Verifying Ice-Sheet Models

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Model verification and validation

Numerical computer codes for ice sheet flow emerge from two stages of effort:

specification of a continuum model (nonlinear PDEs)

modeling errors arise from not solving the right equations $\longrightarrow$ assessment of modeling errors is called model validation.

the numerical approximation of the model (because of the difficulty of solving the above PDEs exactly)

numerical errors arise from not solving the equations right $\longrightarrow$ assessment of numerical errors is called model verification.

Ways to verify the ice sheet models

intercomparison of models - measuring differences among various models' results on the sets of simplified geometry benchmark tests.

- compares models on variety of tests that resemble real ice sheet geometries;
- provides a set of standards for a modelers;
- examples: EISMINT I, EISMINT II, Ross Ice shelf, ISMIP-HOM, ISMIP-HEINO, ISMIP-POLICE, MISMIP.

verification by exact solution - measuring differences between model results and (may be artificially constructed) exact solutions.

- allows modelers check correctness of a code and to *estimate magnitude of numerical errors on a given grid;*
- *allows to measure convergence of numerical methods;*
- *allows tests codes for a variety of cases including different boundary conditions.*

Example of building a manufactured exact solution (Bueler, 2006)

- completely made-up PDE:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u^2$$

  is hard to find any exact solutions
- but one can find such for a slightly more general PDE:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u^2 + f(x,t) \tag{1}$$

- for *example*, let $u(x,t) = x^3 + t$; compute

$$f = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - u^2 = 1 - 6x - \left(x^3 + t\right)^2$$

- with this f, equation (1) has $u = x^3 + t$ as solution.

Contents

Verification of the SIA models

intercomparison of models

- EISMINT I - isothermal nonsliding ice flow (Huybrechts et al., 1996)
- EISMINT II - thermocoupled nonsliding ice flow (Payne et al., 2000)

verification by manufactured exact solutions

- isothermal nonsliding SI (Halfar 1983, Bueler et al 2005)
- thermocoupled nonsliding SI (Bueler et al 2005)

Shallow Ice Approximation (SIA)

Reference: K. Hutter, Theoretical Glaciology. Dordrecht, Kluwer Academic Publishers, 1983.

Assumptions: Longitudinal and transverse stresses are neglected.

Numerics: Quasi-2D model – 1 degree of freedom per node.

Conclusion: Valid only for an ice mass with a small aspect ratio (*ice thickness* $<<$ *ice horizontal dimensions*)

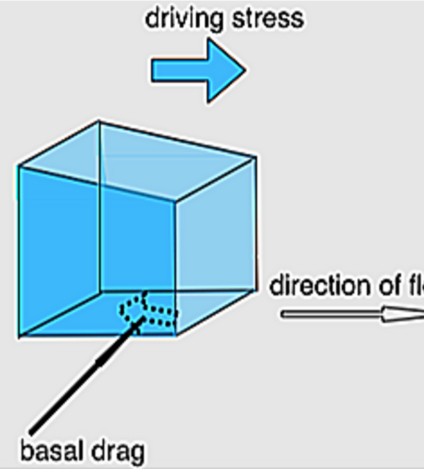


Figure: Force Balance for Shallow-Ice Approximation

Ice Sheet Equations of the SIA

conservation of mass

$$h_t = M - \nabla \cdot (\bar{U} h) \tag{2}$$

get velocity in SIA by vertically-integrating this:

$$\bar{U}(z) = -2(\rho g)^n |\nabla h|^{n-1} \nabla h \int_b^z \left(\frac{h - \xi}{A(T^*)} \right)^n d\xi + \bar{U}(b) \tag{3}$$

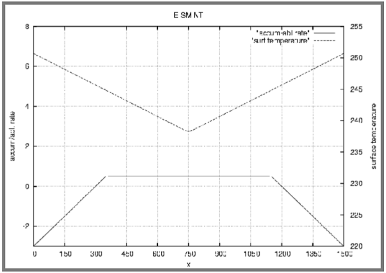
conservation of energy

$$\rho c_p \left(T_t + \bar{U} \cdot \nabla T \right) = k \nabla^2 T + (\sigma_{xz}, \sigma_{yz}) \cdot \frac{\partial \bar{U}}{\partial z} \tag{4}$$

where $\bar{U}$ is vertically averaged horizontal velocity.

