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A STUDY ON FUZZY α - Ext SOBER SPACES

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Abstract:

In this paper, the new concept of Fuzzy α-Exterior Sober Space (briefly, Fα-Ext Sober Space) is introduced. Two extension theorems for Fuzzy α-Ext Sober Space are studied using fuzzy quasi-homeomorphism. In this connection, some equivalent statements for fuzzy α-Ext Sober Spaces are also established.

Key Words: Fuzzy irreducible closed sets, Fuzzy $\tilde{\xi}$ - structure space, Fuzzy quasi-homeomorphism, Fuzzy α-Ext generic set, Fuzzy α-Ext Sober Space

1. Introduction:

In 1965, Zadeh [9] introduced the notion of fuzzy sets and fuzzy set operations. The concept of sobriety in topological spaces and its importance as a separation axiom became known through the book of 1982 by P. Johnstone. Soberity, a special separation property of topological spaces, plays an important role in studying continuous lattices and domains (cf. [4, 5, 6]). During the years 1986-1987, S. Rodabaugh extended the concept soberity to fuzzy topological spaces.

In this paper, the new concept of Fuzzy α-Exterior Sober Space (briefly, Fα-Ext Sober Space) is introduced. Two extension theorems for Fuzzy α-Ext Sober Space are studied using fuzzy quasi-homeomorphism. In this connection, some equivalent statements for fuzzy α-Ext Sober Spaces are also established.

2 Preliminaries:

Definition 2.1 [7] Let X be a set and τ be a family of fuzzy subsets of X. Then τ is called fuzzy topology on X if satisfies the following conditions:

(i) 0_X, $1_X \in \tau$;
(ii) If λ, μ $\in \tau$, then λ ∧ μ $\in \tau$;
(iii) If $\lambda_i \in \tau$ for each i ∈ I, then $\vee\lambda_i \in \tau$.

The ordered pair (X, τ) is said to be a fuzzy topological space (in short, FTS). Moreover, the members of τ are said to be the fuzzy open sets and their complements are said to be the fuzzy closed sets.

Definition 2.2 [7] A fuzzy set $\lambda \in I^X$ in a fuzzy topological space (X, τ) is said to be Fuzzy α-open if $\lambda \leq$ Fint(Fcl(Fint(λ))).

Definition 2.3 [7] Let (X, τ) be a FTS and $\lambda \in I^X$. Then the fuzzy α-interior of λ is denoted by Fα- int(λ) and defined as Fα-int(λ) = $\vee\{\beta \in I^X : \beta \leq \lambda, \beta$ is Fα open}.

Proposition 2.1 [1] Let f be a function from (X, τ) to (Y, σ). Then $f(f^{-1})(\lambda) \leq \lambda$ for any fuzzy set in λ in (Y, σ).

Definition 2.4 [7] Let (X, τ) be FTS and let $\mu \in I^X$. Then fuzzy Exterior of λ is FExt(λ) = Fint(1_X −λ).

Definition 2.5 [7] Let (X, τ) be a FTS and $\lambda \in I^X$ be any fuzzy set in (X, τ). Then fuzzy α-Exterior of λ is denoted by Fα-Ext and defined as Fα-Ext(λ) = Fα-int(1 − λ).

Definition 2.6 [5]A subset C of X is irreducible if it is nonempty and for all closed subsets F, F_0 of X, C ⊂ F ∪ F_0 implies C ⊂ F or C ⊂ F_0. The closure of a point is always an irreducible closed set.

Definition 2.7 [6] A topological space X is called a sober space if every irreducible closed subset is the closure of some unique point in X.

Definition 2.8 [7] Let (X_1, τ_1) and (X_2, τ_2) be any two fuzzy topological spaces and let f : (X_1, τ_1) → (X_2, τ_2). Then f is said to be

(i) a fuzzy continuous function if for each fuzzy open set $\mu \in I^{X_2}$, $f^{-1}(\mu) \in I^{X_1}$ is fuzzy open in (X_1, τ_1).
(ii) a fuzzy homeomorphism iff f is bijective and f and f^{-1} are fuzzy continuous.

Definition 2.9 [8]A fuzzy point μ_p is quasi-coincient with the fuzzy set μ_A iff $\mu_p(p) + \mu_A(p) > 1$.

Definition 2.10 [7] A fuzzy soft topological space (f_E, τ) is said to be fuzzy soft T_0 space if for every of disjoint fuzzy soft points e_h and e_g, ∃ a fuzzy soft open set containing one but not the other.

3 Fuzzy α-Ext Sober Spaces:

Definition 3.1: Let (X, τ) be fuzzy topological space (briefly, FTS). A fuzzy set$\mu \in I^X$ is called fuzzy irreducible if $\mu \neq 0_X$ and for all fuzzy closed sets $\gamma, \delta \in I^X$ with $\mu \leq (\gamma \vee \delta)$, it follows that either $\mu \leq \gamma$ or $\mu \leq \delta$.

