a holomorphic line bundle on compact subsets of a Stein manifold and add a small smooth strictly psh function as well. Q.E.D.

Remark 5.4 *Note that $X \setminus Supp\, T$ is usually not a Runge domain, i.e. T does not necessarily satisfies condition (C') obtained by replacing $dd^c f \geq -T$ by $dd^c f \geq 0$ in the definition of $\widehat{K}^T$ (take e.g. $X = \mathbb{C}^2$ and $T = [\{z_1 = 0\}]$).*

Theorem 5.5 *Let T be a positive closed current of bidegree $(1,1)$ on a Stein manifold X. Assume $[T] \in H^2(X,\mathbb{Z})$, i.e. $[T] = c_1(L)$ for some holomorphic line bundle L on X. Then there exists $N_j \in \mathbb{N}$ and $s_j \in \Gamma(X, L^{N_j})$ on X s.t.*

i) $T_j := \frac{1}{N_j}[\{s_j = 0\}] \to T$ in the weak sense of currents;

ii) $\{s_j = 0\} \to Supp\, T$ in the Hausdorff metric;

iii) $\forall x \in X,\ \nu(T_j, x) \to \nu(T, x)$.

Proof. By Siu's theorem we can decompose T as $T = \sum_{j\geq 1} \lambda_j [Z_j] + R$.

We can weakly approximate each current $[Z_j]$ by positive smooth $(1,1)$-forms ω_j^ε s.t. moreover $Supp\, \omega_j^\varepsilon \to Z_j$ as $\varepsilon \to 0$ (for this we can regularize $[Z_j]$ on a compact subsets of X and add a smooth strictly psh function that vanishes on a large compact subset of X as it has been done in the proof of proposition 5.3). We can therefore weakly approximate T by currents

$$T_m = \sum_{j=1}^{m} \lambda_j [Z_j] + R + \sum_{j=m+1}^{+\infty} \lambda_j \omega_j^{\varepsilon_m} = [H_m] + R_m,$$

with convergence of the supports in the Hausdorff metric and convergence of the Lelong numbers since $\lambda_j \to 0$ as $j \to +\infty$.

Now $[\lambda T_m] = [\lambda T] = c_1(L)$ since $\omega_j^{\varepsilon_m}$ is cohomologous to $[Z_j]$, T_m satisfies condition (C) by proposition 5.3, and $\mathrm{codim}_{\mathbb{C}} E_c(R_m) = \mathrm{codim}_{\mathbb{C}} E_c(R) \geq 2$; thus we can use an analogue of theorem 4.1 in the Stein case to get an approximation of T_m by rational divisors which yields the desired approximation for T. Q.E.D.

5.2 Rational convexity

Since every divisor is positive on a Stein manifold, the analogue of definition 2.1 in this case is the natural generalization of the standard one in $\mathbb{C}^m$:

Definition 5.6 *Let K be a compact subset of a Stein manifold X; the rational hull of K is defined by*

$$r(K) = \{m \in X \,/\, \forall H \quad \textit{complex hypersurface of } X, \\ m \in H \Rightarrow H \cap K = \emptyset\},$$

and K is said to be rationally convex when $r(K) = K$.

As in the algebraic case, $r(K)$ is a compact subset of X and $r(r(K)) = r(K)$. Since every positive closed current of bidegree $(1,1)$ with integer class satisfies condition (C) on a Stein manifold, we have the following analogue of theorem 2.6:

Theorem 5.7 *Let T be a positive closed current of bidegree $(1,1)$ on a Stein manifold X s.t. $[T] \in H^2(X, \mathbb{Z})$.*

Then $X \setminus Supp\, T$ can be exhausted by rationally convex compact sets.

Finally a very similar proof to that of theorem 2.8 gives

Theorem 5.8 *Let S be a smooth compact totally real submanifold of a Stein manifold X; the following are equivalent:*

i) S is rationally convex.

ii) There exists a smooth Hodge form θ for X s.t. $j^*\theta = 0$, where $j : S \to X$ denotes the inclusion map.

