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On Orthogonal Parts of a Solution to a Cauchy BVP over Sobolev Spaces

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Abstract

Let Ω be a smooth and bounded doamin in $\mathbb{R}^n$. Considering three BVPs.

(I) First order: Let $f \in \mathfrak{L}^2(\Omega), g \in W^{\frac{1}{2},2}(\Omega)$. Then the first order Cauchy BVP :

$$\begin{cases} Du = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

has a solution u given as $W^{1,2}(\Omega) \ni u = [u_g] \uplus [u_f]$ where $[u_g]$ to be the part of the solution that is evolved from the trace value g of u, and $[u_f]$ to be the part of the solution that is evolved from the value f of the differential equation over the domain Ω.

(II) Second order: Let $f \in \mathfrak{L}^2(\Omega), g_1 \in W^{\frac{3}{2},2}(\partial\Omega), g_2 \in W^{\frac{1}{2},2}(\partial\Omega)$.

Then the BVP:

$$\begin{cases} -D^2 u = f & \text{in } \Omega \\ \tau u = g & \text{on } \partial\Omega \end{cases}$$

where $\tau u = \left(u_{|\partial\Omega}, Du_{|\partial\Omega}\right) = (g_1, g_2)$ has a solution $u \in W^{2,2}(\Omega)$ with $u = [u]_{(g_1,g_2)} \uplus [u]_f$ and

(III) Higher order: Let $f \in \mathfrak{L}^2(\Omega), g_j \in W^{k-j+\frac{1}{2}}(\partial\Omega), j = 1, ..., k-1$. Then

$$\begin{cases} D^k u = f & \text{in } \Omega \\ \tau u = g & \text{on } \partial\Omega \end{cases}$$

where $\tau u = \left(u_{|\partial\Omega}, Du_{|\partial\Omega}, ..., D^{k-1}u_{|\partial\Omega}\right) = (g_1, g_2, ..., g_{k-1})$, for $k \geq 3$, has a solution $u \in W^{k,2}(\Omega)$ given as $u = [u]_{(g_1,g_2,...,g_{k-1})} \uplus [u]_f$.

The symbol $\uplus$ represents an orthogonal sum of functions that are from orthogonal sum $\oplus$ of function subspaces of a Sobolev space with inner product.

Keywords
Dirac Operator, Orthogonal Sum, Cauchy Problem, Sobolev Space.

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Contents

1. Preliminaries

Question! How are solutions to boundary value problems in an inner product Sobolev space that is orthogonally decomposable behave? What part of the solution falls in to which part of the summands that are closed subspaces of the space? This is an interesting question to study. In this article was investigated such decompositions for first, second and higher order PDEs in a Sobolev space. In fact solutions in this case appear to evolve from the trace values on the boundary and the values of differential equations in the interior of the domain to become orthogonal components. In [1],[4],[6],[7] was successfully developed decomposition results of Hilbert and Sobolev spaces in general. Was developed properties of the inner product that are defined and that of functions in the respective spaces. In [1] there is seen how norm is enlarged and space is expanding when the regularity exponent increases.

In this paper, I will investigate and show, a solution of a partial differential equation in Ω with a boundary value over $\partial\Omega$ over function spaces: $W^{k,2}(\Omega)$ is actually an orthogonal sum of those parts of the solution that evolve from values of the derivative of the solution to a certain specific order in the interior of Ω and from trace value of the solution on the boundary $\partial\Omega$ of the domain.

That is, for $f \in \mathfrak{L}^2(\Omega)$, the BVP:

$$\begin{cases} D^\alpha u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

of order α for $1 < |\alpha| \le d$ with d a positive integer, has a solution u in a certain Sobolev space $W^{k,2}(\Omega)$ of an inner product $\langle .,. \rangle_{W^{k,2}(\Omega)}$. Besides, the solution u be an orthogonal sum of its parts, i.e.,

$$u = [u]_f \uplus [u]_g \tag{1.2}$$

so that

$$(i) \qquad \left\langle [u]_f, [u]_g \right\rangle_{W^{k,2}(\Omega)} = 0$$

$$(ii) \qquad \|u\|^2_{W^{k,2}(\Omega)} = \| [u]_f \|^2_{W^{k,2}(\Omega)} + \| [u]_g \|^2_{W^{k,2}(\Omega)}$$

where $[u]_f$ is the part of the solution that evolves from f and $[u]_g$ is the part that evolves from g.

Was also investigate the maximum principle for the solution of the first order BVP when f is monogenic or Clifford analytic function over the domain.

2. Orthogonality

Let d be a positive integer and Ω be a smooth and bounded domain in $\mathbb{R}^d$ with a non empty boundary $\partial\Omega$. Let $\alpha \in \mathbb{N}$, $1 < p < \infty$.

Definition 1 *(Weak Derivative) For a function f, it is say g is the weak or generalized α^{th} order derivative of f over Ω written as*

$$g = D^\alpha f$$

if

$$\int_\Omega f(x) D^\alpha \psi(x) d\Omega_x = (-1)^\alpha \int_\Omega g(x)\psi(x) d\Omega_x, \forall \psi \in C_0^\infty(\Omega).$$

Clearly a function that is a derivative in the ordinary sense of a function is a weak derivative but not the converse. That is a weakly differentiable function may not be differentiable in ordinary sense.

Example 1 *Consider the function*

$$f(x) = \begin{cases} 0 & 0 \le x \le 1 \\ x-1 & 1 \le x \le 2 \end{cases}.$$

Then f is continuous on $[0,1]$ but not differentiable in the regular sense as the derivative

$$g(x) = \begin{cases} 0 & 0 < x < 1 \\ 1 & 1 < x < 2 \end{cases}$$

is discontinuous at $x = 1$. However g is the first order weakly or generalized derivative of f on $[0,1]$ since

$$\begin{aligned} \int_{[0,2]} f(x)\phi'(x)dx &= \int_{[1,2]} x\phi'(x)dx - \int_{[1,2]} \phi'(x)dx \\ &= x\phi(x)\,|_1^2 - \int_{[1,2]} \phi(x)dx - \phi(2) + \phi(1) \\ &= -\int_{[1,2]} \phi(x)dx \\ &= -\int_{[0,2]} g(x)\phi(x)dx \\ &= -\int_{[0,2]} f'(x)\phi(x)dx \end{aligned}$$

$\forall \phi \in C_0^\infty([0,2])$.