Remark 3.1: Let (X, τ) be a fuzzy topological space. Any $\lambda \in I^X$ is said to be fuzzy irreducible closed iff it is both fuzzy irreducible and fuzzy closed.

Definition 3.2: Let (X, τ) be a fuzzy topological space and let $\lambda, \mu \in I^X$ be such that $\mu \le \lambda$. Then μ is said to be a fuzzy $\boldsymbol{\alpha}$-Exterior generic set of λ (briefly, F$\boldsymbol{\alpha}$-Ext generic set) if Fcl(F$\boldsymbol{\alpha}$-Ext(μ)) = λ.

Definition 3.3: Let (X, τ) be a FTS. Then (X, τ) is said to be a fuzzy $\boldsymbol{\alpha}$-Ext Sober space (F$\boldsymbol{\alpha}$-Ext Sober Space) if for every fuzzy irreducible closed set $\lambda \in I^X$, there exists a unique F$\boldsymbol{\alpha}$-Ext generic set $\mu \in I^X$ of λ such that $\lambda \ge \mu$.

Proposition 3.1:

Let (X, τ) be a fuzzy $\boldsymbol{\alpha}$-Ext Sober space and (X, τ^*) be a fuzzy topological space such that $\tau \subseteq \tau^*$. If $\beta \in I^X$ is a fuzzy irreducible closed set in (X, τ^*), then $\beta \le Fcl_\tau$(F$\boldsymbol{\alpha}$-Ext(γ)) for some $\gamma \in I^X$ with $\gamma \le Fcl_\tau(\beta)$ where $Fcl_\tau(\beta)$ refers the fuzzy closure of β with respect to τ.

Proof:

Let $\beta \in I^X$ be a fuzzy irreducible closed set in (X, τ). Then clearly $Fcl_\tau(\beta)$ is fuzzy irreducible closed. But as a contrary, assume that $Fcl_\tau(\beta)$ is not a fuzzy irreducible closed set in (X, τ). Then $Fcl_\tau(\beta) = \lambda_1 \vee \lambda_2$ where $\lambda_1, \lambda_2 \in I^X$ are fuzzy closed sets in (X, τ) with $Fcl_\tau(\beta) \nless \lambda_1$ and $Fcl_\tau(\beta) \nless \lambda_2$. Since $\beta \in I^X$ is a fuzzy irreducible closed set in (X, τ^*), for $\lambda_1, \lambda_2 \in \tau^*$, $\beta \le \lambda_1 \vee \lambda_2$, implies that definitely $\beta \le \lambda_1$ or $\beta \le \lambda_2$. Thus, $\beta \le (\lambda_1 \wedge \beta) \vee (\lambda_2 \wedge \beta)$, with both $\lambda_1 \wedge \beta$ and $\lambda_2 \wedge \beta$ are fuzzy closed.

From $\beta \le (\lambda_1 \wedge \beta) \vee (\lambda_2 \wedge \beta)$, $\beta < (\lambda_1 \wedge \beta)$ or $\beta < (\lambda_2 \wedge \beta)$. Also, it follows that $\beta < \lambda_1$ or $\beta < \lambda_2$ and so $Fcl_\tau(\beta) < \lambda_1$ or $Fcl_\tau(\beta) < \lambda_2$ which is a contradiction. Therefore, $Fcl_\tau(\beta)$ is a fuzzy irreducible closed set in(X, τ). Since (X, τ) is a Fα-Ext Sober space, there existsFα-Ext generic set $\gamma \in I^X$ of $Fcl_\tau(\beta)$ such that $\gamma \le Fcl_\tau(\beta)$. Since γ is Fα-Ext generic set of $Fcl_\tau(\beta)$, $Fcl_\tau(\beta) = Fcl_\tau$(F$\alpha$-Ext($\gamma$)) for some $\gamma \le Fcl_\tau(\beta)$. Thus $\beta \le Fcl_\tau$(Fα-Ext(γ)).

Definition 3.4: Let (X_1, τ_1) and (X_2, τ_2) be any two fuzzy topological spaces. Let f : (X_1, τ_1) → (X_2, τ_2) be a fuzzy continuous function. Then f is said to be a fuzzy quasi-homeomorphism if f is bijective and for each fuzzy open set $\lambda \in I^{X_1}$, there exists a unique fuzzy open set $\mu \in I^{X_2}$ in (X_2, τ_2) such that $\lambda = f^{-1}(\mu)$.

