PART III:
Beyond secondary creep: anisotropic flow laws and the theory of continuous diversity

1. Fabric and its evolution: Available anisotropic flow laws

2. Continuous diversity of polycrystalline ice masses: the most comprehensive theory to model induced anisotropy

Swedish Polar Research Secretariat / Bert Bolin Center for Climate Research / Stockholm University

1. Fabric and its evolution: Available anisotropic laws

microscopic

v. d. Veen & Whillans
1994 I
dyn. anisotropic viscous power law for indiv. grains

Azuma & Goto-Azuma
1996 H
static anisotropic flow law
no fabric evolution

Duval, Castelnau et al.
1983 ~ 2005 I
VPSC (Visco-Plastic Self Contained)
dyn. anisotr. linear flow law for individual grains

macroscopic

Lliboutry
1993
static anisotropic flow law

Meyssonnier & Philipp
1996 S
dynamic anisotropic flow law (transv. isotropic) based on VPSC and ODF (Orientation Distribution Function) implemented (simplified version) by **Gagliardini & Meyssonnier** 1999/2000

Svendsen/Gödert & Hutter
1996, 1998 S/SI
dyn. anisotr. (transv. isotr./orthotr.) flow law based on ODF/ODF + indiv. grains

Gillet-Chaulet et al.
2005 S
stat. anisotr. (orthotr.) flow law based on ODF and parameters from physical μ-M models, designed for l.s. num. modeling

phemomenological

Morland & Staroszczyk
1998~2001
obtain evolving anisotropy from instantaneous states of deformation without explicit reference to fabric or grain size
(! reversibility of anisotropy)

1. Fabric and its evolution: Available anisotropic laws

H – (homogenization) models: based on averages of individual grains

Schmidt tensors: $\check{S}_{ij} = 1/N \sum^{n}_{g=1} m_i^{(g)} c_j^{(g)}$

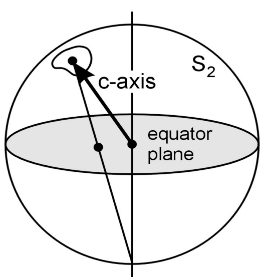
$m_i^{(g)}$: unit vector parallel to resolved shear stress in the basal plane

$c_j^{(g)}$: unit vector parallel to the c-axis orientation

n: Number of grains g

S – (statistical) models: based on an Orientation Distribution Function

ODF: orientation density $f = f(\mathbf{x}, t, \mathbf{n})$, $\mathbf{n}$ vector of unit length in S^2



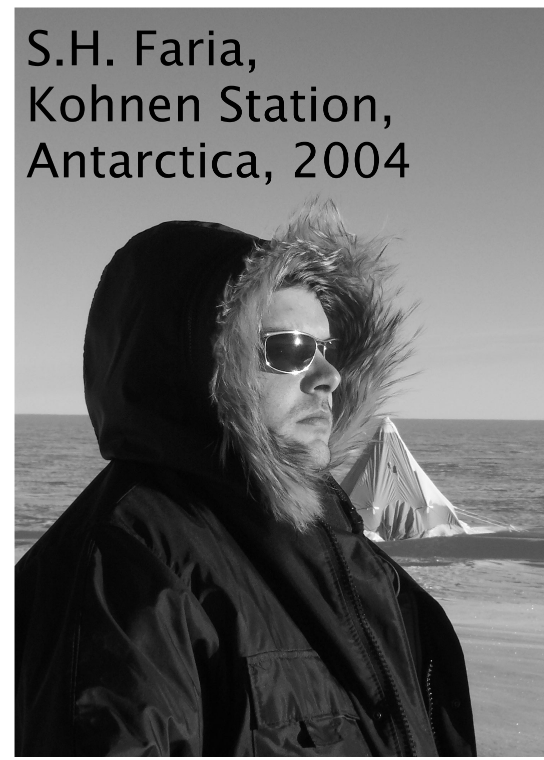
$$\int_{S^2} f(\mathbf{x}, t, \mathbf{n})\, d^2n = 1$$

alignment/structure/anisotropy tensors: $\mathbf{A} := \int_{S^2} f(\mathbf{x}, t, \mathbf{n})\, \mathbf{n} \otimes \mathbf{n}\, d^2n$