Definition 2 *For $1 < p < \infty$, $k \in \mathbb{N} \cup \{0\}$, the Sobolev space $W^{k,p}(\Omega)$ is defined as the set of all functions f in $\mathfrak{L}^p(\Omega)$ such that the α^{th} order weak derivative $D^\alpha f \in \mathfrak{L}^p(\Omega)$ for $0 \le |\alpha| \le k$.*

As I mentioned in the preliminary, these function spaces are ideal spaces to search for solutions to problems of reality unlike regular function spaces such as C^k, where continuity to an order is required.

The particular Sobolev space (or Hilbert space of higher regularity) $W^{k,2}(\Omega)$ where $p=2$, and $k \geq 1$ is an inner product space with inner product

$$\langle f,\ g\rangle_{W^{k,2}(\Omega)} = \int_\Omega \left(\sum \int_{0\leq|\alpha|\leq k} D^\alpha f(x) D^\alpha g(x) \right) d\Omega_x \quad (2.1)$$

with norm given by

$$\|f\|_{W^{k,2}(\Omega)} = \left(\langle f,\ f\rangle_{W^{k,2}(\Omega)} \right)^{\frac{1}{2}}$$

Therefore there is a distance or metric defined in terms of this norm given by

$$\rho_{W^{k,2}(\Omega)}(f,\ g) := \|f-g\|_{W^{k,2}(\Omega)}. \quad (2.2)$$

When $p=2$ and $k=0$, there is the usual Hilbert space

$$W^{0,2}(\Omega) = \mathfrak{L}^2(\Omega)$$

with inner product

$$\langle f,\ g\rangle_{W^{0,2}(\Omega)} = \langle f,\ g\rangle_{\mathfrak{L}^2(\Omega)} = \int_\Omega f(x)\, g(x) d\Omega_x \quad (2.3)$$

.

Definition 3 *It is say two functions $f,\, g \in W^{k,2}(\Omega)$ orthogonal with respect to the inner product defined by 2.1 if*

$$\langle f,\ g\rangle_{W^{k,2}(\Omega)} = 0. \quad (2.4)$$

For more orthogonal functions see [2].

Theorem 1 *For two orthogonal functions f and g of $W^{k,2}(\Omega)$ if $h = f \uplus g$ then*

$$\|h\|^2_{W^{k,2}(\Omega)} = \|f\|^2_{W^{k,2}(\Omega)} + \|g\|^2_{W^{k,2}(\Omega)}.$$

Proof. 1 *Clearly*

$$\begin{aligned}
\|h\|_{W^{k,2}(\Omega)} &= \left(\langle h,\ h\rangle_{W^{k,2}(\Omega)} \right)^{\frac{1}{2}} \\
&= \left(\langle f \uplus g,\ f \uplus g\rangle_{W^{k,2}(\Omega)} \right)^{\frac{1}{2}} \\
&= \left(\langle f,\ f\rangle_{W^{k,2}(\Omega)} + \langle g,\ g\rangle_{W^{k,2}(\Omega)} \right)^{\frac{1}{2}} \\
&= \left(\|f\|^2_{W^{1,2}(\Omega)} + \|g\|^2_{W^{1,2}(\Omega)} \right)^{\frac{1}{2}}
\end{aligned}$$

since $\langle f, g\rangle_{W^{k,2}(\Omega)} = 0.$

$$\implies \quad \|h\|^2_{W^{k,2}(\Omega)} = \|f\|^2_{W^{1,2}(\Omega)} + \|g\|^2_{W^{1,2}(\Omega)}.$$

As indicated above, the interest of the talk is partly on the Sobolev space

$$W^{1,2}(\Omega) = \{f \in \mathfrak{L}^2(\Omega) : Df \in \mathfrak{L}^2(\Omega)\}$$

with inner product

$$\langle f,\ g\rangle_{W^{1,2}(\Omega)} = \int_\Omega (f(x)g(x) + Df(x)Dg(x))\, d\Omega_x. \quad (2.5)$$

there is few preliminary results on norm and orientation for single and pairs of functions.

Theorem 2 *For a function f, if $Df = f$, then*

$$\|f\|_{W^{1,2}(\Omega)} = \sqrt{2}\|f\|_{\mathfrak{L}(\Omega)}$$

in particular

$$\|e^x\|_{W^{1,2}(\Omega)} = \sqrt{2}\|e^x\|_{\mathfrak{L}(\Omega)}.$$

Can be also discuss about angles in inner product spaces, from the fact that

$$\langle f,\ g\rangle_{W^{1,2}(\Omega)} = \|f\|_{W^{1,2}(\Omega)}\|g\|_{W^{1,2}(\Omega)} \cos\theta.$$

Theorem 3 *For $0 < \alpha < \beta < 1$, there is*

$$\rho_{W^{1,2}(\Omega)}(f,\ \alpha f) > \rho_{W^{1,2}(\Omega)}(f,\ \beta f).$$

Proof. 2

$$\begin{aligned}
\rho_{W^{1,2}(\Omega)}(f,\ \alpha f) &= \\
&= \|f - \alpha f\|_{W^{1,2}(\Omega)} \\
&= (\langle f - \alpha f,\ f - \alpha f\rangle)^{\frac{1}{2}} \\
&= \\
\left(\int_\Omega (f(x) - \alpha f(x))^2 + \left(D(f(x) - \alpha f(x))^2 \right) d\Omega_x \right)^{\frac{1}{2}} \\
&= \sqrt{(1-\alpha)^2} \left(\int_\Omega \left(f(x)^2 + Df(x)^2 \right) d\Omega_x \right)^{\frac{1}{2}} \\
&= (1-\alpha)\|f\|_{W^{1,2}(\Omega)} \\
&> (1-\beta)\|f\|_{W^{1,2}(\Omega)} = \rho_{W^{1,2}(\Omega)}(f,\ \beta f)
\end{aligned}$$

since $0 < \alpha < \beta < 1 \Longrightarrow 1 - \alpha > 1 - \beta > 0.$

Corollary 1 *The following are valid,*

(i) $\lim_{\varepsilon \downarrow 0} \rho_{W^{1,2}(\Omega)}(f,\ \varepsilon f) = \|f\|_{W^{1,2}(\Omega)}$

(ii) $\lim_{\varepsilon \longrightarrow 1} \rho_{W^{1,2}(\Omega)}(f,\ \varepsilon f) = 0$

(iii) $\rho_{W^{1,2}(\Omega)}(f,\ -\alpha f) < \rho_{W^{1,2}(\Omega)}(f,\ -\beta f)$ *for* $0 < \alpha < \beta < 1.$