Example 3.1: Let X_1 = { a, b } = X_2. Let $\lambda \in I^{X_1}$ and $\boldsymbol{\alpha} \in I^{X_2}$ be defined as $\lambda(a) = 0.6$, $\lambda(b) = 0.7$, $\boldsymbol{\alpha}(a) = 0.7$ and $\boldsymbol{\alpha}(b) = 0.6$. Then $\tau_1 = \{0_{X_1}, 1_{X_1}, \lambda\}$ and $\tau_2 = \{0_{X_2}, 1_{X_2}, \boldsymbol{\alpha}\}$. Clearly, ($X_1$, τ_1) and (X_2, τ_2) are fuzzy topological spaces respectively. Let f : (X_1, τ_1) → (X_2, τ_2) be fuzzy continuous function defined by f(a) = b, f(b) = a. For each fuzzy open set $\lambda = (0.6, 0.7) \in I^{X_1}$, there exist $\mu = (0.7, 0.6) \in I^{X_2}$ such that $f^{-1}(\mu) = (0.6, 0.7) = \lambda$. Then f is said to be fuzzy quasi-homeomorphism.

Proposition 3.2: Let (X_1, τ_1) and (X_2, τ_2) be any two FTSs and let f : (X_1, τ_1) → (X_2, τ_2) be a fuzzy quasi-homeomorphism. Then for any fuzzy set $\lambda \in I^{X_1}$, $\lambda = f^{-1}(f(\lambda))$.

Proposition 3.3:

Let (X_1, τ_1) and (X_2, τ_2) be any two FTSs and let f : (X_1, τ_1)→ (X_2, τ_2) be a fuzzy continuous function. If $\lambda \in I^{X_2}$ is a fuzzy irreducible set in (X_2, τ_2), then $f^{-1}(\lambda)$ is fuzzy irreducible in (X_1, τ_1).

Proof:

Let $\lambda \in I^{X_2}$be a fuzzy irreducible set in (X_2, τ_2). Let $\boldsymbol{\alpha}, \beta \in I^{X_1}$ be fuzzy closed sets in (X_1, τ_1). Suppose $f^{-1}(\lambda) \le \boldsymbol{\alpha} \vee \beta$. Then $f(f^{-1}(\lambda)) \le \lambda \le f(\boldsymbol{\alpha} \vee \beta)$ which implies that $\lambda \le f(\boldsymbol{\alpha} \vee \beta) = f(\boldsymbol{\alpha}) \vee f(\beta)$. Since λ is fuzzy irreducible, $\lambda \le f(\boldsymbol{\alpha})$ or $\lambda \le f(\beta)$. Thus either $f^{-1}(\lambda) \le \boldsymbol{\alpha}$ or$f^{-1}(\lambda) \le \beta$. Thus $f^{-1}(\lambda)$ is fuzzy irreducible in (X_1, τ_1).

Proposition 3.4:

Let (X_1, τ_1) and (X_2, τ_2) be any two FTSs and let f : (X_1, τ_1) → (X_2, τ_2) be fuzzy quasi-homeomorphism. If for any two fuzzy sets $\lambda, \mu \in I^{X_2}$, $f^{-1}(\lambda) = f^{-1}(\mu)$, then $\lambda = \mu$.

Proposition 3.5:

Let (X_1, τ_1) and (X_2, τ_2) be any two FTSs and let f : (X_1, τ_1) → (X_2, τ_2) be fuzzy quasi-homeomorphism. Then the following properties hold:

(i) if (X_1, τ_1) is a fuzzy T_0-space, then f is injective.
(ii) if (X_1, τ_1)is fuzzy $\boldsymbol{\alpha}$-Ext Sober space and (X_2, τ_2) is a fuzzy T_0-space, then f is a fuzzy homeomorphism.

Proof:

(i) Let $\alpha, \beta \in I^{X_1}$ be such that $f(\alpha) = f(\beta)$. Suppose that $\alpha \ne \beta$, then there exists a fuzzy open set $\sigma \in I^{X_1}$ such that $\alpha \le \sigma$ and $\beta \nleq \sigma$, since (X_1, τ_1) is a fuzzy T_0-space. Also since f is fuzzy quasi-homeomorphism, there exists a fuzzy open set $\lambda \in I^{X_2}$ in (X_2, τ_2) satisfying $f^{-1}(\lambda) = \sigma$.

$$\text{Hence } \alpha \le \sigma = f^{-1}(\lambda) \text{ and } \beta \nleq \sigma = f^{-1}(\lambda);$$
$$\text{Then } f(\alpha) \le \lambda \text{ and also } f(\beta) \nleq \lambda$$

Which is a contradiction, since $f(\boldsymbol{\alpha}) = f(\beta)$. Therefore $\alpha = \beta$. Hence f is injective.

