Manufactured analytical solutions to the steady-state isothermal 2-D flowline and 3-D full-Stokes ice-flow models

1 Introduction

Model verification is crucial in developing a numerical model. Ice-sheet modeling community has been using two tools to verify models, comparison of numerically computed solutions to analytical solutions when possible; and *intercomparison*, that is, measuring differences between various models' results on the sets of simplified geometry benchmark tests.

For *shallow-ice approximation* (SIA) models, the simplified geometry tests as well as the results of intercomparison of different SIA models can be found in [7]. As for the exact solutions for SIA equations, two techniques have been used to generate analytical solutions, *similarity reduction* technique (an approach that identifies equations for which the solution depends on certain groupings of the independent variables rather than depending on each of the independent variables separately [9, 12, 13, 2]) and *manufactured solutions* technique (an approach that chooses a reasonable "solution" functions, say a velocity-field and pressure, substitutes them into the Stokes equations, and determines the body force necessary to make the chosen functions into actual solutions [2, 3, 4]).

For higher-order models and full-Stokes models, the simplified geometry tests and the results of intercomparison of different models can be found in [8]. As for the exact solutions, mathematical work has mainly focused on the flow of *linear* media and a quasi-analytical solutions have been found for the first-order approximation equations for computing the three-dimensional stress and velocity field in grounded glaciers in [1]. Analytical solutions have been found describing transient two dimensional flow [14, 15, 16], three-dimensional steady-state flow [17, 16] and transient evolution flow [10].

All the above solutions give a good physical insight into the flow processes; however, they cannot be easily used to benchmark the numerical solutions. For example, Gudmundsson in [10] obtained the three-dimensional solution of the linearized version of the zeroth-order problem for a linear viscous medium while numerical solutions are not limited by this assumption. Therefore, to use this solution for benchmarking numerical ice sheet models, the exact error estimate has to be done [11].

In this paper, we are offering a manufactured analytical solutions for computing the three-dimensional and two-dimensional stress and velocity fields of the steady-state isothermal 2-D flowline and 3-D full-Stokes ice flow models that can be easily used to benchmark the numerical solutions. The analytical solutions are solutions of the Stokes problem with variable viscosity. For 2-D models, the boundary conditions can be specified as essential Dirichlet conditions or as the periodic boundary conditions similar to boundary conditions of experiment B (ice flow over a rippled bed) [8]. For 3-D models, the boundary conditions are specified as essential boundary conditions. The advantage of the solutions is their simplicity and easy application. Another advantage is that the analytical solutions can be found for different surface and bed geometries. By changing a parameter value, the analytical solutions will allow the modelers to investigate their algorithms for different range of aspect ratios. Finally, the analytical solutions may help the modelers to estimate the numerical error in the case when the effect of the boundary conditions are eliminated, that is, when the exact solutions values are specified as inflow and outflow boundary conditions.

2 Manufactured analytical solutions of the 2-D isothermal steady-state flowline ice-flow model

2.1 Ice-Flow Model

We consider a two-dimensional steady-state flowline model in the Cartesian coordinates $(\tilde{x}, \tilde{z})$ with the domain $0 \leq \tilde{x} \leq L$, $\tilde{b}(\tilde{x}) \leq \tilde{z} \leq \tilde{s}(\tilde{x})$, where $\tilde{s}(\tilde{x})$ defines the surface and $\tilde{b}(\tilde{x})$ defines the base of the glacier. Dimensional variables in this work are denoted with a tilde and non-dimensional variables without.