Proposition 1 *For $1<\alpha<\beta$, and $f\in W^{1,2}(\Omega)$, there is*

$$\rho_{W^{1,2}(\Omega)}(f,\ \alpha f)\ \leq\ \rho_{W^{1,2}(\Omega)}(f,\ \beta f).$$

Proof. 3

$$\rho_{W^{1,2}(\Omega)}(f,\ \alpha f)=\|f-\alpha f\|_{W^{1,2}(\Omega)}$$

$$=\ |\,1-\alpha\,|\ \|f\|_{W^{1,2}(\Omega)}$$

$$\leq\quad |\,1-\beta\,|\ \|f\|_{W^{1,2}(\Omega)}$$

$$=\quad \rho_{W^{1,2}(\Omega)}(f,\beta f)$$

Theorem 4 *For $\alpha\neq 0$, if $\theta=\theta\langle(f,\ g)$ and $\theta^*=\theta\langle(f,\ \alpha g)$, then*

$$\text{either}\ \ \theta=\theta^*\quad\text{or}\quad \theta^*=\pi-\theta.\ \text{I.e.,}\ \theta+\theta^*=\pi$$

where θ and θ^* are angles between the indicated pair of functions.

Proof. 4 *Starting from the inner product,*

$$\begin{aligned}\langle f,\ g\rangle_{W^{1,2}(\Omega)} &= \|f\|_{W^{1,2}(\Omega)}\|g\|_{W^{1,2}(\Omega)}\cos\theta\\ &\Longrightarrow\quad \theta=\cos^{-1}\left(\frac{\langle f,\ g\rangle_{W^{1,2}(\Omega)}}{\|f\|_{W^{1,2}(\Omega)}\|g\|_{W^{1,2}(\Omega)}}\right)\end{aligned}$$

and

$$\langle f,\ \alpha g\rangle_{W^{1,2}(\Omega)}=\ |\,\alpha\,|\ \|f\|_{W^{1,2}(\Omega)}\|g\|_{W^{1,2}(\Omega)}\cos\theta^*$$

which implies

$$\begin{aligned}\theta^* &= \cos^{-1}\left(\frac{\langle f,\ \alpha g\rangle_{W^{1,2}(\Omega)}}{|\,\alpha\,|\ \|f\|_{W^{1,2}(\Omega)}\|g\|_{W^{1,2}(\Omega)}}\right)\\ &= \cos^{-1}\left(\frac{\alpha\langle f,\ g\rangle_{W^{1,2}(\Omega)}}{|\,\alpha\,|\ \|f\|_{W^{1,2}(\Omega)}\|g\|_{W^{1,2}(\Omega)}}\right)\\ &= \cos^{-1}\left(\frac{\alpha}{|\,\alpha\,|}\cos\theta\right).\end{aligned}$$

Now there is two cases to consider:
(i) For

$$\alpha>0\Longrightarrow\frac{\alpha}{|\,\alpha\,|}=1$$

there is

$$\cos^{-1}(\cos\theta)=\theta\Longrightarrow\theta^*=\theta.$$

(ii) For

$$\alpha<0\Longrightarrow\frac{\alpha}{|\,\alpha\,|}=-1$$

there is

$$\begin{aligned}\cos^{-1}(-\cos\theta) &= \pi-\theta\\ &\Longrightarrow\quad \theta^*+\theta=\pi\end{aligned}$$

and hence the angles are supplementary.
These conclude the proof.

3. Orthogonal Decompositions

Looking at how function spaces orthogonally decomposed so that any function in the space is an orthogonal sum of component functions from the orthogonal parts of the space. The spaces was consider are the Sobolev spaces $W^{k-1,2}(\Omega)$, the space for $p=2$ and $k\in\mathbb{N}$. Was stablish the following results along with several properties in [1] and [4].

Theorem 5 *[4] The space $\mathfrak{L}^2(\Omega)$ has an orthogonal decomposition*

$$\mathfrak{L}^2(\Omega)=A^{1,2}(\Omega)\oplus D\left(W_0^{1,2}(\Omega)\right)\tag{3.1}$$

where

$$A^{1,2}(\Omega)=KerD\cap\mathfrak{L}^2(\Omega)$$

and

$$W_0^{1,2}(\Omega)=\{f\in W^{1,2}(\Omega):\left(f_{|\partial\Omega},Df_{|\partial\Omega}\right)=(0,0)\}$$

so that

$$\forall f\in\mathfrak{L}^2(\Omega),\exists g\in A^{1,2}(\Omega)\ \ \text{and}\ \ h\in D\left(W_0^{1,2}(\Omega)\right)$$

such that

$$f=g\uplus h.$$

Theorem 6 *[1] The Sobolev space $W^{1,2}(\Omega)$ has an orthogonal decomposition*

$$W^{1,2}(\Omega)=A^{2,2}(\Omega)\oplus D^2\left(W_0^{3,2}(\Omega)\right)\tag{3.2}$$

where

$$A^{2,2}(\Omega)=KerD^2\cap W^{1,2}(\Omega)$$

and

$$W_0^{3,2}(\Omega)=\{f\in W^{3,2}(\Omega):\left(f_{|\partial\Omega},\ Df_{|\partial\Omega},\ D^2f_{|\partial\Omega}\right)=(0,\ 0,\ 0)\}$$

so that

$$\forall f\in W^{1,2}(\Omega),\exists g\in A^{2,2}(\Omega)\ \ \text{and}\ \ h\in D^2\left(W_0^{3,2}(\Omega)\right)$$

such that

$$f=g\uplus h.$$

These decompositions enable us to give the following main results of our research.

4. Main Results

In this section was consider first and second order boundary value problems and see how the solutions are simply the orthogonal sums of parts that evolve from boundary values of the solution and interior values of the derivative of the solution to the given order. Starting with the first order Cauchy problem.

