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Consistency and Convergence Analysis of an $F(x,y)$ Functionally Derived Explicit Fifth-Stage Fourth-Order Runge-Kutta Method

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***Abstract:** The purpose of this paper is to analyze the consistency and convergence of an explicit fifth-stage fourth-order Runge-Kutta method derived using $f(x,y)$ functional derivatives. The analysis revealed that the method is consistent and convergent. The implementation of this method on initial-value problems was done in a previous paper, and it revealed that the method compared favorably well with the existing classical fourth stage fourth order explicit Runge Kutta method.*

***Key words:** Consistency, Convergence, Explicit, Runge-Kutta Methods, Linear and non- linear equations, Taylor series, Parameters, Initial-value Problems, $f(x,y)$ functional derivatives.*

I. INTRODUCTION

Some of the Runge-Kutta methods derived today do not possess the properties of convergence and consistency, hence, they are not capable enough to handle problems the way they ought to. This paper successfully analyzed the consistency and convergence of a derived fifth stage fourth order explicit Runge-Kutta method on initial value problems.

Runge-kutta methods are numerical (one-step) methods for solving initial value problems of the form:

$$y'(x) = f(x,y), \quad y(x_0) = y_0. \tag{1.1}$$

Also, according to [5], [6], and [11][12][13], in Ordinary Differential Equations, initial value problems are problems with subsidiary conditions which are called initial conditions and are applicable to solving real life problems. This can be used to analyze growth and decay problems in real life situations.

In the works of [8], [7], and [10], Explicit Runge-Kutta methods have proven to be one of the best methods for solving initial value problems in Ordinary Differential Equations. However, the method is subject to improvement, hence more research is still been carried out to get better efficiency and accuracy of the method. Many researchers have worked to improve on the accuracy of the method as can been seen in the work of [1], [3], [4] and [9][14][15][16].

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II. THE FIFTH STAGE FOURTH ORDER METHOD IS WRITTEN BELOW

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 3k_2 - 3k_3 + 4k_4 + k_5)$$
$$k_1 = f(x_n, y_n)$$
$$k_2 = f\left(x_n + \frac{h}{4}, y_n + \frac{h}{4}k_1\right)$$
$$k_3 = f(x_n + \frac{h}{4}, y_n + \frac{h}{4}(-k_1 + 2k_2))$$
$$k_4 = f(x_n + \frac{h}{2}, y_n + \frac{h}{4}(k_2 + k_3))$$
$$k_5 = f(x_n + h, y_n + \frac{h}{2}(-k_1 + k_2 - 2k_3 + 4k_4))$$

III. CONSISTENCY AND CONVERGENCE ANALYSIS OF THE FIFTH STAGE FOURTH ORDER EXPLICIT RUNGE KUTTA METHOD

Theorem 3.0: The explicit fifth-stage fourth-order method is consistent if it converges to the initial value problem $y' = f(x,y), y(x_o) = y_o$.

Proof: Using the exact solution $y(x_n)$ of the initial value problem:

$y' = f(x,y), y(x_o) = y_o$, we have that: $T_n(h^5) = y_{n+1} - y_n = \frac{h}{6}\left(f(x_n, y_n) + 3\left[f\left(x_n + c_2h, y_n + ha_{21}k_1f(x_n, y_n)\right)\right] - \right.$

$$3\left[f\left(x_n + c_3h,\ y_n + h\left(a_{31}f(x_n, y_n) + a_{32}\left(f\left(x_n + c_2h, y_n + ha_{21}f(x_n, y_n)\right)\right)\right)\right)\right] +$$

$$4\left[f\left(x_n + c_4h,\ y_n + h\left(a_{41}f(x_n, y_n) + a_{42}\left(f\left(x_n + c_2h, y_n + ha_{21}f(x_n, y_n)\right) + \right.\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\left. a_{43}\left(f\left(x_n + c_3h, y_n + h(a_{31}f(x_n, y_n) + a_{32}\left(f\left(x_n + c_2h, y_n + ha_{21}f(x_n, y_n)\right)\right)\right)\right)\right)\right)\right)\right)\right] + f(x_n +$$

$$c_5h,\ y_n + h(a_{51}f(x_n, y_n) + a_{52}\left(f\left(x_n + c_2h, y_n + ha_{21}f(x_n, y_n)\right)\right) + a_{53}(f\left(x_n + c_3h,\ y_n +\right.$$

$$h(a_{31}f(x_n, y_n)) + a_{32}\left(f\left(x_n + c_2h, y_n + ha_{21}f(x_n, y_n)\right)\right) + a_{54}\left(f\left(x_n + c_4h, y_n +\right.\right.$$

$$h\left(a_{41}f(x_n, y_n) + a_{42}\left(f\left(x_n + c_2h, y_n + ha_{21}f(x_n, y_n)\right)\right) + a_{43}\left(f\left(x_n + c_3h,\ y_n +\right.\right.\right.$$

$$\left.\left.\left.\left.\left. h(a_{31}f(x_n, y_n)) + a_{32}\left(f\left(\left(x_n + c_2h, y_n + ha_{21}f(x_n, y_n)\right)\right)\right)\right)\right)\right)\right)\right]$$

$$\textit{Dividing all through by } h \textit{ and taking the limit of both side as } h \to 0, \textit{we have}$$

$$h_n(h) = \frac{y_{n-1} - y_n}{h} = \frac{1}{6}\left[f(x_n, y_n) + 3f(x_n, y_n) - 3f(x_n, y_n) + 4f(x_n, y_n) + f(x_n, y_n)\right]$$

$$= \frac{1}{6}[6f(x_n, y_n)] = f(x_n, y_n)$$

$$\emptyset(x_n, y_n, o) = f(x_n, y_n), \qquad y(x_0) = y_0.$$

$$\textit{Hence our method is consistent and convergent }.$$

IV. CONCLUSION

It is clearly seen from the analyses above that the method converges to the initial value problem. Hence, the method is consistent. As such, it will be consistent and convergent in handling initial value problems in ordinary differential equations. These are necessary properties any numerical method should possess.

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