CEGAR-based approach for solving combinatorial optimization modulo quantified linear arithmetics problems
— Technical Appendix —

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Theorem 1 ($\phi \Rightarrow \phi_{\text{approx}}$). *Let ϕ a SAT+qLP problem and ϕ_{approx} its Boolean abstraction. For any model $(x, y) \in \mathbb{B}^n \times \mathbb{R}^m$ of ϕ, there exists $\bar{f} \in \mathbb{B}^{|D|+|E|+|H|}$ such that $(x, \bar{f})$ is a model of ϕ_{approx}.*

Proof. Let $(x, y) \models \phi$ and $\bar{f}$ such that: $\forall c_h \in D \cup E \cup H, \bar{f}_{c_h} = \top \iff x \not\models c_h$. For $(x, \bar{f}) \models \phi_{\text{approx}}$, $(x, \bar{f})$ should satisfy the Eqs. 4.

- **(4a)** As Eq. 1a equals Eq. 4a and $x \models \bigwedge_{c \in C} c(x)$, we have $(x, \bar{f}) \models \bigwedge_{c \in C} c(x)$.
- **(4b)** By definition of $\bar{f}$, $\bar{f}_d = \top$ for each clause $d \in D$ not satisfied by x. Thus, each clause $d \in D$ is satisfied by either x or $\bar{f}$. Therefore, $(x, \bar{f}) \models \bigwedge_{d \in D} \bar{d}(x, \bar{f})$.
- **(4c)** Using same reasoning as for (4b), there are $(x, \bar{f}) \models \bigwedge_{e \in E} \bar{e}(x, \bar{f})$ and $(x, \bar{f}) \models \bigwedge_{h \in H} \bar{h}(x, \bar{f})$. Therefore, $(x, \bar{f}) \models \bigwedge_{e \in E} \bar{e}(x, \bar{f}) \wedge \bigwedge_{h \in H} \bar{h}(x, \bar{f})$.

Therefore $(x, \bar{f}) \models \phi_{\text{approx}}$, and $\phi \implies \phi_{\text{approx}}$.

Models of (g, ϕ) are subsets of models of ϕ and by definition $\not\models (g, \phi)$ if $\not\models \phi$. Hence, $(g, \phi) \iff \phi$, i.e., $(g, \phi) \implies \phi_{\text{approx}}$.

Let $\nu^* = (x, y)$. Suppose that $\nu^* \models (g, \phi)$ with $g(\nu^*)$ its optimal value. By previous statements, $\exists \bar{f}, (x, \bar{f}) \models \phi_{\text{approx}}$. As $g : \mathbb{B}^n \to \mathbb{R}$, then $g((x, \bar{f})) = g(x) = g((x, y))$. □

Corollary 1.1. $(g, \phi) \implies \phi_{approx}$.

Proof. Models of (g, ϕ) are subsets of models of ϕ and by definition $\not\models (g, \phi)$ if $\not\models \phi$. Hence, $(g, \phi) \iff \phi$. Therefore by Theorem 1, $(g, \phi) \implies \phi_{\text{approx}}$. □

Lemma A. *Given C_h a set of hybrid clauses and $x \in \mathbb{B}^n$ a Boolean variables assignment, $y \models \mathcal{C}_x^{C_h} \iff (x, y) \models \bigwedge_{c_h \in C_h} c_h(x, y)$.*