Theorem 7 *Let $f \in \mathfrak{L}^2(\Omega)$ and $g \in W^{\frac{1}{2},2}(\Omega)$. The first order Cauchy problem*

$$\begin{cases} Du = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

has a solution $u \in W^{1,2}(\Omega)$ such that

$$u = [u]_f \uplus [u]_g$$

where $[u]_f$ is the part of the solution that evolves from f and $[u]_g$ is the part of the solution that evolves from g with the following properties

(i)

$$\left\langle [u]_f ,\ [u]_g \right\rangle_{W^{1,2}(\Omega)} = 0$$

(ii)

$$\| u \|^2_{W^{1,2}(\Omega)} = \| [u]_f \|^2_{W^{1,2}(\Omega)} + \| [u]_g \|^2_{W^{1,2}(\Omega)}$$

(iii)

$$\left\langle u,\ [u]_f \right\rangle_{W^{1,2}(\Omega)} = \| [u]_f \|^2_{W^{1,2}(\Omega)}$$

(iv)

$$\left\langle u,\ [u]_g \right\rangle_{W^{1,2}(\Omega)} = \| [u]_g \|^2_{W^{1,2}(\Omega)}$$

Proof. 5 *For two functions $f, g \in C^1(\Omega)$, integration by parts provide*

$$\int_\Omega f(x-y)\, Dg(y) d\Omega_y =$$

$$= \int_{\partial\Omega} f(x-y)\, \upsilon(y)\, g(y)\, d\partial\Omega_y +$$

$$- \int_\Omega D_y f(x-y)\, g(y)\, d\Omega_y.$$

Then by taking $f = \Gamma$ and $g = u$, there is

$$\int_\Omega \Gamma(x-y)\, Du(y) d\Omega_y =$$

$$= \int_{\partial\Omega} \Gamma(x-y)\, \upsilon(y)\, u(y)\, d\partial\Omega_y - \int_\Omega D_y \Gamma(x-y)\, u(y)\, d\Omega_y$$

where Γ is the fundamental solution to the Dirac operator D.

But

$$D_y \Gamma(x-y) = \delta(x-y)$$

the Kronecker delta function and thus,

$$\int_\Omega \Gamma(x-y)\, Du(y) d\Omega_y =$$

$$= \int_{\partial\Omega} \Gamma(x-y)\, \upsilon(y)\, u(y)\, d\partial\Omega_y - \int_\Omega \delta(x-y)\, u(y)\, d\Omega_y$$

$$= \int_{\partial\Omega} \Gamma(x-y)\, \upsilon(y)\, u(y)\, d\partial\Omega_y - u(x)$$

i.e.,

$$u(x) = \int_{\partial\Omega} \Gamma(x-y)\, \upsilon(y)\, u(y)\, d\partial\Omega_y - \int_\Omega \Gamma(x-y)\, Du(y) d\Omega_y \tag{4.2}$$

which provides the integral representation of the solution u to the BVP given by

$$u(x) = \int_{\partial\Omega} \Gamma(x-y)\, \upsilon(y)\, g(y) d\partial\Omega_y + \left(- \int_\Omega \Gamma(x-y)\, f(y) d\Omega_y \right).$$

Seeing that this sum is in fact an orthogonal sum $\uplus$.

Because the solution

$$u \in W^{1,2}(\Omega) = A^{1,2}(\Omega) \oplus D\left(W_0^{3,2}(\Omega) \right)$$

has a unique decomposition as sum of components from the two sub spaces, $A^{1,2}(\Omega)$ and $D\left(W_0^{3,2}(\Omega) \right)$. But the first integral

$$\int_{\partial\Omega} \Gamma(x-y)\, \upsilon(y)\, g(y) d\partial\Omega_y$$

is a monogenic function over Ω and hence an element of

$$Ker D \cap \mathfrak{L}^2(\Omega) = A^{1,2}(\Omega).$$

Needing to verify that the second integral

$$\left(- \int_\Omega \Gamma(x-y)\, f(y) d\Omega_y \right) \in D\left(W_0^{1,2}(\Omega) \right)$$

as well. That is, there is a function $\xi \in W_0^{3,2}(\Omega)$ *so that*

$$-\int_{\Omega}\Gamma(x-y)\,f(y)d\Omega_y = D\xi(x) \quad with \quad \xi_{|\partial\Omega}=0.$$

Clearly from the fact that

$$u(x)=\int_{\partial\Omega}\Gamma(x-y)\,\upsilon(y)\,g(y)d\partial\Omega_y+\left(-\int_{\Omega}\Gamma(x-y)\,f(y)d\Omega_y\right)$$

and $u_{|\partial\Omega}=g$ *there is*

$$\begin{aligned}\tau u_{|\partial\Omega} &= \\ &= \tau\left(\int_{\partial\Omega}\Gamma(x-y)\,\upsilon(y)\,g(y)d\partial\Omega_y+\right.\\ &\left.+\left(-\int_{\Omega}\Gamma(x-y)\,f(y)d\Omega_y\right)_{|\partial\Omega}\right.\\ &= \tau\left(\int_{\partial\Omega}\Gamma(x-y)\,\upsilon(y)\,g(y)d\partial\Omega_y\right)_{|\partial\Omega}+\\ &\quad \tau\left(-\int_{\Omega}\Gamma(x-y)\,f(y)d\Omega_y\right)_{|\partial\Omega}\end{aligned}$$

$$\tau u_{|\partial\Omega}=g$$

$$\Longrightarrow \quad \tau\left(-\int_{\Omega}\Gamma(x-y)\,f(y)d\Omega_y\right)_{|\partial\Omega}=0.$$

Thus $\xi_{|\partial\Omega}=0$. *In an analogous manner of the integral representation of the solution u, get it now the integral representation of* ξ *as*

$$\begin{aligned}\xi(x) &= -\int_{\Omega}\Gamma(x-z)\int_{\Omega}\Gamma(z-y)\,f(y)d\Omega_y d\Omega_z\\ &= -\int_{\Omega}\int_{\Omega}\Gamma(x-z)\Gamma(z-y)\,f(y)d\Omega_y d\Omega_z \in W_0^{1,2}(\Omega).\end{aligned}$$

Therefore, there is a

$$\xi(x)=-\int_{\Omega}\int_{\Omega}\Gamma(x-z)\Gamma(z-y)\,f(y)d\Omega_y d\Omega_z \in W_0^{1,2}(\Omega)$$

so that

$$\begin{aligned}D_x\xi(x) &= D_x\left(-\int_{\Omega}\int_{\Omega}\Gamma(x-z)\Gamma(z-y)\,f(y)d\Omega_y d\Omega_z\right)\\ &= -\int_{\Omega}\Gamma(x-y)\,f(y)d\Omega_y.\end{aligned}$$