2. Continuous diversity of polycrystalline ice masses: the most comprehensive theory to model induced anisotropy

The **theory of mixtures with continuous diversity** (MCD)

- has been developped by S. H. Faria from ~ 2001
- conforms to the principles of Rational Mechanics Modeling of Materials
- is a thermodynamic theory
- is the most comprehensive theory to model hetero-geneity, in particluar induced anisotropy
- has many other applications



S.H. Faria, Kohnen Station, Antarctica, 2004

In the context of ice sheet modeling, MCD allows for the simultaneous modeling of

- texture evolution (rotation of c-axis)
- recrystallization, polygonization, recovery

2. Continuous diversity of polycrystalline ice masses: the most comprehensive theory to model induced anisotropy

Background to the MCD:

Single constituent continua:

5 scalar balance laws for independent primary fields $\rho(x,t)$, $\mathbf{v}(\mathbf{x},t)$, $T(x,t)$

general balance law:

$$\partial (*)/ \partial t + \mathrm{div}\, [(*)\mathbf{v} + \phi)] - s = p$$

(*): additive quantity, **v**: velocity, ϕ, s, p: flux, supply and production of (*)

div: divergence operator in Euclidean space E^3

conservation equation: p=0

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Multiconstituent continua (chemically reacting mixtures, granular media,...):

N discrete constituents, indexed by α, typically $N \leq 3$

5N balance laws for primary fields: $\rho_\alpha(\mathbf{x},t)$ $\mathbf{v}_\alpha(\mathbf{x},t), T_\alpha(\mathbf{x},t)$ $\alpha = 1, \ldots N$

General balance law: $$\partial (*_\alpha) / \partial t + \operatorname{div} [(*_\alpha)\mathbf{v}_\alpha + \phi_\alpha)] - s_\alpha = p_\alpha$$

Non-conservation equations on constituent level: $p_\alpha \neq 0$

Mixture balance laws are derived from the constituent balance laws according to the Rational Mechanics Modeling of Materials approach (Truesdell's third metaphysical principle) and provide homogenization rules:

$\sum^N_{\alpha=1} \rho_\alpha = \rho$ $\sum^N_{\alpha=1} \mathbf{T}_\alpha - \rho_\alpha\ \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha = \mathbf{T}$ $\mathbf{u}_\alpha = \mathbf{v}_\alpha - \mathbf{v}$ diffusion velocity

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Classical continuous mixtures

Countable set of constituents with individual primary variables,

e.g. $\rho_\alpha(\mathbf{x}, t)$

Each constituent has countably many properties distinguishing it from other constituents

Nature shows us often the reverse situation:

Mixtures with continuous diversity

Infinitely many constituents with primary variables depending on the continuously varying species label

$\rho^*(\mathbf{x}, t, \alpha)$

α in $A = [\alpha_{min}, \alpha_{max}]$ species assemblage

Constituents differ from each other only in very few properties (size, orientation, age, ...)

Polycrystalline ice

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Mixtures with continuous diversity

Balance equations for primary fields (note: # does not increase with α) depend on position in i/ Euclidean space and ii/ Species space

$\rho(\mathbf{x}, t, \alpha)$, $\mathbf{v}(\mathbf{x}, t, \alpha)$, $T(\mathbf{x}, t, \alpha)$ α in $A = [\alpha_{min}, \alpha_{max}]$

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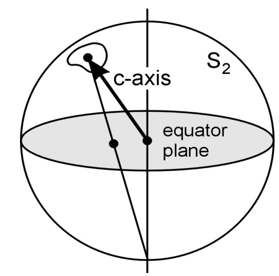
Mixtures with continuous diversity

Balance equations for primary fields (note: # does not increase with α) depend on position in i/ Euclidean space and ii/ Species space

$\rho(\mathbf{x},t,\alpha)$, $\mathbf{v}(\mathbf{x},t,\alpha)$, $T(\mathbf{x},t,\alpha)$ α in $A = [\alpha_{min}, \alpha_{max}]$



Polycrystalline ice: species (single crystals) are identified by their orientation, represented by a unit normal vector **n** in S^2