The field equations for the isothermal steady-state 2-D flowline model consist of the conservation of mass and the conservation of momentum:

$$\vec{v}_{i,j} = 0,$$
$$\frac{\partial \tilde{\mathbf{T}}_{ij}}{\partial j} = \tilde{\rho}\tilde{g}\delta_{ij}.$$

where $\tilde{\mathbf{T}}$ is the stress tensor

$$\tilde{\mathbf{T}} = \begin{bmatrix} 2\tilde{\mu}\dot{\tilde{\epsilon}}_{\tilde{x}\tilde{x}} + \tilde{p} & 2\tilde{\mu}\dot{\tilde{\epsilon}}_{\tilde{x}\tilde{z}} \\ 2\tilde{\mu}\dot{\tilde{\epsilon}}_{\tilde{x}\tilde{z}} & 2\tilde{\mu}\dot{\tilde{\epsilon}}_{\tilde{z}\tilde{z}} + \tilde{p} \end{bmatrix},$$

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$\tilde{\rho}$ is the ice density, $\tilde{g}$ is the gravitational acceleration, $\vec{\tilde{v}} = (\tilde{u}, \tilde{w})$ is the velocity vector, $\tilde{\mu}$ is the effective viscosity, $\tilde{p}$ is the ice pressure, and $\tilde{\epsilon}_{ij}$ are the components of the strain-rate tensor.

These equations can be written in expanded form as:

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{w}}{\partial \tilde{z}} = 0, \tag{1}$$

$$\frac{\partial \left(2\tilde{\mu}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{p}\right)}{\partial \tilde{x}} + \frac{\partial \left(\tilde{\mu}\left(\frac{\partial \tilde{u}}{\partial \tilde{z}} + \frac{\partial \tilde{w}}{\partial \tilde{x}}\right)\right)}{\partial \tilde{z}} = 0, \tag{2}$$

$$\frac{\partial \left(\tilde{\mu}\left(\frac{\partial \tilde{w}}{\partial \tilde{x}} + \frac{\partial \tilde{u}}{\partial \tilde{z}}\right)\right)}{\partial \tilde{x}} + \frac{\partial \left(2\tilde{\mu}\frac{\partial \tilde{w}}{\partial \tilde{z}} + \tilde{p}\right)}{\partial \tilde{z}} = \tilde{\rho}\tilde{g}, \tag{3}$$

where

$$\tilde{\mu} = \frac{\tilde{B}}{2} \left(\frac{1}{2}\left(\frac{\partial \tilde{u}}{\partial \tilde{z}} + \frac{\partial \tilde{w}}{\partial \tilde{x}}\right)^2 - \frac{\partial \tilde{u}}{\partial \tilde{x}}\frac{\partial \tilde{w}}{\partial \tilde{z}}\right)^{\frac{1-n}{2n}}, \tag{4}$$

$\tilde{B}$ is a temperature-independent rate factor, and n is the stress exponent.

2.2 Boundary conditions

The steady-state surface and bed satisfy the kinematic boundary conditions:

$$\tilde{u}(\tilde{x}, \tilde{s}(\tilde{x}))\frac{d\tilde{s}}{d\tilde{x}} - \tilde{w}(\tilde{x}, \tilde{s}(\tilde{x})) = \dot{\tilde{a}}, \tag{5}$$

$$\tilde{u}(\tilde{x}, \tilde{b}(\tilde{x}))\frac{d\tilde{b}}{d\tilde{x}} - \tilde{w}(\tilde{x}, \tilde{b}(\tilde{x})) = 0. \tag{6}$$

The upper surface $\tilde{s}(\tilde{x})$ is stress-free: $\tilde{\mathbf{T}} \cdot \vec{n}_{\tilde{s}} = 0$, $\vec{n}_{\tilde{s}} = \frac{1}{\sqrt{1+\left(\frac{d\tilde{s}}{d\tilde{x}}\right)^2}}\left(-\frac{d\tilde{s}}{d\tilde{x}}, 1\right)$ or

$$\frac{1}{\sqrt{1+\left(\frac{d\tilde{s}}{d\tilde{x}}\right)^2}}\left[-\frac{d\tilde{s}}{d\tilde{x}}\left(2\tilde{\mu}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{p}\right) + \tilde{\mu}\left(\frac{\partial \tilde{u}}{\partial \tilde{z}} + \frac{\partial \tilde{w}}{\partial \tilde{x}}\right)\right] = 0, \tag{7}$$

$$\frac{1}{\sqrt{1+\left(\frac{d\tilde{s}}{d\tilde{x}}\right)^2}}\left[-\frac{d\tilde{s}}{d\tilde{x}}\left(\tilde{\mu}\left(\frac{\partial \tilde{w}}{\partial \tilde{x}} + \frac{\partial \tilde{u}}{\partial \tilde{z}}\right)\right) + \left(2\tilde{\mu}\frac{\partial \tilde{w}}{\partial \tilde{z}} + \tilde{p}\right)\right] = 0. \tag{8}$$