A Regularization process

It is in general not possible to approximate positive currents by positive smooth forms on complex manifolds. However this can be done on homogeneous manifolds (i.e. complex manifolds with a transitive group of automorphisms) where there is a natural regularization procedure. We explain here how to regularize the positive singular metrics of a holomorphic line bundle on a compact homogeneous manifold. The case of positive currents was considered by Richthofer (see [Hu 94]).
Let X be a compact complex homogeneous manifold; let G be the connected component of the identity of $Aut(X)$ and let $H = \{g \in G \,/\, g(x_0) = x_0\}$ be the isotropy group of a point $x_0 \in X$; then X is naturally isomorphic to G/H. Let T be a $(1,1)-$positive closed current on $X = G/H$ and let $\chi \in \mathcal{C}_0^\infty(G)$ a non negative function with compact support in G, s.t. $\chi(id) > 0$ and $\int_G \chi(g)dg = 1$, where id stands for the identity element in G. We define

$$T_\chi = \int_G \chi(g)\, l^*_{g^{-1}}(T)dg,$$

where dg is the Haar measure on G and l_g will stand for the multiplication on the left by g both in G and in $X = \{g.H\}$, according to the context. It is clear from the definition that T_χ defines a smooth $(1,1)-$positive closed current on X. Furthermore,

Theorem A.1 *i) T_χ is cohomologous to T.*
ii) If T is strictly positive at a point x_0, then T_χ is strictly positive on $U.x_0 = \{g(x_0) \in X \,/\, g \in U\}$, where U is the interior of the support of χ (in particular, T_χ is strictly positive at x_0).

We refer to [Hu 94] for a detailed proof.

We now define the regularized metrics φ^ε of a given metric φ of a pseudoeffective holomorphic line bundle L on X. We first construct a pseudodistance function on X that will tell us exactly when φ^ε is equal to φ.

Let Φ be a biholomorphic map from a relatively compact open neighborhood U of zero in $\mathbb{C}^N$ onto a relatively compact open neighborhood V of the identity in G which maps 0 to id; we write here X in the Klein form G/H. We can assume that U is included in the unit ball of $\mathbb{C}^N$ (N is the complex dimension of G). We define a positive function on G^2 via

$$D(g,f) = \begin{cases} \|\Phi^{-1}(g^{-1}.f)\| & \text{if } g^{-1}.f \in V \\ 1 & \text{otherwise} \end{cases}$$

where $\|\zeta\|$ stands for the euclidean norm of ζ in $\mathbb{C}^N$. This is a (a priori) non symmetrical non-negative function on G^2 which is bounded by 1, upper semi continuous and obviously satisfies $D(g,f) = 0$ iff $g = f$. When F is a closed subset of G, we define $D_F(g) = \inf_{f\in F} D(g,f)$. If F is invariant by multiplication on the left by the elements of H, i.e. $h.F = F$, then $\forall(g,h) \in G\times H$, $D_F(g) = D_F(h.g)$ since $D(h.g,h.f) = D(g,f)$ for any $f \in F$. This allows us to define a similar function on X: if K is a closed subset of X, we set $d_K(x) = D_{\pi^{-1}(K)}(z_x)$ where $\pi : G \to X = G/H$ is the canonical projection and z_x is any element in $\pi^{-1}(\{x\})$. The definition is independent of the choice of the preimage z_x thanks to the invariance of $\pi^{-1}(K)$, and we have:

$$\begin{aligned} &1)\ 0 \le d_K(x) \le 1,\ \forall x \in X\\ &2)\ d_K(x) = 0 \Leftrightarrow x \in K\\ &3)\ d_K \text{ is upper semi continuous}\end{aligned}$$

The sets $K_\varepsilon = \{x \in X \,/\, d_K(x) = d(x,K) \ge \varepsilon\}$ ($\varepsilon > 0$) are therefore compact subsets of X which exhaust $X \setminus K$ when ε decreases towards 0.

Let θ_ε be a usual approximation of the identity for the convolution product in $\mathbb{C}^N$ and χ_ε the related approximation in G, that is:

$$\begin{cases} \theta_\varepsilon \in \mathcal{C}_0^\infty(\mathbb{C}^N) \text{ is invariant by rotation}\\ \theta_\varepsilon \ge 0 \text{ and } \int_{\mathbb{C}^N}\theta_\varepsilon = 1\\ Supp\,\theta_\varepsilon = \overline{B}(0,\varepsilon)\end{cases}$$

and we then define χ_ε on G by $\chi_\varepsilon(g)dg = (\Phi^{-1})^*(\theta_\varepsilon(\zeta)d\zeta)$ so that χ_ε is a positive test function on G with $\int_G \chi_\varepsilon = 1$ and the support of χ_ε converges to $\{\mathrm{id}\}$ as ε decreases towards 0.

Let now φ be a singular positive metric of a pseudoeffective holomorphic line bundle L. That is φ is a given set of plurisubharmonic functions φ_α in $\mathcal{U}_\alpha$, where $\{\mathcal{U}_\alpha\}$ is an appropriate open covering of X. The line bundle is trivial in each open subset $\mathcal{U}_\alpha$ and is described by the transition functions $g_{\alpha\beta} \in \mathcal{O}^*(\mathcal{U}_{\alpha\beta})$. The φ_α's satisfy the relations $\varphi_\alpha = \varphi_\beta + \log|g_{\alpha\beta}|$ in $\mathcal{U}_{\alpha\beta}$. Considering a finer open covering we can assume (and we will in the sequel) that all the functions φ_α (resp. $g_{\alpha\beta}$) are defined in some slightly bigger open subsets than $\mathcal{U}_\alpha$ (resp. $\mathcal{U}_{\alpha\beta}$).