Hence

$$\begin{aligned}u(x) &= \\ &\int_{\partial\Omega}\Gamma(x-y)\,\upsilon(y)\,g(y)d\partial\Omega_y \uplus\\ &\left(-\int_{\Omega}\Gamma(x-y)\,f(y)d\Omega_y\right)\\ &= [u]_g \uplus [u]_f\end{aligned}$$

$$u(x)=[u]_g \uplus [u]_f$$

with

$$[u]_g=\int_{\partial\Omega}\Gamma(x-y)\,\upsilon(y)\,g(y)d\partial\Omega_y \quad and$$

$$[u]_f=-\int_{\Omega}\Gamma(x-y)\,f(y)d\Omega_y.$$

(i) *Then since* $W^{1,2}(\Omega)$ *is a space with inner product of an orthogonal decomposition there is*

$$\left\langle\int_{\partial\Omega}\Gamma(x-y)\,\upsilon(y)\,g(y)d\partial\Omega_y,\ -\int_{\Omega}\Gamma(x-y)\,f(y)d\Omega_y\right\rangle_{W^{1,2}(\Omega)} = 0.$$

(ii) *Again from the fact that the sum is an orthogonal sum, the components obey the parallelogram law*

$$\begin{aligned}\|\ u\ \|^2_{W^{1,2}(\Omega)} &= \|\int_{\partial\Omega}\Gamma(x-y)\,\upsilon(y)\,g(y)d\partial\Omega_y\|^2_{W^{1,2}(\Omega)}+\\ &\quad +\|\int_{\Omega}\Gamma(x-y)\,f(y)d\Omega_y\|^2_{W^{1,2}(\Omega)}\\ &= \|\ [u]_g\ \|^2_{W^{1,2}(\Omega)}+\|\ [u]_f\ \|^2_{W^{1,2}(\Omega)}.\end{aligned}$$

(iii) *follows from the fact that*

$$\begin{aligned}&\left\langle u,\ [u]_f\right\rangle_{W^{1,2}(\Omega)} =\\ &= \left\langle\int_{\partial\Omega}\Gamma(x-y)\,\upsilon(y)\,g(y)d\partial\Omega_y+\right.\\ &\left.+\left(-\int_{\Omega}\Gamma(x-y)\,f(y)d\Omega_y\right),\ -\int_{\Omega}\Gamma(x-y)\,f(y)d\Omega_y\right\rangle_{W^{1,2}(\Omega)}\end{aligned}$$

$$= \left\langle -\int_{\Omega}\Gamma(x-y)\,f(y)d\Omega_y,\ -\int_{\Omega}\Gamma(x-y)\,f(y)d\Omega_y\right\rangle_{W^{1,2}(\Omega)}$$

$$= \ \| -\int_{\Omega}\Gamma(x-y)\,f(y)d\Omega_y \|^2_{W^{1,2}(\Omega)}$$

$$\left\langle u,\ [u]_f\right\rangle_{W^{1,2}(\Omega)} = \ \| -\int_{\Omega}\Gamma(x-y)\,f(y)d\Omega_y \|^2_{W^{1,2}(\Omega)}$$

and likewise (iv) *follows from*

$$\left\langle u,\ [u]_g\right\rangle_{W^{1,2}(\Omega)=}$$

$$= \left\langle \int_{\partial\Omega}\Gamma(x-y)\,\upsilon(y)\,g(y)d\partial\Omega_y,\right.$$

$$\left.\int_{\partial\Omega}\Gamma(x-y)\,\upsilon(y)\,g(y)d\partial\Omega_{yW^{1,2}(\Omega)}\ \right\rangle$$

$$= \ \| \int_{\partial\Omega}\Gamma(x-y)\,\upsilon(y)\,g(y)d\partial\Omega_y \|^2_{W^{1,2}(\Omega)}$$

The next result is for second order BVP.

Theorem 8 *Let* $f\in\mathfrak{L}^2(\Omega)$ *and* $g_1\in W^{\frac{3}{2},2}(\partial\Omega)$, $g_2\in W^{\frac{1}{2},2}(\partial\Omega)$, *then the second order BVP*

$$\begin{cases} -D^2u=f & in\ \Omega\\ u=(g_1,\ g_2) & on\ \partial\Omega\end{cases} \tag{4.3}$$

with

$$g_1=\tau u_{|\partial\Omega}\ \ and\ \ g_2=\tau Du_{|\partial\Omega}$$

has a solution $u\in W^{2,2}(\Omega)$ *given by*

$$\begin{aligned} u(x) \ = \ & \int_{\partial\Omega}\Gamma(x-y)\,\upsilon(y)\,g_1(y)\,d\partial\Omega_y \\ & -\int_{\Omega}\int_{\partial\Omega}\Gamma(x-z)\,\Gamma(z-y)\,\upsilon(z)\,g_2(z)\,d\partial\Omega_z d\Omega_y + \\ & +\int_{\Omega}\int_{\Omega}\Gamma(x-z)\,\Gamma(z-y)\,f(z)\,d\Omega_z d\Omega_y\end{aligned}$$

with the following properties
(i)

$$u(x) = (\ \int_{\partial\Omega}\Gamma(x-y)\,\upsilon(y)\,g_1(y)\,d\partial\Omega_y +$$

$$-\int_{\Omega}\int_{\partial\Omega}\Gamma(x-z)\,\Gamma(z-y)\,\upsilon(z)\,g_2(z)\,d\partial\Omega_z d\Omega_y\ )$$

$$\uplus\left(\int_{\Omega}\int_{\Omega}\Gamma(x-z)\,\Gamma(z-y)\,f(z)\,d\Omega_z d\Omega_y\right)$$

$$= [u]_{g_1,g_2}\uplus[u]_f$$

$$u(x) = [u]_{g_1,g_2}\uplus[u]_f\,.$$

(ii)

$$\| u \|^2_{W^{1,2}(\Omega)} =$$

$$= \ \| \ (\textstyle\int_{\partial\Omega}\Gamma(x-y)\,\upsilon(y)\,g_1(y)\,d\partial\Omega_y +$$

$$-\int_{\Omega}\int_{\partial\Omega}\Gamma(x-z)\,\Gamma(z-y)\,\upsilon(z)\,g_2(z)\,d\partial\Omega_z d\Omega_y \|^2_{W^{1,2}(\Omega)}$$

$$+\ \| \int_{\Omega}\int_{\Omega}\Gamma(x-z)\,\Gamma(z-y)\,f(z)\,d\Omega_z d\Omega_y \|^2_{W^{1,2}(\Omega)}$$

$$= \| \ [u]_{g_1,g_2} \|^2_{W^{1,2}(\Omega)} + \| \ [u]_f \|^2_{W^{1,2}(\Omega)}$$