The lower surface $\tilde{b}(\tilde{x})$ may be frozen, that is, Dirichlet conditions are specified at the bed; or the shear stresses may be specified at the bed, that is, Newman boundary

conditions $\tilde{\mathbf{T}} \cdot \vec{n}_{\tilde{b}} = \vec{\tilde{f}} \cdot \vec{n}_{\tilde{b}}$, where $\vec{n}_{\tilde{b}} = \frac{1}{\sqrt{1+\left(\frac{d\tilde{b}}{d\tilde{x}}\right)^2}} \left(\frac{d\tilde{b}}{d\tilde{x}}, -1\right)$ and $\vec{\tilde{f}} = \left(-\tilde{\tau}_{\tilde{b}}, \tilde{\rho}\tilde{g}\tilde{h}\right)$, $\tilde{\tau}_{\tilde{b}} = \beta^2 \tilde{u}(\tilde{x}, \tilde{b})$, are specified at the bed:

$$\frac{1}{\sqrt{1+\left(\frac{d\tilde{b}}{d\tilde{x}}\right)^2}} \left[\frac{d\tilde{b}}{d\tilde{x}} \left(2\tilde{\mu}\frac{\partial\tilde{u}}{\partial\tilde{x}} + \tilde{p} \right) - \tilde{\mu} \left(\frac{\partial\tilde{u}}{\partial\tilde{z}} + \frac{\partial\tilde{w}}{\partial\tilde{x}} \right) \right] = -\tilde{\tau}_{\tilde{b}}, \tag{9}$$

$$\frac{1}{\sqrt{1+\left(\frac{d\tilde{b}}{d\tilde{x}}\right)^2}} \left[\frac{d\tilde{b}}{d\tilde{x}} \left(\tilde{\mu} \left(\frac{\partial\tilde{w}}{\partial\tilde{x}} + \frac{\partial\tilde{u}}{\partial\tilde{z}} \right) \right) - \left(2\tilde{\mu}\frac{\partial\tilde{w}}{\partial\tilde{z}} + \tilde{p} \right) \right] = \tilde{\rho}\tilde{g}\tilde{h}. \tag{10}$$

Along the glacier's upstream and downstream boundaries, either periodic

$$\tilde{u}(0, \tilde{z}) = \tilde{u}(L, \tilde{z}), \quad \frac{\partial\tilde{u}}{\partial\tilde{x}}(0, \tilde{z}) = \frac{\partial\tilde{u}}{\partial\tilde{x}}(L, \tilde{z}); \tag{11}$$

$$\tilde{w}(0, \tilde{z}) = \tilde{w}(L, \tilde{z}), \quad \frac{\partial\tilde{w}}{\partial\tilde{x}}(0, \tilde{z}) = \frac{\partial\tilde{w}}{\partial\tilde{x}}(L, \tilde{z}); \tag{12}$$

$$\tilde{p}(0, \tilde{z}) = \tilde{p}(L, \tilde{z}), \quad \frac{\partial\tilde{p}}{\partial\tilde{x}}(0, \tilde{z}) = \frac{\partial\tilde{p}}{\partial\tilde{x}}(L, \tilde{z}) \tag{13}$$

or Dirichlet boundary conditions

$$\tilde{u}(0, \tilde{z}) = \tilde{u}_{exact}(0, \tilde{z}), \quad \tilde{u}(L, \tilde{z}) = \tilde{u}_{exact}(L, \tilde{z}); \tag{14}$$

$$\tilde{w}(0, \tilde{z}) = \tilde{w}_{exact}(0, \tilde{z}), \quad \tilde{w}(L, \tilde{z}) = \tilde{w}_{exact}(L, \tilde{z}); \tag{15}$$

$$\tilde{p}(0, \tilde{z}) = \tilde{p}_{exact}(0, \tilde{z}), \quad \tilde{p}(L, \tilde{z}) = \tilde{p}_{exact}(L, \tilde{z}) \tag{16}$$

may be specified.