(ii) Let $\lambda \in I^{X_2}$ be a fuzzy closed set in (X_2, τ_2). If λ is fuzzy irreducible in (X_2, τ_2), by Proposition 3.3, $f^{-1}(\lambda)$ is a fuzzy irreducible in (X_1, τ_1). To prove f is surjective. Given that (X_1, τ_1) be Fα-Ext sober space and let $\mu \in I^{X_2}$ be fuzzy irreducible closed in(X_2, τ_2). Since Fcl(Fα-Ext(μ)) is also fuzzy closed in (X_2, τ_2), it is fuzzy irreducible. Then f^{-1}(Fcl(Fα-Ext(μ))) is a fuzzy irreducible closed set of (X_1, τ_1). Since (X_1, τ_1) is fuzzy α-Ext

Sober space, by Proposition 3.1, there exists a Fα-Ext generic set $\sigma \in I^{X_1}$ such that f^{-1}(Fcl(Fα-Ext(μ))) ≤ Fcl(Fα-Ext(σ)). And also σ is Fα-Ext generic set of f^{-1}(Fcl(Fα-Ext(μ))) such that

$$F^{-1}(Fcl(F\alpha\text{-Ext}(\mu))) \geq \sigma$$
$$Fcl(F\alpha\text{-Ext}(\mu)) \geq f(\sigma)$$
$$f(\sigma) \leq Fcl(F\alpha\text{-Ext}(\mu)).$$

Let f^{-1}(Fcl(Fα-Ext(f(σ)))) ≤ f^{-1}(Fcl(Fα-Ext(Fcl(Fα-Ext(μ))))) ≤ f^{-1}(Fcl(Fα-Ext(μ))). Therefore f^{-1}(Fcl(Fα-Ext(μ))) ≥ f^{-1}(Fcl(Fα-Ext(f(σ)))).
It is known that f^{-1}(Fcl(Fα-Ext(μ))) ≤ Fcl(Fα-Ext(σ)) = Fcl(Fα-Ext(f^{-1}f(σ))) = Fcl(f^{-1}(Fα-Ext(f(σ))) ≤ f^{-1}(Fcl(Fα-Ext(f(σ)))). Thusf^{-1}(Fcl(Fα-Ext(μ))) ≤ f^{-1}(Fcl(Fα-Ext(f(σ)))).
Therefore f^{-1}(Fcl(Fα-Ext(f(σ)))) = f^{-1}(Fcl(Fα-Ext(μ))). Since f is fuzzy quasi-homeomorphism, by Proposition 3.4, Fcl(Fα-Ext(f(σ))) = Fcl(Fα-Ext(μ)). Since (X_2, τ_2) is a FT_0-space, f(σ) = μ. Thus f is surjective map and so it is bijective. Since any bijective fuzzy quasi-homeomorphism is fuzzy homeomorphism, q is fuzzy homeomorphism.

Definition 3.5: Let (X, τ) be a FTS and let S(X) be the set of all fuzzy irreducible closed sets in (X, τ). Let$\alpha \in I^X$ be a fuzzy open set in (X, τ). Then the collection ξ = {σ ∈ S(X) : α q σ}. Then the collection ξ which is finer than the fuzzy topology τ on X is said to be a fuzzy ξ-structure on S(X). Then S(X) with fuzzy ξ-structure denoted by (S(X), ξ) is said to be a fuzzy ξ structure space. A fuzzy ξ-structure on S(X) together with 0_X is said to be a fuzzy $\tilde{\xi}$-structure on S(X). Then (S(X), $\tilde{\xi}$) is called a fuzzy $\tilde{\xi}$-structure space. Each member of $\tilde{\xi}$ is said to be fuzzy $\tilde{\xi}$-structure open set and the complement of each fuzzy $\tilde{\xi}$-structure open set is said to be fuzzy $\tilde{\xi}$-structure closed.

Definition 3.6: Let (X, τ) be a FTS and (S(X), $\tilde{\xi}$) be a fuzzy $\tilde{\xi}$-structure space and let η_X : (X, τ) → (S(X), $\tilde{\xi}$). If λ ∈ I^X is a fuzzy set in (X, τ) and $\eta_X(\lambda)$ = Fcl(Fα-Ext(λ), then η_X is said to be a fuzzy quasi-homeomorphism with respect to be fuzzy $\tilde{\xi}$-structure space.

Remark 3.2: Here (S(X), $\tilde{\xi}$) is a fuzzy α-Ext Sober Space, by Definition 3.3.

Proposition 3.6:

If f : (X_1, τ_1) → (X_2, τ_2) and η_{X_2} : (X_2, τ_2) → (S(X_2), $\tilde{\xi}_2$) are fuzzy quasi-homeomorphisms, then $\eta_{X_2} \circ$ f is also a fuzzy quasi-homeomorphism.

Proof:

Let $\lambda \in I^{X_1}$, $\mu \in I^{X_2}$, σ ∈ S(X_2) be any three fuzzy open sets in (X_1, τ_1),(X_2, τ_2) and (S(X_2), $\tilde{\xi}_2$) respectively. Since f is fuzzy quasi-homeomorphism, λ = f^{-1}(μ). Also since η_{X_2} is fuzzy quasi-homeomorphism, μ = $\eta_{X_2}^{-1}$(σ). To prove $\eta_{X_2} \circ$ f is fuzzy quasi-homeomorphism,

$$(\eta_{X_2} \circ f)^{-1}(\sigma) = (f^{-1} \circ \eta_{X_2}^{-1})(\sigma) = f^{-1}(\eta_{X_2}^{-1}(\sigma)) = f^{-1}(\mu) = \lambda.$$

Hence $\eta_{X_2} \circ$ f is fuzzy quasi-homeomorphism.