Proof. 6 *Clearly since the input function* $f\in\mathfrak{L}^2(\Omega)$, *which is the weekly second order derivative of the solution u, there is u to be in* $W^{2,2}(\Omega)$. *This is because the Dirac operator D is a regularity exponent diminishing operator, between Sobolev spaces.*

The repeated application of the integral representation given in (4.2) will be used. Let $v(x)=Du(x)$, *then there is a first order BVP*

$$\begin{cases} -Dv=f & in\ \Omega\\ v=g_2 & on\ \partial\Omega\end{cases}$$

whose solution is given by

$$v(x) = \int_{\partial\Omega}\Gamma(x-y)\,\upsilon(y)\,g_2(y)d\partial\Omega_y + \left(-\int_{\Omega}\Gamma(x-y)\,f(y)d\Omega_y\right).$$

But $Du=v$ *and hence there is again a first order BVP*

$$\begin{cases} -Du=f & in\ \Omega\\ u=g_1 & on\ \partial\Omega\end{cases}$$

with a solution

$$u(x) =$$

$$= \int_{\partial\Omega}\Gamma(x-y)\,\upsilon(y)\,u(y)\,d\partial\Omega_y - \int_{\Omega}\Gamma(x-y)\,Du(y)d\Omega_y$$

$$= \int_{\partial\Omega}\Gamma(x-y)\,\upsilon(y)\,g_1(y)\,d\partial\Omega_y +$$

$$-\int_{\Omega}\Gamma(x-y)\begin{pmatrix} \int_{\partial\Omega}\Gamma(y-z)\,\upsilon(z)\,g_2(z)d\partial\Omega_z \\ +(-\int_{\Omega}\Gamma(y-z)\,f(z)d\Omega_z)\end{pmatrix} d\Omega_y$$

$$= \int_{\partial\Omega}\Gamma(x-y)\,\upsilon(y)\,g_1(y)\,d\partial\Omega_y +$$

$$-\int_{\Omega}\int_{\partial\Omega}\Gamma(x-y)\,\Gamma(y-z)\,\upsilon(z)\,g_2(z)d\partial\Omega_z d\Omega_y +$$

$$+\int_{\Omega}\int_{\Omega}\Gamma(x-y)\,\Gamma(y-z)\,f(z)d\Omega_z d\Omega_y$$

It need to show that this sum is again an orthogonal sum from the orthogonal decomposition

$$W^{1,2}(\Omega) = A^{2,2}(\Omega) \oplus D^2\left(W_0^{3,2}(\Omega)\right)$$

proven in [1]. Clearly

$$\int_{\partial\Omega} \Gamma(x-y)\,\upsilon(y)\,g_1(y)\,d\partial\Omega_y +$$

$$-\int_{\Omega}\int_{\partial\Omega} \Gamma(x-y)\,\Gamma(y-z)\,\upsilon(z)\,g_2(z)d\partial\Omega_z d\Omega_y$$

is annihilated by D^2 since

$$D^2\left(\int_{\partial\Omega} \Gamma(x-y)\,\upsilon(y)\,g_1(y)\,d\partial\Omega_y +\right.$$

$$\left.-\int_{\Omega}\int_{\partial\Omega} \Gamma(x-y)\,\Gamma(y-z)\,\upsilon(z)\,g_2(z)d\partial\Omega_z d\Omega_y\right.$$

$$= D^2\left(\int_{\partial\Omega} \Gamma(x-y)\,\upsilon(y)\,g_1(y)\,d\partial\Omega_y\right) +$$

$$-D^2\left(\int_{\Omega}\int_{\partial\Omega} \Gamma(x-y)\,\Gamma(y-z)\,\upsilon(z)\,g_2(z)d\partial\Omega_z d\Omega_y\right)$$

$$= -D\left(\int_{\partial\Omega} \Gamma(y-z)\,\upsilon(z)\,g_2(z)d\partial\Omega_z d\Omega_y\right) = 0.$$

Thus

$$\int_{\partial\Omega} \Gamma(x-y)\,\upsilon(y)\,g_1(y)\,d\partial\Omega_y +$$

$$-\int_{\Omega}\int_{\partial\Omega} \Gamma(x-y)\,\Gamma(y-z)\,\upsilon(z)\,g_2(z)d\partial\Omega_z d\Omega_y \in A^{2,2}(\Omega).$$

Next, it need to show that $\exists \xi \in W_0^{3,2}(\Omega)$ such that

$$\int_{\Omega}\int_{\Omega} \Gamma(x-y)\,\Gamma(y-z)\,f(z)d\Omega_z d\Omega_y = D^2\xi(x).$$

Clearly

$$\xi(x)_{|\partial\Omega} = \left(\xi(x)_{|\partial\Omega}, D\xi(x)_{|\partial\Omega}\right) = (0,0).$$

By applying the result of integration by parts above twice and the fact that

$$\xi_{|\partial\Omega} = 0$$

there is $\xi(x) =$

$$= \{\int_{\Omega}\int_{\Omega}\int_{\Omega}\int_{\Omega} \Gamma(x-y)\,\Gamma(y-z)\,\Gamma(z-w)\,\Gamma(w-q)$$

$$f(q)\,d\Omega_q d\Omega_w d\Omega_z d\Omega_y \in W_0^{3,2}(\Omega)\}.$$

Setting

$$[u]_{g_1,g_2} = \int_{\partial\Omega} \Gamma(x-y)\,\upsilon(y)\,g_1(y)\,d\partial\Omega_y +$$

$$-\int_{\Omega}\int_{\partial\Omega} \Gamma(x-y)\,\Gamma(y-z)\,\upsilon(z)\,g_2(z)d\partial\Omega_z d\Omega_y$$

and

$$[u]_f = \int_{\Omega}\int_{\Omega} \Gamma(x-y)\,\Gamma(y-z)\,f(z)d\Omega_z d\Omega_y$$

there is

$$u = [u]_{g_1,g_2} \uplus [u]_f$$

which proves (i).

(ii) follows from the fact that

$$u = [u]_{g_1,g_2} \uplus [u]_f.$$

Theorem 9 *For the BVP given in Theorem* **7,** *if the input function f is monogenic or Clifford analytic over Ω, then the following holds*

(i)

$$u(x) = \frac{1}{\| B(x,\rho) \|}\int_{B(x,\rho)} u(y)\,d\Omega_y, \forall x \in \Omega \ \text{ and } \ \rho > 0$$

(ii)

$$_{x\in\overline{\Omega}} \mid u(x) \mid \ \leq \ \sup_{x\in\partial\Omega} \mid g(x) \mid.$$

Proof. 7 *The input function f is monogenic and hence u is harmonic, since*

$$\Delta u = -D^2 u = -D(f) = 0.$$

But then harmonic functions satisfy the mean value theorem and that proves (i).