Proof. **(→)** Let $y \in \mathbb{R}^m$ such that $y \models \mathcal{C}_x^{C_h}$. By *reductio ad absurdum*, suppose that $\exists c_h \in C_h, (x, y) \not\models \bigwedge_{c_h \in C_h} c_h(x, y)$. The hybrid constraint $c_h(x, y)$ is of the form $\bigwedge_i x_i \bigwedge \neg x_j \wedge f_{c_h}(y) \leq 0$. Thus, there are $x \not\models \bigwedge_i x_i \bigwedge \neg x_j$ and $y \not\models f_{c_h}(y) \leq 0$. By definition of $\mathcal{C}_x^{C_h}$, if $x \not\models \bigwedge_i x_i \bigwedge \neg x_j$ then $f_{c_h}(y) \leq 0 \in \mathcal{C}_x^{C_h}$. As $y \models \mathcal{C}_x^{C_h}$, then $y \models f(y) \leq 0$. Otherwise, $x \models \bigwedge_i x_i \bigwedge \neg x_j$. This contradicts the hypothesis that $\exists c_h \in C_h, (x, y) \not\models \bigwedge_{c_h \in C_h} c_h(x, y)$. Therefore, $(x, y) \models \bigwedge_{c_h \in C_h} c_h(x, y)$.
(←) Let $(x, y) \models \bigwedge_{c_h \in C_h} c_h(x, y)$. Thus, $\forall c_h \in C_h$ either $x \models \bigwedge_i x_i \bigwedge_j \neg x_j$ or $y \models f_{c_h}(y) \leq 0$. Therefore, by definition of $\mathcal{C}_x^{C_h}$, $y \models \mathcal{C}_x^{C_h}$. □

Lemma B. *Given C_h a set of hybrid clauses, $\hat{c}_h$ a hybrid clause and $x \in \mathbb{B}^n$, $\not\models \mathcal{C}_x^{C_h} \vee f^*_{\hat{c}_h}(\mathcal{C}_x^{C_h}) \leq 0$ if and only if $x \models \forall y \in \mathbb{R}^m, \bigwedge_{c_h \in C_h} c_h(x, y) \implies \hat{c}_h(x, y)$.*

Proof. **(→)** If $\not\models \mathcal{C}_x^{C_h}$, then $\forall y \in \mathbb{R}^m, (x, y) \not\models \bigwedge_{c_h \in C_h} c_h(x, y)$. Therefore, $x \models \forall y \in \mathbb{R}^m, \bigwedge_{c_h \in C_h} c_h(x, y) \implies \hat{c}_h(x, y)$. Otherwise, $\exists y \in \mathbb{R}^m, y \models \mathcal{C}_x^{C_h}$ and $f^*_{\hat{c}_h}(\mathcal{C}_x^{C_h}) \leq 0$. Thus, $\exists y \in \mathbb{R}^m, \bigwedge_{c_h \in C_h} c_h(x, y)$. By *reductio ad absurdum*, suppose that $\exists y' \in \mathbb{R}^m, (x, y') \models \bigwedge_{c_h \in C_h} c_h(x, y')$ and $f(y') > 0$. Thus, $f^*_{\hat{c}_h}(\mathcal{C}_x^{C_h}) < f(y')$. However, by definition of $f^*_{\hat{c}_h}(\mathcal{C}_x^{C_h})$, $\forall y \in \mathbb{R}^m, y \models \mathcal{C}_x^{C_h} \implies f(y) \leq f^*_{\hat{c}_h}$. It contradicts the hypothesis that $f^*_{\hat{c}_h}(\mathcal{C}_x^{C_h}) \leq 0$. Therefore, $\not\models \mathcal{C}_x^{C_h} \vee f^*_{\hat{c}_h}(\mathcal{C}_x^{C_h}) \leq 0$ implies that $x \models \forall y \in \mathbb{R}^m, \bigwedge_{c_h \in C_h} c_h(x, y) \implies \hat{c}_h(x, y)$.
(←) Suppose that $x \models \forall y \in \mathbb{R}^m, \bigwedge_{c_h \in C_h} c_h(x, y) \implies \hat{c}_h(x, y)$. By *reductio ad absurdum*, suppose that $\exists y \in \mathbb{R}^m$ such that $y \models \mathcal{C}_x^{C_h}$ and $f^*_{\hat{c}_h}(\mathcal{C}_x^{C_h}) > 0$. Thus, $\exists y \in \mathbb{R}^m, (y \models \mathcal{C}_x^{C_h}) \wedge f(y) > 0$. By definition of $x, \forall y' \in \mathbb{R}^m, (y \models \mathcal{C}_x^{C_h}) \implies f(y) \leq 0$. This contradicts that $\exists y \in \mathbb{R}^m, (y \models \mathcal{C}_x^{C_h}) \wedge f_{\hat{c}_h}(y) > 0$. Therefore, $x \models \forall y \in \mathbb{R}^m, \bigwedge_{c_h \in C_h} c_h(x, y) \implies \hat{c}_h(x, y)$ implies that $\not\models \mathcal{C}_x^{C_h} \vee f^*_{\hat{c}_h}(\mathcal{C}_x^{C_h}) \leq 0$. □

