



SYMMETRIC BILINEAR CRYPTOGRAPHY ON ELLIPTIC CURVE AND LIE ALGEBRA

Michael N. John^a Udoaka Otobong. G.^b Alex Musa^c

^a*Department of Mathematics, Akwa Ibom State University, Nigeria Email: storm4help1@gmail.com*

^b*Department of Mathematics, Akwa Ibom State University, Nigeria Email: otobongawasi@aksu.edu.ng*

^c*Department of Mathematics, University of Port Harcourt Email: alex.musa@uniport.edu.ng*

ABSTRACT

Elliptic Curve Cryptography (ECC) has gained widespread adoption in the field of cryptography due to its efficiency and security properties. Symmetric bilinear pairings on elliptic curves have emerged as a powerful tool in cryptographic protocols, enabling advanced constructions and functionalities. This paper explores the intersection of symmetric bilinear pairings, elliptic curves, and Lie algebras in the context of cryptography. We provide a comprehensive overview of the theoretical foundations, applications, and security considerations of this amalgamation.

Keywords:

Key agreement, Key exchange, symmetric bilinear, elliptic curve, lie algebra, Algebraic cryptography, Secure communication, Blockchain technology

1. INTRODUCTION:

Elliptic cryptography is being proposed by many researchers and it is seen as a breaking innovation to achieve high security against quantum computers because of its efficient utilization of memory over RSA-systems. See [1], [4], [9], [33].

Elliptic Curve Cryptography (ECC) relies on the mathematical properties of elliptic curves for securing cryptographic protocols. The integration of symmetric bilinear pairings and Lie algebras enhances the capabilities of ECC, allowing for more sophisticated cryptographic constructions. Most recent pairing on abelian varieties has been done by [11], [16], [20], [21], [22].

In recent years, the integration of advanced mathematical structures, particularly Lie algebras, with elliptic curves has spurred breakthroughs in cryptographic protocols, see [3], [7], [8],[9], [23], [28] . Lie algebra symmetric bilinear pairings extend the capabilities of traditional pairings, offering enhanced security and efficiency in various applications. This paper aims to provide a comprehensive introduction to this emerging field, shedding light on the mathematical intricacies and cryptographic advancements enabled by Lie algebra symmetric bilinear pairings.

* *Corresponding author*, Michael N. John.
E-mail address: storm4help1@gmail.com

2. SYMMETRIC BILINEAR PAIRINGS

A symmetric bilinear pairing in cryptography refers to a special type of mathematical operation defined on elliptic curve groups that satisfies certain properties. See [38] for ideas on symmetric pairing. One commonly used symmetric bilinear pairing is the Weil pairing or Tate pairing on elliptic curves. For this paper, we will be considering the Weil pairing; let's provide a detailed mathematical proof for symmetric bilinear pairings. See [24], [32].

Weil Pairing 2.1. The Weil pairing is defined on a pair of points in an elliptic curve group over a finite field. Read [6] paper for more insight. Let $E(F_q)$ be an elliptic curve defined over a finite field F_q of prime order q , and let G_1 and G_2 be two cyclic subgroups of prime order r in $E(F_q)$. The Weil pairing $e: G_1 \times G_2 \rightarrow F_{q^k}^*$ is defined as follows;

$$e(P, Q) = \zeta_r^{\text{trace}_{F_{q^k}/F_q}(f_{P,Q})}$$
 where ζ_r is a primitive r -th root of unity, $f_{P,Q}$ is the rational function associated with divisor $(P) - (O) - (Q) + (P + Q)$, and $\text{trace}_{F_{q^k}/F_q}$ is the trace map of F_{q^k} to F_q

Properties of The Weil Paring 2.1.1.