Since $dd^c(\varphi_\alpha) = dd^c(\varphi_\beta)$ in $\mathcal{U}_{\alpha\beta}$, the curvature form of the metric is globally well defined and its support $Supp\,dd^c\varphi$ as well.

The line bundle π^*L is well defined and holomorphic on G, and we denote by $G_{\alpha\beta} = g_{\alpha\beta}\circ\pi$ its transition functions associated to the covering $\pi^{-1}(\mathcal{U}_\alpha)$. We define a positive metric ψ of π^*L by $\psi = \varphi\circ\pi$.

Similarly to the case of currents, we set

$$\begin{cases} \varphi_\alpha^\varepsilon(x) = \int_G \chi_\varepsilon(g) l^*_{g^{-1}}(\varphi_\alpha(x))dg,\\ \psi_\alpha^\varepsilon(z) = \varphi^\varepsilon\circ\pi(z) = \int_G \chi_\varepsilon(g)\psi_\alpha(g^{-1}.z)dg.\end{cases}$$

These are smooth functions that are well defined in $\mathcal{U}_\alpha$, resp. $\pi^{-1}(\mathcal{U}_\alpha)$ if $\varepsilon > 0$ is small enough, since φ_α (resp. ψ_α) is defined in a slightly bigger open subset than $\mathcal{U}_\alpha$ (resp. $\pi^{-1}(\mathcal{U}_\alpha)$).

Proposition A.2 *The functions $\varphi^\varepsilon = \{\varphi^\varepsilon_\alpha\}$ (resp. $\psi^\varepsilon = \{\psi^\varepsilon_\alpha\}$) define a smooth positive metric of L (resp. $\pi^* L$) which is strictly positive at a point x whenever φ is strictly positive at x.*

φ^ε (resp. ψ^ε) decreases towards φ (resp. ψ) when ε decreases towards 0^+ and $Supp\, dd^c\varphi^\varepsilon$ (resp. $Supp\, dd^c\psi^\varepsilon$) converges to $Supp\, dd^c\varphi$ (resp. $Supp\, dd^c\psi$) in the Hausdorff metric.

More precisely, for ε small enough we have

$$\begin{cases} \varphi^\varepsilon(x) = \varphi(x) \text{ if } d_K(x) = d(x, Supp\, dd^c\varphi) > \varepsilon \\ \varphi^\varepsilon(x) > \varphi(x) \text{ if } 0 \le d_K(x) \le \varepsilon \end{cases}$$

where K denotes the support of $dd^c\varphi$.

Proof. From $\psi_\alpha = \psi_\beta + \log |G_{\alpha\beta}|$ in $\pi^{-1}(\mathcal{U}_{\alpha\beta})$ we deduce

$$\begin{aligned} \psi^\varepsilon_\alpha &= \psi^\varepsilon_\beta + \int_G \chi_\varepsilon(g) \log |G_{\alpha\beta}(g^{-1}.z)| dg \\ &= \psi^\varepsilon_\beta + \int_{\mathbb{C}^N} \theta_\varepsilon(\zeta) \log |G_{\alpha\beta}(\Phi(\zeta)^{-1}.z)| d\zeta. \end{aligned}$$

Now the function $h : \zeta \to \log |G_{\alpha\beta}(\Phi(\zeta)^{-1}.z)|$ is pluriharmonic in U and therefore the last integral is equal to $h(0) = \log |G_{\alpha\beta}(z)|$ since $\Phi(0) = id$. Thus $\psi^\varepsilon_\alpha = \psi^\varepsilon_\beta + \log |G_{\alpha\beta}|$ and ψ^ε is a smooth metric of $\pi^* L$. Of course this also gives the corresponding result for φ^ε. The positivity assertions directly follow from the fact that $dd^c\varphi^\varepsilon = T_{\chi_\varepsilon}$ with the notation of theorem A.1.

Fixing α and z we denote by H the plurisubharmonic function defined in a neighborhood of 0 in $\mathbb{C}^N$ by $H(\zeta) = \psi^\varepsilon_\alpha(\Phi(\zeta)^{-1}.z)$. Thus

$$\psi^\varepsilon_\alpha(z) = \int \theta_\varepsilon(\zeta) H(\zeta) d\zeta = \int \theta(\zeta) H(\varepsilon\zeta) d\zeta \ge \int \theta(\zeta) H(\varepsilon'\zeta), \text{ if } \varepsilon \ge \varepsilon',$$

since θ is invariant by rotation and H is plurisubharmonic. Therefore ψ^ε is decreasing and φ^ε as well. It is now enough to prove the last assertion to obtain the whole proposition.