2.3   Dimensionless Equations

To non-dimensionalize variables, let's choose the following typical values: Z - the mean thickness of the ice-sheet, L - the length of ice-sheet, U - a typical velocity in the horizontal direction, W - a typical velocity in the vertical direction, P - the mean pressure and introduce the following non-dimensional variables (variables without

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tilde):

$$\tilde{z} = Zz, \tilde{s} = Zs, \tilde{b} = Zb$$
$$\tilde{x} = Lx$$
$$\tilde{u} = U u \qquad (17)$$
$$\tilde{w} = W w,$$
$$\tilde{p} = P p,$$
$$\tilde{\mu} = \frac{\tilde{B}}{2} \left(\frac{U}{L}\right)^{\frac{1-n}{n}} \mu.$$

To simplify equations, let's introduce aspect ratio parameter δ:

$$\delta = \frac{Z}{L} \qquad (18)$$

and have scale factors L, U, W, and $\overline{P}$ satisfy the following relationships:

$$\frac{\tilde{B}}{2} \left(\frac{U}{L}\right)^{\frac{1}{n}} = \tilde{\rho}\tilde{g}Z = \overline{P}, \quad \frac{WL}{UZ} = 1, \quad [\dot{\tilde{a}}] = W, \qquad (19)$$

then the nondimensional steady-state conservation of mass and momentum equations are as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \qquad (20)$$

$$\delta\frac{\partial \left(2\mu\frac{\partial u}{\partial x} + p\right)}{\partial x} + \frac{\partial \left(\mu \left(\frac{1}{\delta}\frac{\partial u}{\partial z} + \delta\frac{\partial w}{\partial x}\right)\right)}{\partial z} = 0, \qquad (21)$$

$$\delta\frac{\partial \left(\mu \left(\delta\frac{\partial w}{\partial x} + \frac{1}{\delta}\frac{\partial u}{\partial z}\right)\right)}{\partial x} + \frac{\partial \left(2\mu\frac{\partial w}{\partial z} + p\right)}{\partial z} = 1, \qquad (22)$$

where

$$\mu = \left(\frac{1}{2}\left(\frac{1}{\delta}\frac{\partial u}{\partial z} + \delta\frac{\partial w}{\partial x}\right)^2 - \frac{\partial u}{\partial x}\frac{\partial w}{\partial z}\right)^{\frac{1-n}{2n}}. \qquad (23)$$

The kinematic boundary conditions are invariant under the chosen set of scalings:

$$u(x, s(x))\frac{ds}{dx} - w(x, s(x)) = \dot{a}, \qquad (24)$$

$$u(x, b(x))\frac{db}{dx} - w(x, b(x)) = 0, \qquad (25)$$

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where

$$
\dot{a} = \begin{cases} 0 & \text{if } \tilde{a} = 0 \\ 1 & \text{otherwise} \end{cases} \tag{26}
$$

The stress-free boundary conditions at the upper surface $s(x,t)$ become as follows:

$$
\frac{1}{\sqrt{1 + \delta^2 \left(\frac{ds}{dx}\right)^2}} \left[-\delta \frac{ds}{dx} \left(2\mu \frac{\partial u}{\partial x} + p \right) + \mu \left(\frac{1}{\delta} \frac{\partial u}{\partial z} + \delta \frac{\partial w}{\partial x} \right) \right] = 0, \tag{27}
$$

$$
\frac{1}{\sqrt{1 + \delta^2 \left(\frac{ds}{dx}\right)^2}} \left[-\delta \frac{ds}{dx} \left(\mu \left(\delta \frac{\partial w}{\partial x} + \frac{1}{\delta} \frac{\partial u}{\partial z} \right) \right) + \left(2\mu \frac{\partial w}{\partial z} + p \right) \right] = 0. \tag{28}
$$

while the Newman boundary conditions at the lower surface $b(x,t)$ become as follows:

$$
\frac{1}{\sqrt{1 + \delta^2 \left(\frac{db}{dx}\right)^2}} \left[\delta \frac{db}{dx} \left(2\mu \frac{\partial u}{\partial x} + p \right) - \mu \left(\frac{1}{\delta} \frac{\partial u}{\partial z} + \delta \frac{\partial w}{\partial x} \right) \right] = -\frac{\tau_b}{P}, \tag{29}
$$

$$
\frac{1}{\sqrt{1 + \delta^2 \left(\frac{db}{dx}\right)^2}} \left[\delta \frac{db}{dx} \left(\mu \left(\delta \frac{\partial w}{\partial x} + \frac{1}{\delta} \frac{\partial u}{\partial z} \right) \right) - \left(2\mu \frac{\partial w}{\partial z} + p \right) \right] = 1. \tag{30}
$$

In scaled units, the glacier thickness and length are equal to unity. The only change in upstream and downstream boundary conditions is the fact that they are specified at the domain boundaries $x = 0$ and $x = 1$.

2.4   Deriving an exact solution

To satisfy the kinematic boundary conditions (24)-(25), let's assume that in the domain $s > b$

$$
w(x,z) = u(x,z) \left(\frac{db}{dx} \frac{s-z}{s-b} + \frac{ds}{dx} \frac{z-b}{s-b} \right) - \dot{a} \frac{z-b}{s-b}. \tag{31}
$$

From (31), it follows that

$$
\frac{\partial w}{\partial z} = \frac{\partial u}{\partial z} \left(\frac{db}{dx} \frac{s-z}{s-b} + \frac{ds}{dx} \frac{z-b}{s-b} \right) + u \frac{\frac{ds}{dx} - \frac{db}{dx}}{s-b} - \frac{\dot{a}}{s-b}. \tag{32}
$$

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If we substitute (32) into the incompressibility equation (20), we get the following equation containing only variable u and its derivatives:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\left(\frac{db}{dx}\frac{s-z}{s-b} + \frac{ds}{dx}\frac{z-b}{s-b}\right) + u\frac{\frac{ds}{dx}-\frac{db}{dx}}{s-b} - \frac{\dot{a}}{s-b} = 0. \tag{33}$$

Equation (33) is a first-order quasi linear partial differential equation with two independent variables (x and z) and one dependent variable (u). The system of ordinary differential equations

$$\frac{dx}{1} = \frac{dz}{\frac{db}{dx}\frac{s-z}{s-b} + \frac{ds}{dx}\frac{z-b}{s-b}} = -\frac{du}{u\frac{\frac{ds}{dx}-\frac{db}{dx}}{s-b} - \frac{\dot{a}}{s-b}} \tag{34}$$

is called the *characteristic system* of equation (33). If we find two independent particular solutions of this system, which are called the *integrals of system* (34), in the form

$$\phi(x, z, u) = c_1, \quad \psi(x, z, u) = c_2, \tag{35}$$

where c_1 and c_2 are arbitrary constants, then the general solution of equation (33) can be written as

$$\theta(\phi, \psi) = 0, \tag{36}$$

where θ is an arbitrary function of two variables. With equation (36) solved for ϕ, the general solution can be written in the form

$$\phi = \vartheta(\psi), \tag{37}$$

where ϑ is an arbitrary function of one variable.

Thus, to solve equation (33), we have to find integrals ϕ and ψ of the system (34). The first integral of the system (34) can be found by solving equation

$$\frac{dx}{1} = -\frac{du}{u\frac{\frac{ds}{dx}-\frac{db}{dx}}{s-b} - \frac{\dot{a}}{s-b}}. \tag{38}$$

Equation (38) can be re-written as follows:

$$\frac{du}{dx} + \frac{\frac{ds}{dx}-\frac{db}{dx}}{s-b}u = \frac{\dot{a}}{s-b}. \tag{39}$$

If we multiply both sides of equation (39) by $s - b$ and recognize that the left side of the equation is now the following product rule, $(s-b)\frac{du}{dx} + (\frac{ds}{dx} - \frac{db}{dx})u = \frac{d[u(s-b)]}{dx}$. After replacing the left side of the equation with this product rule, we get an equation:

$$\frac{d\,[u(s-b)]}{dx} = \dot{a}.\tag{40}$$

Equation (40) has a solution

$$u \cdot (s - b) = \dot{a}x + c_1, \quad \text{or} \quad c_1 = u \cdot (s - b) - \dot{a}x,\tag{41}$$

where c_1 is a constant.