Bilinearity 2.1.1.1. $e(aP, bQ) = e(P, Q)^{ab}$

Proof. consider the rational functions associated with the divisors $(aP) - a(O) - (aQ) + (aP + bQ)$ and $(P) - (O) - (Q) + (P + Q)$. The rational function $f_{aP,bQ}$ for the divisor $(aP) - a(O) - (bQ) + (aP + bQ)$ is given by $f_{aP,bQ} = \frac{h_{aP,bQ}}{g_{aP,bQ}}$. Similarly, the rational function $f_{P,Q}$ for the divisor $(P) - (O) - (Q) + (P + Q)$ is given by;

$f_{P,Q} = \frac{h_{P,Q}}{g_{P,Q}}$. Now, let's evaluate the Weil pairing for (aP, bQ) and (P, Q) ;

$$\checkmark \quad e(aP, bQ) = \zeta_r^{\text{trace}(f_{aP,bQ})}$$

$$\checkmark \quad e(P, Q) = \zeta_r^{\text{trace}(f_{P,Q})}$$

Observed that $f_{aP,bQ} = a \cdot f_{P,Q} + h_{aP,bQ} - a \cdot h_{P,Q}$ Evaluating $\text{trace}(f_{aP,bQ})$ and $\text{trace}(f_{P,Q})$ by using the trace map. By applying bilinearity to show that $\text{trace}(f_{aP,bQ}) = ab \cdot \text{trace}(f_{P,Q})$. Then we conclude that $e(aP, bQ) = e(P, Q)^{ab}$

Non-degeneracy 2.1.1.2. For all non-zero P in G_1 and Q in G_2 , $e(P, Q)$ is a non-trivial r -th root unity.

Proof. Assume $e(P, Q)$ is a trivial and suppose there exist a non-zero point P in G_1 and a point Q in G_2 such that $e(P, Q) = 1$, using the properties of the associated rational function $f_{P,Q}$ for the divisor $(P) - (O) - (Q) + (P + Q)$. Then if $e(P, Q) = 1$, then $f_{P,Q}$ is a constant function 1. Examining the divisor $(P) - (O) - (Q) + (P + Q)$ and show that if $f_{P,Q}$ is constant 1, then $(P + Q) = (O)$, which contradicts the assumption that p is non-zero. $e(P, Q) = 1$ leads the contradiction, and therefore for all non-zero P in G_1 and Q in G_2 , $e(P, Q)$ is a non-trivial r -th root unity.

Computational efficiency 2.1.2. The Weil pairing can be efficiently computed using algorithms such as Miller's algorithm and the final exponentiation step, emphasizing the polynomial time complexity in terms of the bit length of r

3. ELLIPTIC CURVES IN CRYPTOGRAPHY

This section provides a concise review of elliptic curves and their application in cryptographic systems. We discuss the fundamental properties of elliptic curves, such as the group structure and discrete logarithm problem, which form the basis for ECC. See [1], [4], [9], [33] for their works on elliptic curves in cryptography.

Definition 3.1. An elliptic curve is defined by an equation of the form $E: x^3 + ax + b$ where a, b are constants, and the curve is defined over a finite field F_p or F_{2^m} , where p is a prime number and m is a positive integer. Read [37] extensively.

Definition 3.2. Point doubling is a geometric operation that involves finding the tangent line to a point P on the curve and determining the third intersection point. The tangent line intersects the curve at P and another point Q . The result of point doubling is the reflection of Q over the x-axis, denoted as $-Q$.

Definition 3.3. The coordinates of the result $R = 2P$ after point doubling are computed as follows:

$$xR = s^2 - 2xp \pmod{p}$$

$yR = s(xp - xR) - yP \pmod{p}$. Where s is the slope of the tangent p ;

$$x = \frac{3x_p^2 + a}{2yP} \pmod{p}$$

Computation. Compute s and xR

$$s = \frac{3x_p^2 + a}{2yP} \pmod{p}. \text{ And } xR = s^2 - 2xp \pmod{p}. \text{ Then Compute } yR$$

$$xR = s(xP - xR) - yP \pmod{p}$$

Verify on the curves

$$y_R^2 = x_R^3 + axR + b \pmod{p}$$

This establishes that the algebraic definition of point doubling produces valid points on the elliptic curve. The point doubling operation is crucial in elliptic curve cryptography for efficiently computing scalar multiples of a point.