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Mixtures with continuous diversity

Balance equations for primary fields (note: # does not increase with α) depend on position in i/ Euclidean space and ii/ Species space

$\rho(\mathbf{x},t,\alpha)$, $\mathbf{v}(\mathbf{x},t,\alpha)$, $T(\mathbf{x},t,\alpha)$ α in $A = [\alpha_{min}, \alpha_{max}]$

Primary fields amended by dislocation density ρ_D and c-axis spin velocity $\mathbf{s}$:

$\rho(\mathbf{x},t,\mathbf{n})$, $\rho_D(\mathbf{x},t,\mathbf{n})$, $\mathbf{s}(\mathbf{x},t,\mathbf{n})$, $\mathbf{v}(\mathbf{x},t,\mathbf{n})$, $T(\mathbf{x},t,\mathbf{n})$

General Balance Equation

$$\partial(*)/\partial t + \mathrm{div}_{E^3}[(*)\mathbf{v} + \phi)] + \mathrm{div}_{S^2}[(*)w + \psi] - s = p$$

w interspecies transition rate ψ interspecies flux

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Species balance equation for polycrystals modeled as mixtures with continuous diversity (Faria, 2006, Proc. R. Soc. Lond. A)

Balance of mass:	includes recrystallization
Balance of dislocation density:	includes interspecies flux density of dislocations and production rate of dislocations
Balance of linear momentum:	includes interspecies stress and high-angle interaction force
Balance of lattice spin velocity:	includes polygonization tensor (interspecies couple stress) and high-angle interaction couple
Balance of internal energy:	includes dissipative contributions associated with all new interspecies quantities

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Homogenization of species balance equations:

explores Rational Mechanics Modeling of Materials approach, is of type

$$\rho = \int_{S^2} \rho(\mathbf{x}, t, \mathbf{n}) \, d^2n$$

$$\mathbf{T} = \int_{S^2} \Big(\mathbf{T}(\mathbf{x}, t, \mathbf{n}) - \rho(\mathbf{x}, t, \mathbf{n}) \, [\mathbf{v}(\mathbf{x}, t, \mathbf{n}) - \mathbf{v}(\mathbf{x}, t)] \otimes [\mathbf{v}(\mathbf{x}, t, \mathbf{n}) - \mathbf{v}(\mathbf{x}, t)] \Big) \, d^2n$$

continuous mixtures:

$\sum^{N}_{\alpha=1} \rho_\alpha = \rho$ $\qquad$ $\sum^{N}_{\alpha=1} \mathbf{T}_\alpha - \rho_\alpha \, \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha = \mathbf{T}$

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Constitutive theory:

Work in progress

In Part III: Simplified reduced model

PROCEEDINGS OF THE ROYAL SOCIETY A

Proc. R. Soc. A (2006) **462**, 1493–1514
doi:10.1098/rspa.2005.1610
Published online 8 February 2006

Creep and recrystallization of large polycrystalline masses. I. General continuum theory

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PROCEEDINGS OF THE ROYAL SOCIETY A

Proc. R. Soc. A (2006) **462**, 1699–1720
doi:10.1098/rspa.2005.1635
Published online 15 February 2006

Creep and recrystallization of large polycrystalline masses. II. Constitutive theory for crystalline media with transversely isotropic grains

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PROCEEDINGS OF THE ROYAL SOCIETY A

Proc. R. Soc. A (2006) **462**, 2797–2816
doi:10.1098/rspa.2006.1698
Published online 12 April 2006

Creep and recrystallization of large polycrystalline masses. III. Continuum theory of ice sheets

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This work sets forth the first thermodynamically consistent constitutive theory for ice sheets undergoing strain-induced anisotropy, polygonization and recrystallization effects.

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The simplified model presented in Faria's Part III is still more general than all other anisotropic flow laws.