The second integral of the system (34) can be found by solving equation

$$\frac{dx}{1} = \frac{dz}{\frac{db}{dx}\frac{s-z}{s-b} + \frac{ds}{dx}\frac{z-b}{s-b}}\tag{42}$$

Equation (42) can be re-written as:

$$\frac{dz}{dx} - \frac{s' - b'}{s - b}z = -\frac{s'b - b's}{s - b}\tag{43}$$

This is a first-order ordinary differential equation of the form

$$\frac{dz}{dx} + p(x)z = q(x)\tag{44}$$

where

$$p(x) = -\frac{s'(x) - b'(x)}{s(x) - b(x)},\tag{45}$$

$$q(x) = -\frac{s'(x)b(x) - b'(x)s(x)}{s(x) - b(x)}.\tag{46}$$

Equation (44) can be solved by finding an *integrating factor* $\nu = \nu(x)$ such that

$$\begin{aligned}\frac{d(\nu z)}{dx} &= \nu\frac{dz}{dx} + z\frac{d\nu}{dx}\\ &= \nu q(x).\end{aligned}\tag{47}$$

Dividing through by νz yields

$$\frac{1}{z}\frac{dz}{dx} + \frac{1}{\nu}\frac{d\nu}{dx} = \frac{q(x)}{z}.\tag{48}$$

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Comparing (44) and (48), we can determine the appropriate ν for arbitrary p and q by taking

$$p(x) = \frac{1}{\nu}\frac{d\nu}{dx} \tag{49}$$

Integrating both sides of (49) and using $p(x) = -\frac{s'-b'}{s-b}$ gives

$$\nu(x) = e^{\int p(x)dx} = e^{-\int \frac{s'-b'}{s-b}dx} = e^{-\int \frac{d(s-b)}{s-b}} = e^{-ln(s-b)+ln(c)} = \frac{c}{s-b} \tag{50}$$

Substituting (50) and (46) into (47) gives an equation:

$$\frac{d\frac{cz}{s(x)-b(x)}}{dx} = -c\frac{s'(x)b(x) - b'(x)s(x)}{(s(x)-b(x))^2}, \tag{51}$$

or

$$\frac{d\frac{z}{s(x)-b(x)}}{dx} = -\frac{\left(\frac{s(x)}{b(x)}\right)'}{\left(\frac{s(x)}{b(x)} - 1\right)^2}. \tag{52}$$

Equation (52) has a solution

$$\frac{z}{s-b} = \frac{b}{s-b} + c_2, \quad \text{or } c_2 = \frac{z-b}{s-b}, \tag{53}$$

where c_2 is a constant.

Thus, the general solution of equation (20) can be written as

$$\theta\left(u \cdot (s(x) - b(x)) - \dot{a}x, \frac{z-b(x)}{s(x)-b(x)}\right) = 0, \tag{54}$$

where θ is an arbitrary function of two variables. With equation (54) solved for u, the general solution can be written in the form

$$u(x,z) = \frac{1}{s(x)-b(x)}\vartheta\left(\frac{z-b(x)}{s(x)-b(x)}\right) + \frac{\dot{a}x}{s(x)-b(x)}, \tag{55}$$

where ϑ is an arbitrary function of one variable.

The formula (55) shows that the functions satisfying the kinematic boundary conditions (24)-(25) and the conservation of mass equation (20), derived under assumption (31), depend on the form of the function ϑ and ice surface and bed curves.

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Let's, for simplicity, assume that function ϑ is as follows:

$$\vartheta(x) = x^\lambda, \tag{56}$$

where λ is a constant.