Definition 3.7: Let (S(X_1), $\tilde{\xi}_1$) and (S(X_2), $\tilde{\xi}_2$) be any two fuzzy $\tilde{\xi}$ structure spaces. A function S(f) : (S(X_1), $\tilde{\xi}_1$) → (S(X_2), $\tilde{\xi}_2$) is said to be a fuzzy $\tilde{\xi}$-structure continuous function if for each fuzzy $\tilde{\xi}$-structure open set λ ∈ $I^{S(X2)}$, $S(f)^{-1}$(λ) is fuzzy $\tilde{\xi}$-structure open set in(S(X_1), $\tilde{\xi}_1$).

Definition 3.8: Let (S(X_1), $\tilde{\xi}_1$), (S(X_2), $\tilde{\xi}_2$) be any two fuzzy $\tilde{\xi}$ structure spaces and let S(f) :(S(X_1), $\tilde{\xi}_1$) → (S(X_2), $\tilde{\xi}_2$). Then S(f) is said to be fuzzy homeomorphism if S(f) is bijective and S(f) and $S(f)^{-1}$ are fuzzy $\tilde{\xi}$-structure continuous functions.

Proposition 3.7:

Let (X_1, τ_1) and (X_2, τ_2) be any two FTSs and let (S(X_1), $\tilde{\xi}_1$), (S(X_2), $\tilde{\xi}_2$) be any two fuzzy $\tilde{\xi}$-structure spaces. Let f : (X_1, τ_1) → (X_2, τ_2) be a fuzzy continuous function and S(f) : (S(X_1), $\tilde{\xi}_1$) → (S(X_2), $\tilde{\xi}_2$

). Let η_{X_1} : (X_1, τ_1) → ($S(X_1)$, $\tilde{\xi}_1$) and η_{X_2} : (X_2, τ_2) → ($S(X_2)$, $\tilde{\xi}_2$) be any two fuzzy quasi-homeomorphism. Then the following statements are equivalent:

(i) f is a fuzzy onto quasi-homeomorphism,
(ii) S(f) is a fuzzy homeomorphism.

Proof:

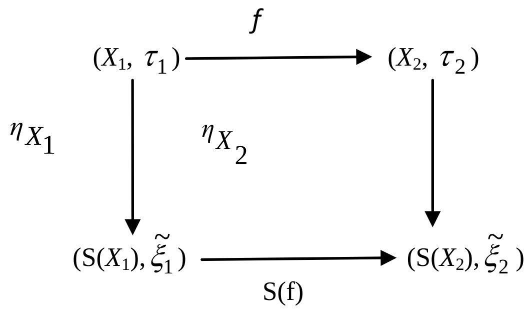

Figure 1

(i) ⇒ (ii) Given η_{X_1}, η_{X_2}, f are fuzzy quasi-homeomorphisms, by Proposition 3.6, $\eta_{X_2} \circ f$ is fuzzy quasi-homeomorphism and $\eta_{X_2} \circ f$: (X_1, τ_1) → ($S(X_2)$, $\tilde{\xi}_2$). Since S(f) : ($S(X_1)$, $\tilde{\xi}_1$) → ($S(X_2)$, $\tilde{\xi}_2$) and S(f) ∘ η_{X_1} : (X_1, τ_1) → ($S(X_2)$, $\tilde{\xi}_2$), by Figure 1. Therefore

$\eta_{X_2} \circ f = S(f) \circ \eta_{X_1}$. Since $\eta_{X_2} \circ f$ is fuzzy quasi-homeomorphism, S(f) ∘ η_{X_1} is also fuzzy quasi-homeomorphism. Hence S(f) is fuzzy quasi-homeomorphism. It is enough to prove that S(f) is bijective.

To prove S(f) is onto: Let λ $\in I^{X_2}$ and also let $\eta_{X_2}(\lambda) \in S(X_2)$. Given that f is onto. Then there exists μ $\in I^{X_1}$ such that λ = f(μ). Thus Fcl(Fα-Ext(λ)) = Fcl(Fα-Ext(f(μ))). Hence by Definition 3.6, $\eta_{X_2}(\lambda) = \eta_{X_2}(f(\mu))$. Since $\eta_{X_2} \circ f = S(f) \circ \eta_{X_1}$, $\eta_{X_2}(\lambda) = S(f)(\eta_{X_1}(\mu))$. Therefore S(f) is onto.