Theorem 2. *Let ϕ be a SAT+qLP formula and ϕ_{approx} its Boolean abstraction. Given $x \in \mathbb{B}^n$ and $y \in \mathbb{R}^m$, $(x, y) \models \phi$ if and only if the following three conditions hold: **(C1)** $\exists \bar{f}, (x, \bar{f}) \models \phi_{approx}$; **(C2)** $y \models \mathcal{C}_x^D$; **(C3)** $\not\models \mathcal{C}_x^E \vee \bigwedge_{h \in \mathcal{C}_x^H} f^*_h(\mathcal{C}_x^E) \leq 0$.*

Proof. **(→)** Suppose that $(x, y) \models \phi$. By Theorem 1, $\phi \implies \phi_{\text{approx}}$. Thus, *C1* holds. As $(x, y) \models \phi$, then $(x, y) \models \bigwedge_{d \in D} d(x, y)$. Thus, Lemma A concludes that *C2* holds. As $(x, y) \models \phi$, then $x \models \forall z \in \mathbb{R}^p, \bigwedge_{e \in E} e(x, z) \implies \bigwedge_{h \in H} h(x, z)$. Thus, $\forall h \in$

$\mathcal{C}_x^H, x \models \forall z \in \mathbb{R}^p, \bigwedge_{e \in E} e(x,z) \implies h(x,z)$. Lemma B concludes that *C3* holds. Therefore, $(x,y) \models \phi$ implies *C1*, *C2* and *C3*.

($\leftarrow$) Suppose that all three conditions hold. By *C1*, $\exists \bar{f}, (x,\bar{f}) \models \phi_{\text{approx}}$. Thus, $x \models \bigwedge_{c \in C} c(x)$ (Eq. 1a). *C2* and Lemma A concludes for Eq. 1b. *C3* and Lemma B conclude for Eq. 1c. Therefore, *C1*, *C2* and *C3* implies $(x,y) \models \phi$. □

Corollary 2.1. *Given $x \in \mathbb{B}^n$ and $y \in \mathbb{R}^m$ a real-valued variables assignment, if **(C1')** $\exists \bar{f}, (x,\bar{f}) \models (g, \phi_{approx})$, **C2** and **C3** hold, then $(x,y) \models (g,\phi)$.*

Proof. As $(x,\bar{f}) \models (g, \phi_{\text{approx}})$, then $(x,\bar{f}) \models \phi_{\text{approx}}$. Thus, *C1* holds. Moreover, $\forall (x', \bar{f}') \models \phi, g(x) \leq g(x')$. By Corollary 1.1, $(g,\phi) \implies \phi_{\text{approx}}$. Therefore, *C1*, *C2*, *C3* hold and x is minimal according to g. Therefore, *C1'*, *C2* and *C3* implies $(x,y) \models (g,\phi)$. □

Lemma 3. $\phi \implies \phi_{approx} \wedge \phi_r^{\exists}(x)$.

Proof. Let $\bar{f}'$ such that $\forall c_h \in D \cup E \cup H, \bar{f}'_{c_h} = \top \iff x' \not\models c_h$. By Theorem 1, we have $(x', \bar{f}') \models \phi_{\text{approx}}$. If $(x,\bar{f})$ satisfies *C2* then $\phi_r^{\exists}(x)$ does not generate new constraints. Thus, $\phi_r^{\exists}(x) \wedge \phi_{\text{approx}} = \phi_{\text{approx}}$. Otherwise, *C2* does not hold for $(x,\bar{f})$. Let $\mathcal{C}_{\text{unsat}}$ be an unsatisfiable core of $\mathcal{C}_x^D$. By *reductio ad absurdum*, suppose that $\exists (x', \bar{f}') \not\models \phi_r^{\exists}(x)$. Thus, $\forall f \in \mathcal{C}_{\text{unsat}}, \bar{f}' = \top$. By definition of $\bar{f}'$ and $\mathcal{C}_{x'}^D$, it means that $\mathcal{C}_{\text{unsat}} \subseteq \mathcal{C}_{x'}^D$. Hence, $(x', f') \models \phi_r^{\exists}(x) \wedge \phi_{\text{approx}}$. Therefore, $\phi \implies \phi_{\text{approx}} \wedge \phi_r^{\exists}(x)$. □