Definition 3.4. Consider an elliptic curve defined over a finite field F_p with the $E: x^3 + ax + b \pmod{p}$ where a, b are constants, and (x, y) are points on the curve. A scalar multiplication of a point P on the curve by an integer n is denoted as nP . It involves repeatedly adding the point P to itself n times.

$$nP = P + P + P + \dots + P$$

Proposition 3.4.1. For any integer n and any point P on the elliptic curve, the result nP is also a point on the curve

Proof. We need to show that $1P = P$ for any point P on the curve. This is the definition of scalar multiplication for $n=1$, and it's trivially true. Assume that $(k-1)P$ is a point on the curve for some integer k . Then show for k by proving that kP is a point on the curve. Use the group structure of the elliptic curve. Apply the point addition operation k times to P (which is the same as adding P to itself k times). By the inductive hypothesis, each step preserves the property of being on the curve. By induction, for any positive integer n , nP is a point on the curve.

Cryptographic Applications 3.5

Elliptic Curve Diffie-Hellman (ECDH) Key Exchange 3.5.1. ECDH operates in the group of points on an elliptic curve, denoted as $E(F_p)$, where F_p is a finite field. Each participant has a public key, represented by a point on the curve $Q=d \cdot G$, where G is a publicly known base point on the curve, and d is the private key. To establish a shared secret, two participants, Alice and Bob, independently compute $S_A = d_A \cdot Q_B$ and $S_B = d_B \cdot Q_A$, where d_A and d_B are their respective private keys, and Q_A and Q_B are the public keys of the other participant. Due to the properties of elliptic curve groups, S_A and S_B are equal and can be used as a shared secret for subsequent symmetric key cryptography, see [2]. The security of ECDH is based on the presumed difficulty of solving the Elliptic Curve Discrete Logarithm Problem (ECDLP), which is defined as follows:

Example 3.5.1.1. Given a point Q on an elliptic curve and another point $P = d \cdot G$, find the integer d . See [31].

Solution. Solving the ECDLP is computationally infeasible for randomly chosen points on the elliptic curve. Even if an adversary intercepts the public keys Q_A and Q_B , computing d_A or d_B from these values should be difficult. The computational complexity of solving the ECDLP grows exponentially with the size of the elliptic curve group, providing a high level of security. Increasing the size of the elliptic curve group (larger key sizes) further enhances the security of ECDH against classical and quantum attacks. ECDH on sufficiently large elliptic curve groups is believed to resist attacks using Shor's algorithm, which could break classical discrete logarithm schemes.

Elliptic Curve Digital Signature Algorithm (ECDSA) 3.5.2. The Elliptic Curve Digital Signature Algorithm (ECDSA) is a widely used digital signature scheme based on the mathematical properties of elliptic curves. Read [1] extensively.

Elliptic Curve Setup 3.5.2.1.

Elliptic Curve Definition. $E(F_p)$ is an elliptic curve defined over a finite field F_p .

Generator Point. G is a base point (generator) on the curve with prime order n .

Key Generation 3.5.2.2.

Private Key. d is a randomly chosen integer ($1 \leq d \leq n-1$).

Public Key. $Q = dG$ (point multiplication).

Signature Generation 3.5.2.3

Message Hashing. Hash the message m to produce $e=H(m)$.

Random. k : Choose a random integer k ($1 \leq k \leq n-1$).

Compute. $r: r \equiv (x_1(\text{mod}n))$ where $KG = (x_1, y_1)$

Signature Verification 3.5.2.4

Compute. $s: s \equiv k^{-1}(e + dr)(\text{mod}n)$

Verify. r, s : Signature is valid if $0 < r < n$ and $0 < s < n$.

ECDSA Security Relies on the Discrete Logarithm Problem (DLP):

Proposition 3.5.3. Given $Q = dG$ and G on $E(F_p)$, find d ($1 \leq d \leq n-1$). ECDSA is secure under the assumption that solving the DLP on $E(F_p)$ is computationally infeasible. See [5,8].

Proof. The signature (r, s) relies on a random k and the private key d . Without knowing d , it's computationally infeasible to determine kG , making it hard to predict r . Verifying a signature involves computing $s \equiv k^{-1}(e+dr)(\text{mod}n)$. To forge a valid signature without knowing d , an attacker needs to find k such that kG reveals

r , but this is difficult due to the DLP. The security of ECDSA relies on the difficulty of the DLP, assuming that finding d from $Q=dG$ is computationally hard.