To prove S(f) is one-to-one: Let $\eta_{X_1}(\mu)$, $\eta_{X_1}(\mu') \in S(X_1)$ be such that $S(f)(\eta_{X_1}(\mu)) = S(f)(\eta_{X_1}(\mu'))$. Since $\eta_{X_2} \circ f = S(f) \circ \eta_{X_1}$, $\eta_{X_2}(f(\mu)) = \eta_{X_2}(f(\mu'))$. Hence Fcl(F$\alpha$-Ext(f(μ))) = Fcl(F$\alpha$-Ext(f(μ′))), by Definition 3.6. To proveFcl(Fα-Ext(μ)) = Fcl(Fα-Ext(μ′)). It is sufficient to show that Fcl(Fα-Ext(μ)) ≤ Fcl(Fα-Ext(μ′)). Let δ $\in I^{X_1}$ be a fuzzy open set in (X_1, τ_1) with μ ≤ δ and σ $\in I^{X_2}$ be a fuzzy open set in (X_2, τ_2). Since f : (X_1, τ_1) → (X_2, τ_2) is fuzzy quasi-homeomorphism, δ = f^{-1}(σ). Since μ ≤ δ and δ = f^{-1}(σ), μ ≤ f^{-1}(σ). Then f(μ) ≤ σ. It follows that f(μ′) ≤ σ implies that μ′ ≤ f^{-1}(σ). Thus μ′ ≤ δ, since δ = f^{-1}(σ). Therefore S(f) is a bijective and fuzzy quasi-homeomorphism. Since bijective fuzzy quasi-homeomorphism is fuzzy homeomorphism, S(f) is fuzzy homeomorphism.

(ii) ⇒ (i) Assume that S(f) is fuzzy homeomorphism and η_{X_1}, η_{X_2} are fuzzy quasi-homeomorphism. Since S(f) ∘ $\eta_{X_1} = \eta_{X_2} \circ f$ is commutative, by Proposition 3.6, f is fuzzy quasi-homeomorphism. It remains to show that f is onto. Let λ $\in I^{X_2}$. Since S(f) is onto, there exists μ $\in I^{X_1}$ such that $S(f)(\eta_{X_1}(\mu)) = \eta_{X_2}(\lambda)$. Thus $\eta_{X_2}(f(\mu)) = \eta_{X_2}(\lambda)$. Therefore Fcl(F$\alpha$-Ext(λ)) = Fcl(F$\alpha$-Ext(f(μ))). Therefore f is onto. Hence f is fuzzy onto quasi-homeomorphism.

Proposition 3.8: [First Extension Theorem for Fα-Ext Sober space]

Let (X_1, τ_1), (X_2, τ_2) and (X_3, τ_3) be any three FTSs and let ($S(X_1)$, $\tilde{\xi}_1$), ($S(X_2)$, $\tilde{\xi}_2$) and ($S(X_3)$, $\tilde{\xi}_3$) be any three fuzzy $\tilde{\xi}$ structure spaces. Also let η_{X_1} : (X_1, τ_1) → ($S(X_1)$, $\tilde{\xi}_1$) and η_{X_2} : (X_2, τ_2) →($S(X_2)$, $\tilde{\xi}_2$). Then the following statements are equivalent:

(i) (X_3, τ_3) is a Fα-Ext Sober space;
(ii) for each fuzzy quasi-homeomorphism q : (X_1, τ_1) → (X_2, τ_2) and each fuzzy continuous function f : (X_1, τ_1) → (X_3, τ_3), there exists one and only one fuzzy continuous function F : (X_2, τ_2) → (X_3, τ_3) such that F ∘ q = f

Proof:

(i) ⇒ (ii) Assume that such $F : (X_2, \tau_2) \to (X_3, \tau_3)$ exists. Then $F \circ q = f$ which implies $S(F) \circ S(q) = S(f)$ where $S(F) : (S(X_2), \tilde{\xi}_2) \to (S(X_3), \tilde{\xi}_3)$ and $S(q) : (S(X_1), \tilde{\xi}_1) \to (S(X_2), \tilde{\xi}_2)$. Given that q is fuzzy quasi-homeomorphism, by Proposition 3.7, S(q) is a fuzzy homeomorphism. Hence $S(F) = S(f) \circ (S(q))^{-1}$. Also $\eta_{X_3} \circ F : (X_2, \tau_2) \to (X_2, \tau_2) \to (S(X_3), \tilde{\xi}_3)$. Therefore $\eta_{X_3} \circ F : (X_2, \tau_2) \to (S(X_3), \tilde{\xi}_3)$. Similarly $S(F) \circ \eta_{X_2} : (X_2, \tau_2) \to (S(X_2), \tilde{\xi}_2) \to (S(X_3), \tilde{\xi}_3)$. Therefore $S(F) \circ \eta_{X_2} : (X_2, \tau_2) \to (S(X_3), \tilde{\xi}_3)$. By Figure 2, $\eta_{X_3} \circ F = S(F) \circ \eta_{X_2}$ commutes. Consequently,

$$F = (\eta_{X_3})^{-1} \circ S(F) \circ \eta_{X_2} = (\eta_{X_3})^{-1} \circ S(f) \circ (S(q))^{-1} \circ \eta_{X_2} \quad \{\because S(F) = S(f) \circ (S(q))^{-1}\}$$

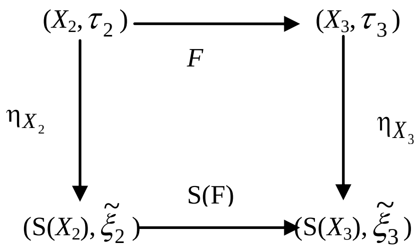

Figure 2

Hence to verify $F : (X_2, \tau_2) \to (X_3, \tau_3)$.