Property 4. *Given a linear objective function $\boldsymbol{f}$ and two linear optimization problems $(\boldsymbol{f}, \mathcal{C}_1)$ and $(\boldsymbol{f}, \mathcal{C}_2)$, $\mathcal{C}_1 \subseteq \mathcal{C}_2 \implies \boldsymbol{f}^*(\mathcal{C}_1) \geq \boldsymbol{f}^*(\mathcal{C}_2)$.*

Proof. By *reductio ad absurdum*, suppose that $\mathcal{C}_1 \subseteq \mathcal{C}_2$ and $f^*(\mathcal{C}_1) < \mathcal{C}_2$. Let $y = \text{argmax}_{y \models \mathcal{C}_2} \boldsymbol{f}(y)$. As $\mathcal{C}_1 \subseteq \mathcal{C}_2$, then $y \models \mathcal{C}_1$. Since $f(y) = f^*(\mathcal{C}_2)$ and $y \models \mathcal{C}_1$, its contradicts $f^*(\mathcal{C}_1) < \mathcal{C}_2$. Therefore, $\mathcal{C}_1 \subseteq \mathcal{C}_2 \implies f^*(\mathcal{C}_1) \geq \mathcal{C}_2$. □

Lemma 5. $\phi \implies \phi_{approx} \wedge \phi_r^{\forall}(x)$

Proof. Let $\bar{f}'$ such that $\forall c_h \in D \cup E \cup H, \bar{f}'_{c_h} = \top \iff x' \not\models c_h$. By Theorem 1, we have $(x', \bar{f}') \models \phi_{\text{approx}}$. If $(x,\bar{f})$ satisfies *C3* then $\phi_r^{\forall}(x)$ does not generate new constraints. Thus, $\phi_r^{\forall}(x) \wedge \phi_{\text{approx}} = \phi_{\text{approx}}$. Otherwise, *C3* does not hold for $(x,\bar{f})$. By *reductio ad absurdum*, suppose that $\exists (x', \bar{f}') \not\models \phi_r^{\forall}(x)$. Let $h \in H$ such that $f_h \in \mathcal{C}_x^H$ and $f_h^*(\mathcal{C}_x^E) > 0$. Such h exists as *C3* does not hold for x. By definition of $(x', y') \models \phi$ and $\bar{f}'$, there are either $\bar{f}'_h = \bot$ or $\bar{f}'_h = \top \wedge f_h^*(\mathcal{C}_{x'}^E) \leq 0$. For the first case, $\bar{f}'_h$ satisfies the constraint of $\phi_r^{\exists}(x)$ associated with h. For the second case, suppose that $\bar{f}'_h = \top \wedge f_h^*(\mathcal{C}_{x'}^E) \leq 0$. Let $\mathcal{C}_{\text{opt}}^{f_h}$ be an optimal core of $(f_h, \mathcal{C}_x^E)$. Thus, $\forall e \in E, f_e \notin \mathcal{C}_{\text{opt}}^{f_h} \implies \bar{f}'_e = \bot$ and $\bar{f}_h = \top$. By definition of $\bar{f}'$ and $\mathcal{C}_{x'}^E$, it means that $\mathcal{C}_{x'}^E \subseteq \mathcal{C}_{\text{opt}}^{f_h}$. However, we have that $f_h^*(\mathcal{C}_{x'}^E) < f_h^*(\mathcal{C}_x^E)$. This contradicts property 4. Hence, $(x', f') \models \phi^{\forall} r(x) \wedge \phi_{\text{approx}}$. Therefore, $\phi \implies \phi_{\text{approx}} \wedge \phi_r^{\forall}(x)$. □