Let's also assume zero accumulation/ablation rate, $\dot{a} = 0$, then the steady-state velocity satisfying the kinematic boundary conditions and the conservation of mass equation are:

$$u(x, z) = \frac{1}{s(x) - b(x)} \left(\frac{z - b(x)}{s(x) - b(x)} \right)^\lambda, \tag{57}$$

$$w(x, z) = u(x, z) \left(\frac{db}{dx} \frac{s - z}{s - b} + \frac{ds}{dx} \frac{z - b}{s - b} \right), \tag{58}$$

Choice (56) of function ϑ generates a frozen bed solutions. If we assume that function ϑ is as follows:

$$\vartheta(x) = x^\lambda + c_b, \tag{59}$$

where c_b is a constant, then we generate solutions with non-zero basal velocities:

$$u(x, z) = \frac{1}{s(x) - b(x)} \left(\frac{z - b(x)}{s(x) - b(x)} \right)^\lambda + \frac{c_b}{s(x) - b(x)}, \tag{60}$$

$$w(x, z) = u(x, z) \left(\frac{\partial b}{\partial x} \frac{s - z}{s - b} + \frac{\partial s}{\partial x} \frac{z - b}{s - b} \right), \tag{61}$$

The constructed velocities satisfy the surface and bed kinematic boundary conditions (24) - (25) and the mass conservation equation (20). They do not necessarily satisfy the conservation of momentum equations (21) - (22) and the basal and surface boundary conditions (27)-(28) and (29)-(30). To make the chosen velocity functions into exact solutions of these equations, we substitute them into the equations and calculate the right-hand side functions that accommodate the solutions. This can be done when specific surface $s(x)$ and bed $b(x)$ are chosen.

Equations (57)-(58) and (60)-(61)are solutions of flow with general surface $s(x)$ and bed $b(x)$. Below are specific solutions for a particular case of a linear ice surface and a sinusoidal bed, similar to experiment B in [8].

2.5 An exact solution for a flow with a linear sloping surface and a sinusoidal bed

To generate a particular solution, let's assume zero accumulation/ablation rate, $\dot{a} = 0$, and the steady-state flow's linear sloping surface and a sinusoidal bed in the manner they are defined in ([8]) as follows:

$$s(x) = -x \cdot \tan(\alpha), \tag{62}$$

$$b(x) = s(x) - 1 + \frac{1}{2}\sin(2\pi x). \tag{63}$$

If we substitute the above functions for bed and surface into (57)-(58), then the corresponding steady-state flow's velocities are as follows:

$$u(x,z) = \frac{1}{1 - \frac{1}{2}\sin(2\pi x)}\left(\frac{z + x\tan(\alpha) + 1 - \frac{1}{2}\sin(2\pi x)}{1 - \frac{1}{2}\sin(2\pi x)}\right)^\lambda, \tag{64}$$

$$w(x,z) = u(x,z)\left(\frac{db}{dx}\frac{s-z}{s-b} + \frac{ds}{dx}\frac{z-b}{s-b}\right). \tag{65}$$

As can be seen from (64)-(65), if $\lambda > 0$ then

$$\text{for } z = b, \quad u(x,b) = 0, \ \ w(x,b) = 0;$$

$$\text{for } z = s, \quad u(x,s) = \frac{1}{s-b} = \frac{1}{h}, \ \ w(x,s) = \frac{\frac{ds}{dx}}{h}.$$

The last expression shows the conservation of mass flux, $q = hu = 1$. This anti-correlated relationship between horizontal velocity and ice thickness is consistent with the simulation of a similar problem, Experiment B in [8], by *all* flowline full-Stokes models.

We should note that for a flow down an infinite plane with a mean inclination $\tan(\alpha)$, the periodic boundary conditions for a function f are defined as follows: $f(0, z + \tan(\alpha)) = f(1, z)$ and the analytical solutions (64) - (65) satisfy these conditions.

Figures (1) and (2) show the velocities corresponding to the linear sloping surface with a slope $\alpha = 0.5°$ and a sloping sinusoidal bed. The constant in (64) is chosen as $\lambda = 2.0$. Figures (1) correspond to velocity field with frozen bed (57-58) and (2) correspond to velocity field with sliding bed (60-61). Figures (3) show the norm of velocities.