The proof of the second result follows from the maximum principle for harmonic functions over a domain Ω. Clearly *the solution function to the BVP is harmonic and its integral representation is given by*

$$u(x)=\int_{\partial\Omega}\Gamma(x-y)\,\upsilon(y)\,g(y)\,d\partial\Omega_y-\int_{\Omega}\Gamma(x-y)\,f(y)\,\partial\Omega_y$$

and this function satisfies the maximum principle. That is u attains its extreme values on the boundary of the domain. This follows from

$$\begin{aligned}\tau u(x)_{\partial\Omega} &= \tau(\int_{\partial\Omega}\Gamma(x-y)\,\upsilon(y)\,g(y)\,d\partial\Omega_y+\\ &\quad -\int_{\Omega}\Gamma(x-y)\,f(y)\,d\Omega_y)_{\partial\Omega}\\ &= \tau\left(\int_{\partial\Omega}\Gamma(x-y)\,\upsilon(y)\,g(y)\,d\partial\Omega_y\right)_{\partial\Omega}+\\ &\quad -\tau\left(\int_{\Omega}\Gamma(x-y)\,f(y)\,d\Omega_y\right)_{\partial\Omega}\\ \tau u(x)_{\partial\Omega} &= \tau\left(\int_{\partial\Omega}\Gamma(x-y)\,\upsilon(y)\,g(y)\,d\partial\Omega_y\right)_{\partial\Omega}\end{aligned}$$

since

$$\tau\left(\int_{\Omega}\Gamma(x-y)\,f(y)\,d\Omega_y\right)_{\partial\Omega}=0.$$

Again $\tau_{\partial\Omega}$ is a trace operator that acts as a left inverse operator of the boundary integral operator

$$\int_{\partial\Omega}\Gamma(x-y)\,\upsilon(y)\,g(y)\,d\partial\Omega_y$$

in such a way that

$$\tau_{\partial\Omega}\circ\left(\int_{\partial\Omega}\Gamma(x-y)\,\upsilon(y)\,g(y)\,d\partial\Omega_y\right)=g.$$

Therefore

$$\tau u(x)_{\partial\Omega}=g(x).$$

Hence

$$_{x\in\overline{\Omega}}\mid u(x)\mid\ \le\ \sup_{x\in\partial\Omega}\mid g(x)\mid.$$

Corollary 2 *If Ω is compactly embedded in $\mathbb{R}^n$, then for the solution u of the BVP of Theorem* **7**, *$\exists x_0,x_1\in\partial\Omega$:*

(i)

$$\mid u(x)\mid\ \le\ \mid g(x_1)\mid\ \forall x\in\overline{\Omega}$$

(ii)

$$\mid g(x_0)\mid\ \le\ \mid u(x)\mid\ \forall x\in\overline{\Omega}.$$

Corollary 3 *If f is monogenic over Ω, then the the Cauchy problem*

$$\begin{cases} Du=f & \text{in } \Omega\\ u=0 & \text{on } \partial\Omega\end{cases}$$

has a solution $u\equiv 0$ and further more $f\equiv 0$.

Proof. 8 *Because f is monogenic over Ω, there is that u is harmonic over Ω since*

$$\Delta u=-D^2u=-Df=0.$$

Then the maximum value principles guarantees that $\mid u(x)\mid$ has maximum values on the boundary of the domain. But the boundary value of u is zero and hence follows that $u\equiv 0$ over $\overline{\Omega}$ which then implies $f\equiv 0$.

Remark 1 *These results can be extended to higher order boundary value problems which is our next focus.*

Theorem 10 *The Sobolev Space $W^{(k-1),2}(\Omega)$ (for $k\ge 1$) has an orthogonal decomposition given by*

$$W^{(k-1),2}(\Omega)=A^{k,2}(\Omega)\oplus D^k\left(W_0^{(2k-1),2}(\Omega)\right)\tag{4.4}$$

where D^k is the k^{th} order Dirac operator and

$$A^{k,2}(\Omega)=KerD^k\cap W^{(k-1),2}(\Omega).$$

From this fundamental theorem, there is a higher order BVP whose solution has orthogonal components.

Theorem 11 *(Higher order BVP) Let*

$f\in^2(\Omega)$, $g_j\in W^{k-j-\frac{1}{2}}(\partial\Omega)$ for $j=0,1,2,3,...k-1$ and $k=1,2,3,....$ Then the k^{th}-order BVP

$$\begin{cases} D^ku=f & \text{in } \Omega\\ \tau u=g & \text{on } \partial\Omega\end{cases}\tag{4.5}$$

with

$$g=\left(g_0\left(=u_{|\partial\Omega}\right),\ g_1\left(=Du_{|\partial\Omega}\right),\ g_2\left(=D^2u_{|\partial\Omega}\right),...,\right.$$

$$\left....,\ g_j\left(=D^ju_{|\partial\Omega}\right)...,\ g_{k-1}\left(=D^{k-1}u_{|\partial\Omega}\right)\right)$$

has a solution $u\in W^{(k-1),2}(\Omega)$ given as an orthogonal sum

$$u=[u]_{(g_0,\,g_1,\,g_2,\,g_3,...,\,g_{k-1})}\uplus[u]_f$$

where

$$[u]_{(g_0,\,g_1,\,g_2,\,g_3,...,\,g_{k-1})}=\sum\int_{j=0}^{k-1}\zeta_{\Omega}^{j}\left(\zeta_{\partial\Omega}g_j\right)\ \text{ and }\ [u]_f=\zeta_{\Omega}^{k}f.$$

That is

(i)

$$\left\langle\sum\int_{j=0}^{k-1}\zeta_{\Omega}^{j}\left(\zeta_{\partial\Omega}g_j\right),\ \zeta_{\Omega}^{k}f\right\rangle_{W^{(k-1),2}(\Omega)}=0$$

.

(ii) $\| u \|^2_{W^{(k-1),2}(\Omega)} =$

$$\| \sum \int_{j=0}^{k-1} \zeta_\Omega^j \left(\zeta_{\partial\Omega} g_j\right) \|^2_{W^{(k-1),2}(\Omega)} +$$

$+ \| \zeta_\Omega^k f \|^2_{W^{(k-1),2}(\Omega)}$.