$F = (\eta_{X_3})^{-1} \circ S(f) \circ (S(q))^{-1} \circ \eta_{X_2} : (X_2, \tau_2) \to (S(X_2), \tilde{\xi}_2) \to (S(X_1), \tilde{\xi}_1) \to (S(X_3), \tilde{\xi}_3) \to (X_3, \tau_3)$

$F = (\eta_{X_3})^{-1} \circ S(f) \circ (S(q))^{-1} \circ \eta_{X_2} : (X_2, \tau_2) \to (X_3, \tau_3)$

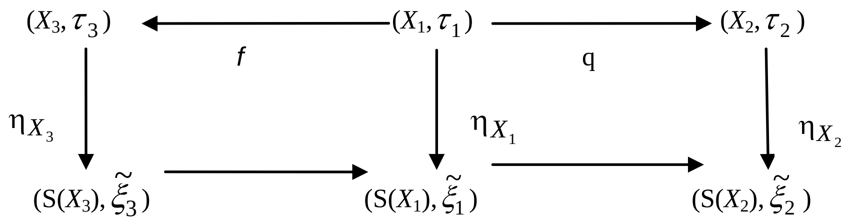

Figure 3

Also the diagram Figure 3 is commutative. Using F, we have to prove $F \circ q = f$. Hence

$$F \circ q = (\eta_{X_3})^{-1} \circ S(f) \circ (S(q))^{-1} \circ \eta_{X_2} = (\eta_{X_3})^{-1} \circ S(f) \circ ((S(q))^{-1} \circ S(q)) \circ \eta_{X_1}$$
$$= (\eta_{X_3})^{-1} \circ S(f) \circ \eta_{X_1} = (\eta_{X_3})^{-1} \circ \eta_{X_3} \circ f = f.$$

(ii) ⇒ (i) There exists a fuzzy continuous function $g : (S(X_3), \tilde{\xi}_3) \to (X_3, \tau_3)$ such that

$g \circ \eta_{X_3} = (X_3, \tau_3) \to (S(X_3), \tilde{\xi}_3) \to (X_3, \tau_3)$

$g \circ \eta_{X_3} = (X_3, \tau_3) \to (X_3, \tau_3) = I_{X_3}$

where I_{X_3} is the identity function in (X_3, τ_3). Therefore, $g \circ \eta_{X_3} = I_{X_3}$ is commutative, by Figure 4. Also Figure 5 is commutative. Similarly,

$\eta_{X_3} \circ g = (S(X_3), \tilde{\xi}_3) \to (X_3, \tau_3) \to (S(X_3), \tilde{\xi}_3)$

$\eta_{X_3} \circ g = (S(X_3), \tilde{\xi}_3) \to (S(X_3), \tilde{\xi}_3) = I_{S(X3)}$

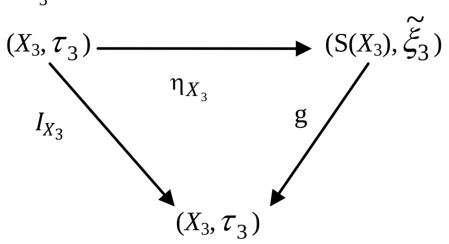

Figure 4:

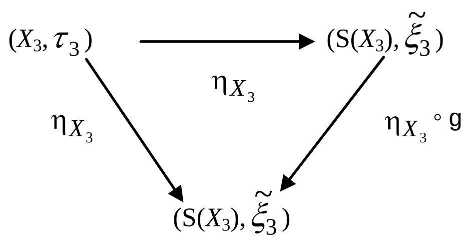

Figure 5:

where $I_{S(X3)}$ is the identity function in $(S(X_3), \tilde{\xi}_3)$. Hence $\eta_{X_3} \circ g = I_{S(X3)} = g \circ \eta_{X_3}$, by (i) ⇒ (ii). Therefore, η_{X_3} is a fuzzy homeomorphism. Therefore (X_3, τ_3) is a Fα-Ext Sober Space.

Proposition 3.9: [Second Extension Theorem for Fα-Ext Sober space]

Let (X_1, τ_1),(X_2, τ_2) and (X_3, τ_3) be any three FTSs and let $(S(X_1), \tilde{\xi}_1)$,$(S(X_2), \tilde{\xi}_2)$ and $(S(X_3), \tilde{\xi}_3)$ be any three fuzzy $\tilde{\xi}$ structure spaces. Letq : $(X_1, \tau_1) \to (X_2, \tau_2)$ be fuzzy continuous function. If for each Fα-Ext Sober Space (X_3, τ_3) and each fuzzy continuous function f : $(X_1, \tau_1) \to (X_3, \tau_3)$, there exists one and only one fuzzy continuous function F : $(X_2, \tau_2) \to (X_3, \tau_3)$ such that $F \circ q = f$. Then q is a fuzzy quasi-homeomorphism.