$\mathbf{T} = -p\,1 + \mu^{(4)}\mathbf{D}^E$ with $\mu^{(4)} = \mu^{(4)}(\rho_D, \mathbf{n}, \ldots)$

It encompasses the previously suggested models by

- Svendsen/Gödert/Hutter
- Azuma/Goto-Azuma
- and the CAFFE model [Continuum mechancial Anisotropic Flow model based on an anisotropic Flow Enhancement factor]
  (cf. Placidi & Hutter, 2005, Seddik et al. 2008, Greve et al. (in print), Faria 2008)

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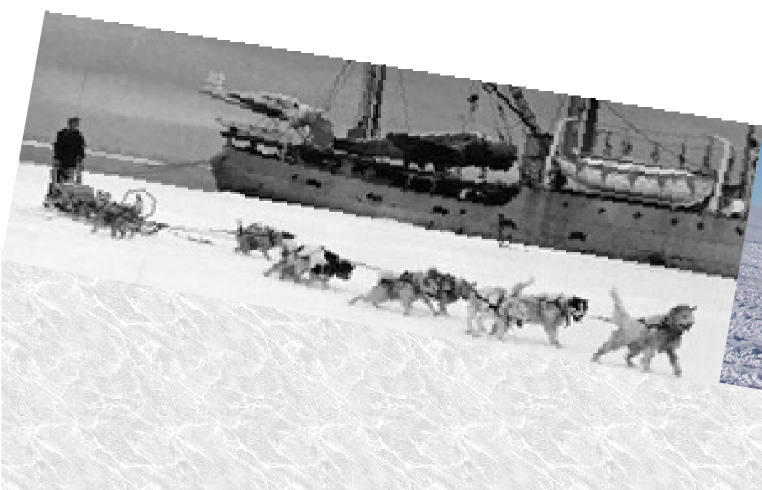
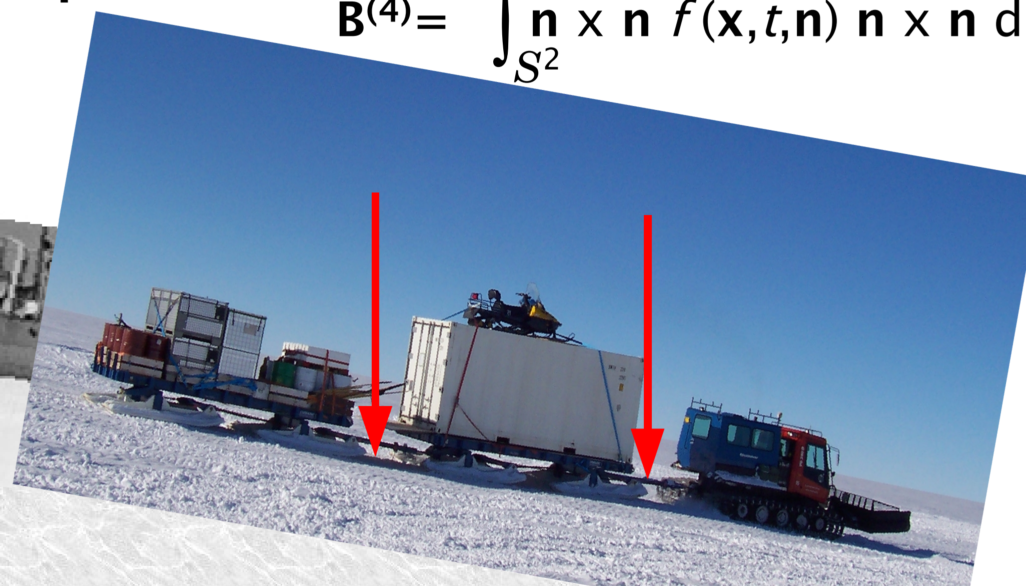
The flow law in the CAFFE model:

$$\mathbf{D} = E(\mathbf{T}^D, \mathbf{A}^{(2)}, \mathbf{B}^{(4)})\, A(T)\, \sigma^{n-1}\, \mathbf{T}^D$$

$$\mathbf{A}^{(2)} = \int_{S^2} f(\mathbf{x}, t, \mathbf{n})\, \mathbf{n} \times \mathbf{n}\, d^2n$$

$$\mathbf{B}^{(4)} = \int_{S^2} \mathbf{n} \times \mathbf{n}\, f(\mathbf{x}, t, \mathbf{n})\, \mathbf{n} \times \mathbf{n}\, d^2n$$





The CAFFE model is implemented in Elmer/Ice @ CSC Finland (Th. Zwinger)