EISMINT II (European Ice Sheet Modeling Initiative)

- intercomparison of 10 ice-sheet models on a series of experiments.
- a circular ice sheet is used and steady states and responses to stepped changes in climate are investigated.
- Exp. A: starting from zero ice, ice accumulation/ablation rate and ice-surface temperature are fixed as functions of geographical position:



$$M(x,y) = min\left[M_{max}, S_b\left(R_{el} - \sqrt{(x - x_{summit})^2 + (y - y_{summit})^2}\right)\right]$$

$$T_{surface}(x,y) = T_{min} + S_T\sqrt{(x - x_{summit})^2 + (y - y_{summit})^2}$$

where M_{max} is the maximum accumulation rate; R_{el} is a distance from the summit (x_{summit}, y_{summit}) where the accumulation rate becomes zero; S_b is the gradient of accum. rate change with horizontal distance; T_{min} is the minimum surface temperature; S_T is the gradient of air temperature change with horizontal distance.

EISMINT: Experiment A

- wide range of results;
- how to estimate magnitude of numerical errors for a particular model?

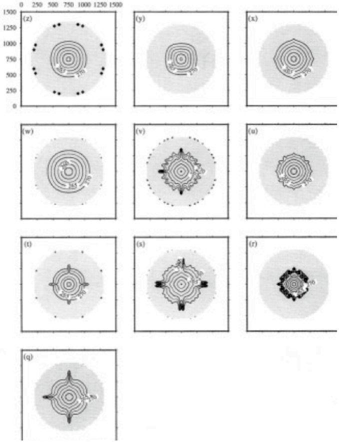


Figure: Predicted Steady-state basal temperatures in Exp. A (from EISMINT II)

EISMINT: Experiment B

- Use as initial condition the final steady-state ice sheet of Exp. A (constant $T_{min} = 238.15K$) and
- surface temperature experiences $5K$ warming $(T_{min} = 243.15K)$.

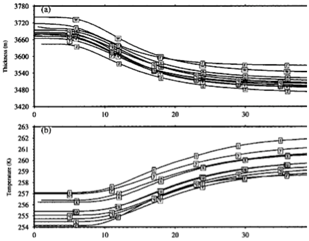


Figure: from EISMINT II: Time series of thickness and basal temperature

EISMINT: Experiment F

- Use as initial condition the final steady-state ice sheet of Exp. A (constant $T_{min} = 238.15K$) and
- surface temperature experiences $15K$ cooling $(T_{min} = 223.15K)$
- Are the spokes (in exp. F) just numerical errors? No, they reflect a sensitivity of the continuum equations to perturbation in some geometry/temperature regimes (Bueler, Hindmarsh).

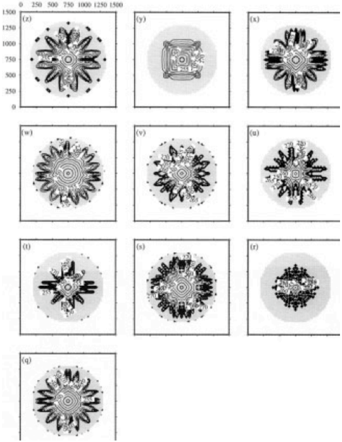
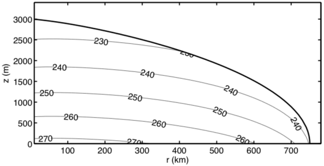


Figure: Predicted Steady-state basal temperatures in Exp. F (from EISMINT II).

Verification of the SIA model using manufactured solution (Bueler)

Exact solution to thermocoupled SIA



h, T chosen (circular ice caps like EISMINT) $\longrightarrow$ compute accumulation, velocity, etc. which satisfy all equations.