Proof:

To prove q is a fuzzy quasi-homeomorphism, by Proposition 3.7, it is enough to show that S(q) : $(S(X_1), \tilde{\xi}_1) \to (S(X_2), \tilde{\xi}_2)$ is a fuzzy homeomorphism.

Let $\eta_{X_2} : (X_2, \tau_2) \to (S(X_2), \tilde{\xi}_2)$, $\widetilde{\eta_{X_1}} : (X_2, \tau_2) \to (S(X_1), \tilde{\xi}_1)$ and $g : (S(X_2), \tilde{\xi}_2) \to (S(X_1), \tilde{\xi}_1)$ be such that the Figures 6 commutes. Hence

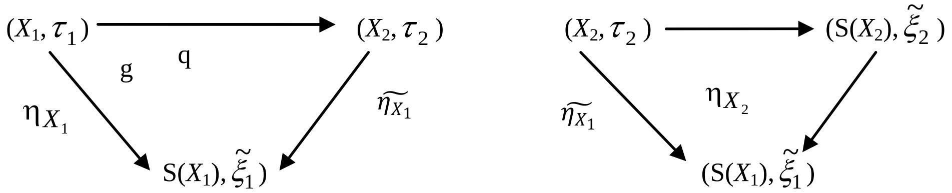

Figure 6

$$g \circ \eta_{X_2} \circ q : (X_1, \tau_1) \to (X_2, \tau_2) \to (S(X_2), \tilde{\xi}_2) \to (S(X_1), \tilde{\xi}_1)$$

$$g \circ \eta_{X_2} \circ q : (X_1, \tau_1) \to (S(X_1), \tilde{\xi}_1)$$

$$\eta_{X_1} : (X_1, \tau_1) \to (S(X_1), \tilde{\xi}_1)$$

$$\therefore\ g \circ \eta_{X_2} \circ q = \eta_{X_1}$$

On the other hand, the rectangle Figure 1 is commutative. Thus $(g \circ S(q)) \circ \eta_{X_1} = g \circ (S(q) \circ \eta_{X_1}) = g \circ (\eta_{X_2} \circ q) = \eta_{X_1}$. Hence $(g \circ S(q)) \circ \eta_{X_1} = \eta_{X_1}$. Thus $g \circ S(q) = I_{S(X1)}$ where $I_{S(X1)}$ is the identity function in $(S(X_1), \tilde{\xi}_1)$. Similarly

$$(S(q) \circ g) \circ (\eta_{X_2} \circ q) = S(q) \circ (g \circ \eta_{X_2} \circ q) = S(q) \circ \eta_{X_1}$$

$$(S(q) \circ g) \circ (\eta_{X_2} \circ q) = \eta_{X_2} \circ q (\because\ S(q) \circ \eta_{X_1} = \eta_{X_2} \circ q)$$

$$\therefore\ S(q) \circ g = I_{S(X2)}$$

Where $I_{S(X2)}$ is the identity function in $(S(X_2), \tilde{\xi}_2)$. To prove $\eta_{X_2} \circ q$ is fuzzy quasi-homeomorphism (i.e., Figure 7). Since η_{X_2} is fuzzy quasi-homeomorphism, it is enough to show that q is fuzzy quasi-homeomorphism.

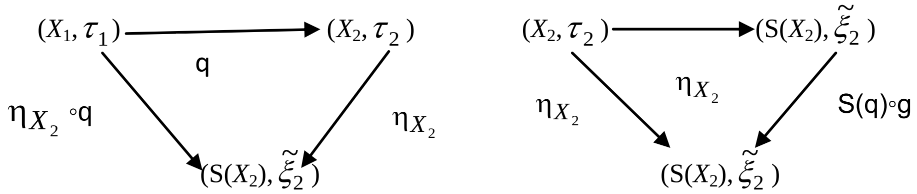

Figure 7

$$(S(q) \circ g) \circ \eta_{X_2} : (X_2, \tau_2) \to (S(X_2), \tilde{\xi}_2) \to (S(X_1), \tilde{\xi}_1)$$

$$(S(q) \circ g) \circ \eta_{X_2} : (X_2, \tau_2) \to (S(X_2), \tilde{\xi}_2)$$

$$\eta_{X_2} : (X_2, \tau_2) \to (S(X_2), \tilde{\xi}_2).$$

Therefore (S(q) ∘ g) ∘ η_{X_2} = η_{X_2} (By Figure 7). Since η_{X_2} is fuzzy quasi-homeomorphism and it is known that S(q) ∘ g = $I_{S(X2)}$, S(q) is fuzzy homeomorphism. By Proposition 3.7, q is fuzzy quasi-homeomorphism.

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