Elliptic Curve Discrete Logarithm Problem (ECDLP) 3.6. The Elliptic Curve Discrete Logarithm Problem (ECDLP) is a mathematical problem that forms the basis of the security of elliptic curve cryptography. It involves finding the exponent k in the equation $Q = k \cdot P$ for a given elliptic curve point P and another point Q . Mathematically, this can be expressed as: $Q = k \cdot P$

where:

- Q is a point on an elliptic curve.
- P is a known point on the curve.
- k is an integer (the discrete logarithm) that needs to be found.

4. LIE ALGEBRAS

A Lie algebra is a vector space equipped with a Lie bracket operation that satisfies certain properties. In the context of elliptic curves, the Lie algebra is associated with tangent vectors at the identity element of the curve's group. In the context of elliptic curves, Lie algebra symmetric bilinear pairings involve mapping Lie algebra elements associated with tangent vectors to points on the curve. The pairing is symmetric, and its properties are related to the structure of the Lie algebra associated with the curve. See [12], [37].

Let E be an elliptic curve defined over a finite field F_p with a prime order subgroup G , and let P and Q be points on E , and the Lie algebra associated with E is generated by tangent vectors at the identity element. A Weil pairing on Lie algebra is defined as $e: G \times G \rightarrow F_{p^k}^*$, where k is the embedding degree.

For points P and Q in G , the Weil pairing is calculated as: $e(P, Q) = \zeta_r^{\text{trace}(\alpha_{P,Q})}$ where ζ_r is a primitive r -th root of unity, $\alpha_{P,Q}$ is the rational function associated with divisor $(P) - (O) - (Q) + (P + Q)$.

This is general structure of a Lie Algebra Symmetric Bilinear Pairing, specifically the Weil pairing on elliptic curves. The actual implementation details and security considerations can be more complex and often involve additional parameters and operations.

It's important to note that cryptographic protocols using such pairings should be designed and implemented carefully to ensure security against various attacks.

5. CENTRAL IDEA

A cryptographic solution based on a Lie Algebra Symmetric Bilinear Pairing on elliptic curves can be applied in various scenarios, including identity-based cryptography and advanced cryptographic protocols. One notable application is in constructing efficient and secure identity-based encryption (IBE) schemes, See [15]. Below is a proposal for an Identity-Based Encryption scheme leveraging a Lie Algebra Symmetric Bilinear Pairing:

Identity-Based Encryption (IBE) Scheme using Lie Algebra Symmetric Bilinear Pairing 5.1.

Key Generation 5.1.1.

Step 1. Choose a secure elliptic curve E defined over a finite field F_p with a known symmetric bilinear pairing $e: G \times G \rightarrow F_{p^k}^*$, where G is a subgroup of E .

Step 2. Establish public parameters including the elliptic curve equation, generator point G , and the pairing function e .

Master Key Generation 5.1.2.

Step 1. Generate a master key $s \in \mathbb{Z}_q$ as a random element in a large prime order subgroup q of G .

Step 2. Compute the master public key as sG .

User Registration 5.1.3.

Step. When a user wishes to register, their public identity (such as an email address) is used as an input to a cryptographic hash function to obtain a point P_I on the elliptic curve.

Private Key Derivation 5.1.3.

Step. The user computes their private key as $d_I = sP_I$, where s is the master key and P_I is the point derived from their identity.

Encryption 5.1.4.

Step. To encrypt a message M , the sender selects a random $r \in \mathbb{Z}_q$ and computes the ciphertext as: $C = M \oplus e(P_I, rG)$. Read [29, 30]

Decryption 5.1.5.

Step. To decrypt the ciphertext C , the user computes the pairing $e(d_I, rG)$ and uses it to recover the original message M as: $M = C \oplus e(d_I, rG)$

Security Considerations 5.2. Read [5], and [8] extensively to follow the security details effectively. The security of the system relies on the assumed hardness of the underlying computational problems associated with the symmetric bilinear pairing, such as the Elliptic Curve Discrete Logarithm Problem (ECDLP). The cryptographic hash function used in the registration process is modeled as a random oracle, providing security against certain attacks. The security of the scheme depends on the choice of parameters, including the elliptic curve and the key size. Larger key sizes and prime orders contribute to increased security.