(iii)

$$\left\langle u,\ \sum \int_{j=0}^{k-1} \zeta_\Omega^j \left(\zeta_{\partial\Omega} g_j\right) \right\rangle_{W^{(k-1),2}(\Omega)} =$$

$$= \| \sum \int_{j=0}^{k-1} \zeta_\Omega^j \left(\zeta_{\partial\Omega} g_j\right) \|^2_{W^{(k-1),2}(\Omega)}$$

.

(iv) $\langle u,\ \zeta_\Omega^k f \rangle_{W^{(k-1),2}(\Omega)} = \| \zeta_\Omega^k f \|^2_{W^{(k-1),2}(\Omega)}$

where $\zeta_{\partial\Omega}$, ζ_Ω *are integral transform operators given by*

$$\zeta_{\partial\Omega}\eta\left(x\right) = \int_{\partial\Omega} \Gamma\left(x-y\right) \upsilon\left(y\right) \eta\left(y\right) d\partial\Omega_y$$

and

$$\zeta_\Omega \eta\left(x\right) = \int_\Omega \Gamma\left(x-y\right) \eta\left(y\right) d\Omega_y.$$

Proof. 9 *Clearly since* f *is in* $^2(\Omega)$, *the solution* u *is in* $W^{(k-1),2}(\Omega)$. *This is because the Dirac operator* D *is a regularity diminishing operator by* 1 *and so its* k^{th} *order reduces a regularity of* u *by* k *and to be in* $^2(\Omega)$. *Using induction on* k. *For* $k = 1,2$ *there is the Dirac operator* D *and* D^2, *the case of a first and second order Cauchy problems, which there is shown in Theorems* [7] *and* [8]. *Now*

$$D^k u = f \ \text{ on } \Omega \text{ with } u_{|\partial\Omega} = \left(g_0,\ g_1,\ g_2, ...,\ g_{k-1}\right)$$

$$\Longrightarrow \quad u = \zeta_{\partial\Omega}\left(g_0\right) + \zeta_\Omega\left(\zeta_{\partial\Omega}\left(g_1\right)\right) + \zeta_\Omega\zeta_\Omega\left(\zeta_{\partial\Omega}\left(g2\right)\right) + \zeta_\Omega\zeta_\Omega\zeta_\Omega\left(\zeta_{\partial\Omega}\left(g_3\right)\right) + ... + \underbrace{\zeta_\Omega ... \zeta_\Omega}_{(k-1)\text{-compositions}} \left(\zeta_{\partial\Omega}\left(g_{k-1}\right)\right) + \underbrace{\zeta_\Omega ... \zeta_\Omega}_{k\text{- compositions}} \left(f\right).$$

That is

$$u = \sum \int_{j=0}^{k-1} \zeta_\Omega^j \left(\zeta_{\partial\Omega} g_j\right) + \zeta_\Omega^k f.$$

But the function

$$\eta = \sum \int_{j=0}^{k-1} \zeta_\Omega^j \left(\zeta_{\partial\Omega} g_j\right) \in KerD^k \cap W^{k,2}\left(\Omega\right) = A^{k,2}\left(\Omega\right)$$

since

$$D^k \eta\left(x\right) = D^k \left(\sum \int_{j=0}^{k-1} \zeta_\Omega^j \left(\zeta_{\partial\Omega} g_j\right)\right) = D\left(\zeta_{\partial\Omega} g_j\right) = 0$$

and $D\zeta_\Omega g = g$ *and also* $\zeta_{\partial\Omega} g$ *is monogenic and hence annihilated by* D. *Now what is left is to show that*

$$\zeta_\Omega^k f \in D^k \left(W_0^{(2k-1),2}\left(\Omega\right)\right)$$

that is there is to find an $h \in W_0^{(2k-1),2}(\Omega)$ *such that*

$$\zeta_\Omega^k f = D^k h \ \text{ with } \ h_{|\partial\Omega} = 0.$$

But this is achieved by considering $h = \zeta_\Omega^{2k} f$.

Clearly

$$h = \zeta_\Omega^{2k} f \in W_0^{(2k-1),2}\left(\Omega\right)$$

and

$$D^k h = D^k \zeta_\Omega^{2k} f = \zeta_\Omega^k f.$$

From the fact that

$$\left(D^k \left(W_0^{(2k-1),2}\left(\Omega\right)\right)\right)^\perp = A^{k,2}\left(\Omega\right)$$

Then proveding the claim that the solution is the orthogonal sum of functions that evolve from the boundary values g_j, *and value* f *of the differential operator inside the domain*

$$\begin{aligned} u &= \sum \int_{j=0}^{k-1} \zeta_\Omega^j \left(\zeta_{\partial\Omega} g_j\right) \uplus \zeta_\Omega^k f \\ &= [u]_{(g_1, g_2, ..., g_{k-1})} \uplus [u]_f \end{aligned}$$

as required.

Corollary 4 *The solution* u *to* 4.5 *satisfies the following orthogonality properties*

(i)

$$\begin{aligned} &\left\langle u,\ [u]_{(g_0,\, g_1,\, g_2, ...,\, g_{k-1})} \right\rangle_{W^{(k-1),2}(\Omega)} = \\ &= \| [u]_{(g_0,\, g_1,\, g_2, ...,\, g_{k-1})} \|^2_{W^{(k-1),2}(\Omega)} \\ &= \| \sum \int_{j=0}^{k-1} \zeta_\Omega^j \left(\zeta_{\partial\Omega} g_j\right) \|^2_{W^{(k-1),2}(\Omega)} . \end{aligned}$$

(ii)

$$\begin{aligned} \left\langle u,\ [u]_f \right\rangle_{W^{(k-1),2}(\Omega)} &= \| [u]_{(g_0,\, g_1,\, g_2, ...,\, g_{k-1})} \|^2_{W^{(k-1),2}(\Omega)} \\ &= \| \zeta_\Omega^k f \|^2_{W^{(k-1),2}(\Omega)} . \end{aligned}$$

Corollary 5 *The higher order Cauchy problem* 4.5, *with* $g \equiv 0$, *i.e.,* $g_j = 0, \forall j = 0, ..., k-1$, *has a solution* u *given by*

$$u = \zeta_\Omega^k f$$

and hence

$$u = 0 \uplus \zeta_\Omega^k f.$$

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