Exact solution formulas

$$h(r,t) = h(r) + \phi(r)\gamma(t), \quad T(r,t,z) = T(r)\frac{\nu(t,r) + h(r,t)}{\nu(t,r) + z}, \text{ where}$$

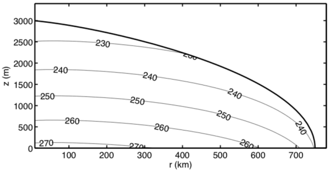
$$h(r) = \frac{h_0}{\left(1 - \frac{1}{n}\right)^{\frac{n}{2n+2}}} \left[\left(1 + \frac{1}{n}\right) s - \frac{1}{n} + (1-s)^{1+\frac{1}{n}} - s^{1+\frac{1}{n}} \right]^{\frac{n}{2n+2}},$$

$$\phi(r) = cos^2\left(\frac{\pi(r - 0.6L)}{0.6L}\right), \quad \gamma(r) = A_r sin\left(\frac{2\pi t}{t_p}\right),$$

$$\nu(t,r) = \frac{kT(r)}{2G}\left(1 + \sqrt{1 + 4\frac{h(t,r)G}{kT(r)}}\right), \quad s = r/L.$$

Verification of the SIA model using manufactured solution (Bueler)

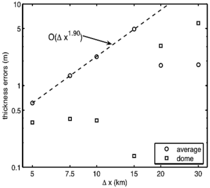
Exact solution to thermocoupled SIA

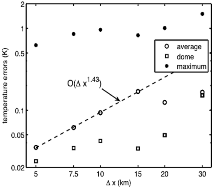


h, T chosen (circular ice caps like EISMINT) $\longrightarrow$ compute accumulation, velocity, etc. which satisfy all equations.

Useful for

- calculation of numerical errors (h and T) and
- measuring convergence rate under grid refinement





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Higher-order (HO) and full-Stokes (FS) 3-D models

Reference: F. Pattyn, Investigating the stability of subglacial lakes with a full-Stokes ice-sheet model, ... 2008.
R.C.A.Hindmarsh, A numerical comparison of approximation to the Stokes equations used in ice sheet and glacier modeling, ... 2004.

Assumptions: Higher-order – variational stresses are neglected, full-Stokes – all stresses are included.

Numerics: Higher-order – 2D models; full-Stokes – 3D models.

Conclusion: Better predictions but computationally intensive.

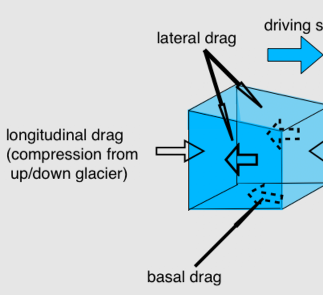


Figure: Force Balance

Verification of the full-Stokes and HO models

intercomparison of models

- ISMIP-HOM - isothermal flow (Pattyn et al, 2008)

verification by exact solution

- quasi-analytical solutions for the 1-st order approximation equations (Blatter 1995)
- analytical solutions for transient 2-D flow (Hutter 1980, Hutter 1983)
- 3-D solution of the linearized 0-th order problem (Gudmundsson 2003)
- manufactured solutions of a steady-state isothermal 2-D and 3-D flow (Fastook and Sargent)

ISMIP-HOM

- Intercomparison of 28 full-Stokes and HO models.
- 6 experiments (2-D and 3-D geometries): 5 - steady-state (Glen-type flow law), 1 - time-dependent (constant viscosity);
- 1 experiment with data from Haut Glacier d'Arolla;
- Isothermal ice mass;
- Periodic lateral boundary conditions.

ISMIP-HOM: Experiment B

- 2-D: steady-state ice flow over a rippled bed;
- Boundary conditions: frozen bed, stress-free surface, periodic lateral;
- The surface elevation and the bed topography are defined as:

$$s(x,y) = -x \cdot \tan\alpha,$$
$$b(x,y) = s(x,y) - 1000 + 500\sin(\omega x),$$

where $\omega = 2\pi/L$, L is the ice length.