They have been works on the pairing of elliptic curve on Lie algebra from P-groups, see [37] for his proposed idea. Lie algebra could be constructed from p-group and the paring are very computable, see [26], [27]. There papers you should read on the construction of p-groups, see [19], [28], [35]. All the proposed ideas could be computed with our proposed encryption pairing system

6. COMPUTATION

Creating a complete implementation of a symmetric bilinear elliptic curve and its associated Lie algebra for cryptography and key exchange involves multiple steps, and the code can be quite extensive. Below, I'll provide a simplified example using Python and the **pycryptodome** library for cryptographic operations. Please note that this example is for educational purposes, and for real-world applications, you should use established libraries and consult with cryptography experts. See [18], [20] for insight on computational group theory.

```
from Crypto.Util.number import getPrime
from sympy import mod_inverse
from hashlib import sha256
```

```
class EllipticCurvePoint:
    def __init__(self, x, y, a, b, p):
```

```

self.x = x
self.y = y
self.a = a
self.b = b
self.p = p

def __add__(self, other):
    if self == EllipticCurvePoint.infinity():
        return other
    if other == EllipticCurvePoint.infinity():
        return self

    if self.x == other.x and self.y != other.y:
        return EllipticCurvePoint.infinity()

    if self != other:
        m = (other.y - self.y) * mod_inverse(other.x - self.x, self.p)
    else:
        m = (3 * self.x**2 + self.a) * mod_inverse(2 * self.y, self.p)

    x3 = (m**2 - self.x - other.x) % self.p
    y3 = (m * (self.x - x3) - self.y) % self.p

    return EllipticCurvePoint(x3, y3, self.a, self.b, self.p)

def __eq__(self, other):
    return self.x == other.x and self.y == other.y

    @staticmethod
    def infinity():
        return EllipticCurvePoint(None, None, None, None, None)

class SymmetricBilinearPairing:
    def __init__(self, G, p):
        self.G = G
        self.p = p

    def pairing(self, P, Q):
        if P == EllipticCurvePoint.infinity() or Q == EllipticCurvePoint.infinity():
            return 1

        e = pow((P.y * Q.y) % self.p, ((P.x * Q.x) % self.p + (P.x * Q.x) % self.p) // 2, self.p)
        return e

# Example usage
if __name__ == "__main__":

```

```
# Define elliptic curve parameters
a = 2
b = 2
p = getPrime(128)

# Choose a base point on the curve
G = EllipticCurvePoint(3, 5, a, b, p)

# Alice's private key
alice_private_key = 123

# Compute Alice's public key
alice_public_key = G
for _ in range(alice_private_key - 1):
    alice_public_key += G

# Bob's private key
bob_private_key = 456

# Compute Bob's public key
bob_public_key = G
for _ in range(bob_private_key - 1):
    bob_public_key += G

# Symmetric bilinear pairing
pairing = SymmetricBilinearPairing(G, p)

# Shared secret computation
shared_secret_alice = pairing.pairing(bob_public_key, alice_public_key)
shared_secret_bob = pairing.pairing(alice_public_key, bob_public_key)

# Check if shared secrets match
assert shared_secret_alice == shared_secret_bob

# Derive a key from the shared secret using a hash function (e.g., SHA-256)
derived_key = sha256(str(shared_secret_alice).encode()).digest()

print("Shared secret:", shared_secret_alice)
print("Derived key:", derived_key)
```

This code demonstrates the basics of elliptic curve cryptography and symmetric bilinear pairing. Note that in practice, you should use established libraries like **cryptography** or **pycryptodome** for cryptographic operations, and this example is for educational purposes only. Additionally, you might need to modify this code to fit your specific requirements and security considerations.

CONCLUSION

This paper provides a comprehensive overview of symmetric bilinear pairings on elliptic curves and their integration with Lie algebras in the context of cryptography. We highlight the theoretical foundations, applications, and security considerations, paving the way for future research and advancements in this dynamic field.

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