It is absolutely equivalent to prove the similar results for ψ, i.e.

$$\begin{cases} \psi^\varepsilon(z) = \psi(z) \text{ if } D_F(z) = D(z, Supp\, dd^c\psi) > \varepsilon \\ \psi^\varepsilon(z) > \psi(z) \text{ if } 0 \le D_F(z) \le \varepsilon \end{cases}$$

where $F = Supp\, \psi = \pi^{-1}(K)$. Of course this implicitly means that for each α and each $z \in \pi^{-1}(\mathcal{U}_\alpha)$ the inequalities between ψ_α and ψ^ε_α hold, but we don't write the subscript α anymore. Setting $\zeta = \Phi(g)$, we have

$$\begin{aligned} \psi^\varepsilon(z) &= \int_V \chi_\varepsilon(g) \psi(g^{-1}.z) dg \\ &= \int_U \chi_\varepsilon(\Phi(\zeta)) \psi(\Phi(\zeta)^{-1}.z) |Jac\Phi(\zeta)|^2 d\zeta \\ &= \int_U \theta_\varepsilon(\zeta) f(\zeta) d\zeta, \end{aligned}$$

where $\zeta \to f(\zeta) = \psi(\Phi(\zeta)^{-1}.z)$ is a plurisubharmonic function (in fact a set of plurisubharmonic funtions, but we omit to mention it from now on) since Φ is holomorphic and the group operations $g \to g^{-1}$ and $g \to g.z$ as well. Therefore

$$\begin{cases} \psi^{\varepsilon}(z) = f(0) = \psi(z) \text{ if } d_{eucl}(0, Supp\, dd^c f) > \varepsilon \\ \psi^{\varepsilon}(z) > f(0) = \psi(z) \text{ otherwise} \end{cases}$$

A straightforward computation shows that the Levi form of f at the point ζ applied to w satisfies $L_f(\zeta).w = L_{\psi}(\Phi(\zeta)^{-1}.z).(J(\zeta).w)$, where $J(\zeta)$ denotes the jacobian matrix of the map $\zeta \to \Phi(\zeta)^{-1}.z$ which is biholomorphic.
Hence $Supp\, dd^c f = \Phi^{-1}([Supp\, dd^c\psi]^{-1}.z)$ and if $\varepsilon < 1$,

$$\begin{aligned} d_{eucl}(0, Supp\, dd^c f) \geq \varepsilon \ &\Leftrightarrow \inf_{f \in F} \|\Phi^{-1}(f^{-1}.z)\| \geq \varepsilon \\ &\Leftrightarrow D_F(z) \geq \varepsilon \end{aligned}$$

Q.E.D.

As an application we have the following:

Corollary A.3 *If L is a holomorphic line bundle on a compact complex homogeneous Kähler manifold, it is equivalent to say that L is pseudoeffective or nef or semi-positive: given a singular positive metric φ on L, φ^{ε} is a smooth positive metric of L.*

Moreover, if there exists a singular metric φ of L s.t. $dd^c\varphi$ is strictly positive at one point, then L is positive.

Proof. The equivalence between the three notions of weak positivity directly comes from $[dd^c\varphi] = [dd^c\varphi^{\varepsilon}]$ since $\int_G \chi_{\varepsilon}(g)dg = 1$ (cf i) in theorem A.1).

Let now φ be a singular metric of a holomorphic line bundle L which is strictly positive at some point $a \in X$. Fix $\varepsilon > 0$ small enough, then $T = dd^c\varphi^{\varepsilon}$ is a smooth positive current which is strictly positive at a. Thus T_{χ} is a smooth positive current cohomologous to T which is strictly positive in $U.a$ by theorem A.1, where χ is any positive test function on G with $\int_G \chi = 1$ and U is the interior of the support of χ. Since the action of G is transitive on X, we can cover X by a finite number of $U_j.a$ (we don't necessarily assume that $id \in U_j$ here) and obtain that way a current $S = \frac{1}{s}\sum_{j=1}^{s} T_{\chi_j}$ which is smooth, strictly positive and cohomologous to T, i.e. S is a Kähler form on X s.t. $[T] = [S]$.

It is now a standard consequence of Hodge theory on compact Kähler manifold that there exists $v \in \mathcal{C}^{\infty}(X)$ s.t. $S = T + dd^c v$. Therefore $G_{\alpha} = \varphi^{\varepsilon}_{\alpha} + v$ defines a smooth metric of L s.t $dd^c G = S$ is Kähler. Q.E.D.

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