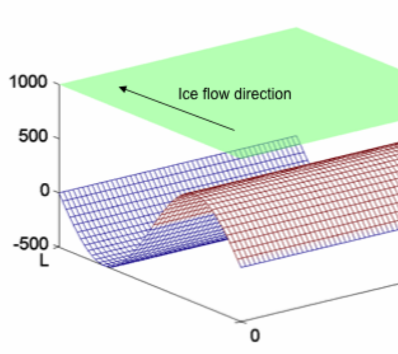


Figure: Experiment B - from Pattyn, 2008.

ISMIP-HOM: Conclusions (from Pattyn, 2008)

- Benchmarks work better for longer length scales than for smaller;
- However, interesting features appear at smaller length scales ($L = 5$): distinction between FS and HO models;
- Differences between models are due to either physical approximations or numerical problems/inaccuracies.

▸ FS verification

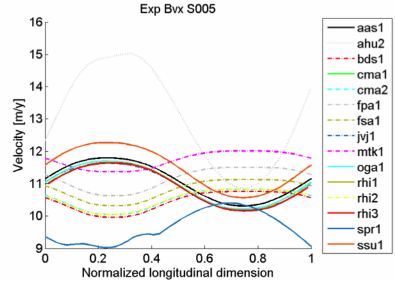


Figure: Experiment B: clear distinction in behavior between HO and FS models (Pattyn, 2008).

Steady-state isothermal 2-D flowline model

conservation of mass

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{5}$$

kinematic boundary conditions

$$u(x, s(x))\frac{ds}{dx} - w(x, s(x)) = \dot{a}, \tag{6}$$
$$u(x, b(x))\frac{db}{dx} - w(x, b(x)) = 0, \tag{7}$$

Steady-state isothermal 2-D flowline model

conservation of momentum

$$\frac{\partial\left(2\mu\frac{\partial u}{\partial x}+p\right)}{\partial x}+\frac{\partial\left(\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right)}{\partial z}=0, \tag{8}$$

$$\frac{\partial\left(\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right)}{\partial x}+\frac{\partial\left(2\mu\frac{\partial w}{\partial z}+p\right)}{\partial z}=\rho g, \tag{9}$$

boundary conditions

- stress-free surface;
- frozen bed or sliding bed;
- lateral bc: periodic or Dirichlet.

constitutive relation (Glen's ice flow law)

$$\mu=\frac{B}{2}\left(\frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2-\frac{\partial u}{\partial x}\frac{\partial w}{\partial z}\right)^{\frac{1-n}{2n}}, \tag{10}$$

Derivation of a manufactured exact solution

Let's assume that in the domain $s(x) > b(x)$:

$$w(x,z) = u(x,z)\left(\frac{db}{dx}\frac{s-z}{s-b} + \frac{ds}{dx}\frac{z-b}{s-b}\right) - \dot{a}\frac{z-b}{s-b}. \tag{11}$$

then

1. kinematic boundary conditions are satisfied and
2. conservation of mass equation is reduced to the equation of one variable $u(x,z)$:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\left(\frac{db}{dx}\frac{s-z}{s-b} + \frac{ds}{dx}\frac{z-b}{s-b}\right) + u\frac{\frac{ds}{dx} - \frac{db}{dx}}{s-b} - \frac{\dot{a}}{s-b} = 0. \tag{12}$$

This equation has a solution:

$$u(x,z) = \frac{1}{s(x)-b(x)}\vartheta\left(\frac{z-b(x)}{s(x)-b(x)}\right) + \frac{\dot{a}x}{s(x)-b(x)}, \tag{13}$$

where ϑ is an arbitrary function of one variable.