Theorem 6. *Given $(x,\bar{f}) \models \phi_{approx}$, $\phi \implies \phi_r^{\exists}(x) \wedge \phi_r^{\forall}(x) \wedge \phi_{approx}$.*

Proof. By Lemma 3, we have $\phi \implies \phi_r^{\exists}(x) \wedge \phi_{\text{approx}}$. By Lemma 5, we have $\phi \implies \phi_r^{\forall}(x) \wedge \phi_{\text{approx}}$. As the constraints generated by $\phi_r^{\exists}(x)$ and $\phi_r^{\forall}(x)$ impact disjoint sets of variables $\bar{f}$, then $\phi \implies \phi_r^{\exists}(x) \wedge \phi_r^{\forall}(x) \wedge \phi_{\text{approx}}$. □

Corollary 6.1. $(g,\phi) \implies \phi_r^{\exists}(x) \wedge \phi_r^{\forall}(x) \wedge \phi_{approx}$.

Proof. By definition, $\phi \iff (g,\phi)$. Therefore by Theorem 6, $(g,\phi) \implies \phi_r^{\exists}(x) \wedge \phi_r^{\forall}(x) \wedge \phi_{\text{approx}}$. □

Corollary 6.2. $\forall \nu^* \models (g,\phi) \implies \exists \nu' \models \phi_r^{\exists}(x) \wedge \phi_r^{\forall}(x) \wedge \phi_{approx}, g(\nu') = g(\nu^*)$.

Proof. Suppose that $(x', y') \models (g,\phi)$ with $g((x',y'))$ its optimal value. By definition, $\exists \bar{f}', (x', \bar{f}') \models \phi_r^{\exists}(x) \wedge \phi_r^{\forall}(x) \wedge \phi_{\text{approx}}$. As $g : \mathbb{B}^n \to \mathbb{R}$, then $g((x', \bar{f}')) = g(x') = g((x', y'))$. □

Lemma 7. $\exists y \in \mathbb{R}^m, \mathcal{C}_x^D \iff \bigwedge_{\mathcal{P}_i \in \mathcal{P}_x^D} y \models \mathcal{P}_i$.

Proof. ($\rightarrow$) Suppose that $\exists y \in \mathbb{R}^m, \mathcal{C}_x^D$ and it exists $\mathcal{P}_i \in \mathcal{P}_x^D$ unsatisfiable. We know that $\mathcal{P}_x^D$ is a partition of $\mathcal{C}_x^D$, hence $\mathcal{P}_i \subseteq \mathcal{C}_x^D$. If $\mathcal{P}_i$ is unsatisfiable, so is $\mathcal{C}_x^D$. Therefore, it could not exists $\mathcal{P}_i \in \mathcal{P}_x^D$ unsatisfiable if $\mathcal{C}_x^D$ is satisfiable.

($\leftarrow$) Suppose that $\forall \mathcal{P}_i \in \mathcal{P}_x^D, y_i \models \mathcal{P}_i$. We know that $\mathcal{P}_x^D$ is a partition of $\mathcal{C}_x^D$ such that no variables are shared among the constraints of different partitions. Hence $y = y_{i_i} \models \bigwedge_{\mathcal{P}_i \in \mathcal{P}_x^D} \mathcal{P}_i$, and $\bigwedge_{\mathcal{P}_i \in \mathcal{P}_x^D} y \models \mathcal{P}_i$. As y is a model of all the subsets in the partition $\mathcal{P}_x^D$, $y \models \mathcal{C}_x^D$. □

Lemma 8. *If $\mathcal{C}_x^E$ is satisfiable, then $f_h^*(\mathcal{C}_x^E) = f_h^*(\mathcal{P}'^E_x)$.*

Proof. By definition of $\mathcal{P}$, we have that all the linear constraints that can have an impact on the variables involved in f_h are in P'. Therefore, linear constraints in the other subsets will not impact the variables involved in f_h. These constraints can only impact the satisfiability of the problem, however, we supposed that $\mathcal{C}_x^E$ is satisfiable. Hence, the optimum of $(f_h, \mathcal{C}_x^E)$ depends only of the constraints in $\mathcal{P}'^E_x$. □