Satisfying the conservation of momentum equation and the stress-free surface boundary conditions

Substitution of the manufactured solution to the conservation of momentum equations and the surface boundary conditions will result in additional terms in the PDEs:

conservation of momentum

$$\frac{\partial\left(2\mu\frac{\partial u}{\partial x}+p\right)}{\partial x}+\frac{\partial\left(\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right)}{\partial z}=\Sigma_x, \quad (14)$$

$$\frac{\partial\left(\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right)}{\partial x}+\frac{\partial\left(2\mu\frac{\partial w}{\partial z}+p\right)}{\partial z}-\rho g=\Sigma_z, \quad (15)$$

boundary conditions

- stress-free surface bc (add an artificial term to the RHS);
- frozen bed bc – satisfied automatically;
- periodic side bc – satisfied automatically.

Verification of the FS model using manufactured solution

Let's, for simplicity, assume that function ϑ is as follows:

$$\vartheta(x) = x^{\lambda} + c_b, \tag{16}$$

where $\lambda \geq 2$ and $c_b \geq 0$ are constants; $c_b = 0$ for frozen-bed solutions; then

Exact solutions are

$$\begin{aligned} u(x,z) &= \frac{1}{s(x)-b(x)}\left(\frac{z-b(x)}{s(x)-b(x)}\right)^{\lambda} + \frac{c_b}{s(x)-b(x)}, \\ w(x,z) &= u(x,z)\left(\frac{\partial b}{\partial x}\frac{s-z}{s-b} + \frac{\partial s}{\partial x}\frac{z-b}{s-b}\right) \end{aligned}$$

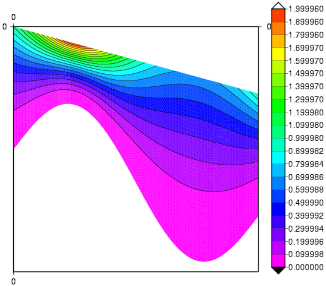
where $s(x)$ and $b(x)$ are the ice surface elevation and bed topography curves.

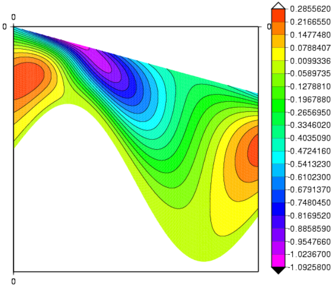
Verification of the FS model using manufactured solution: EISMINT Experiment B type solution

If surface elevation and bed topography are defined as in Exp. B of ISMIP-HOM:

$$s(x,y) = -x \cdot \tan\alpha,$$
$$b(x,y) = s(x,y) - 1000 + 500\sin(\omega x),$$

then the horizontal and vertical velocities are as follows:





Verification of the FS using manufactured solution: How realistic is the solution?

- conservation of mass flux: $q = hu = 1$ is satisfied:

$$\text{for } z = b, \qquad u(x,b) = 0, \quad w(x,b) = 0;$$

$$\text{for } z = s, \qquad u(x,s) = \frac{1}{s-b} = \frac{1}{h}, \quad w(x,s) =$$

- This anti-correlated relationship between u and h is consistent with the simulation of a Exp. B in ISMIP-HOM by *all* flowline full-Stokes models.

▸ Experiment B, EISMINT

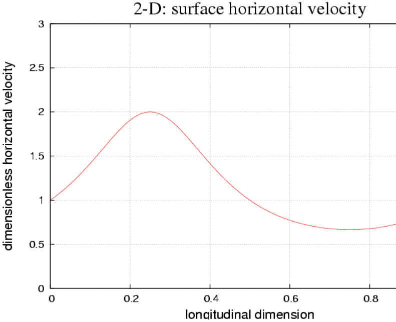


Figure: Surface horizontal velocity.

Verification of the FS using manufactured solution

Summary:

- Manufactured analytical solutions for 2-D steady-state isothermal flowline models are derived:
  - variable viscosity;
  - solutions can be specified for different surface and bed geometries;
  - solutions are periodic;
  - solutions are easy to use.
- Similar manufactured solutions derived for 3-D full-Stokes ice flow model.

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Thank you for your attention!