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Appendix A shows calculation of artificial force functions which make functions (64-65) satisfy the conservation of momentum equations and its boundary conditions, that is, make them the actual solutions of the flowline model.
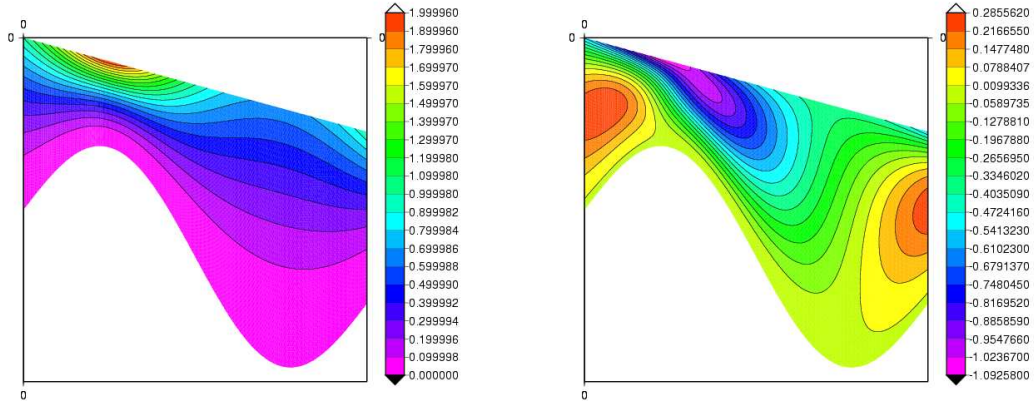

Figure 1: Velocities corresponding to a linear sloping surface and a sinusoidal frozen bed. Left: horizontal velocity, right: vertical velocity.
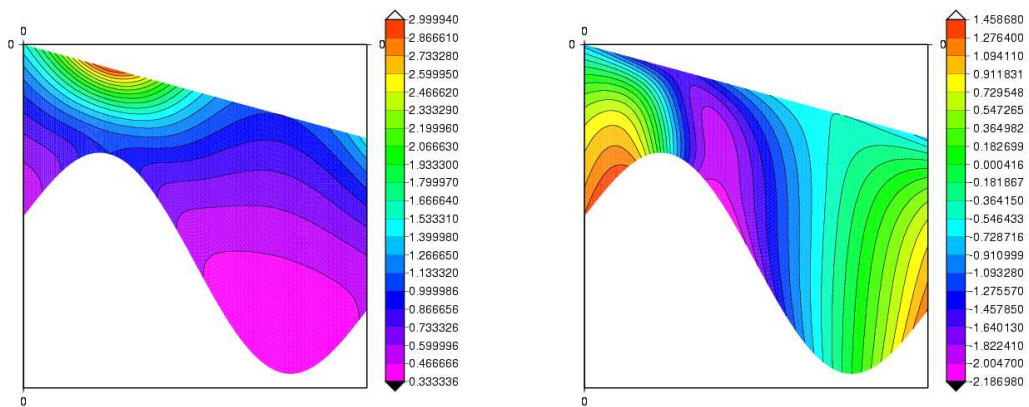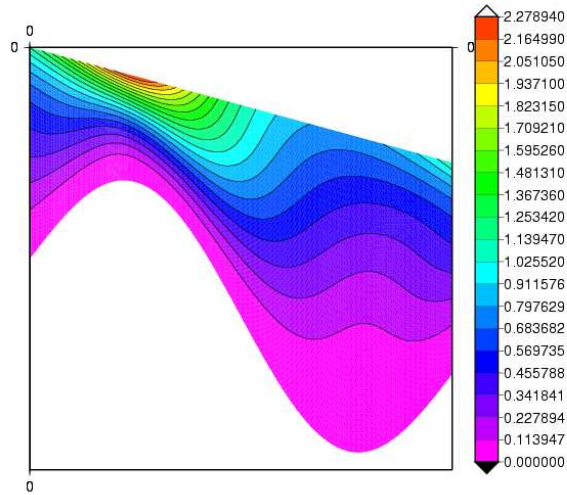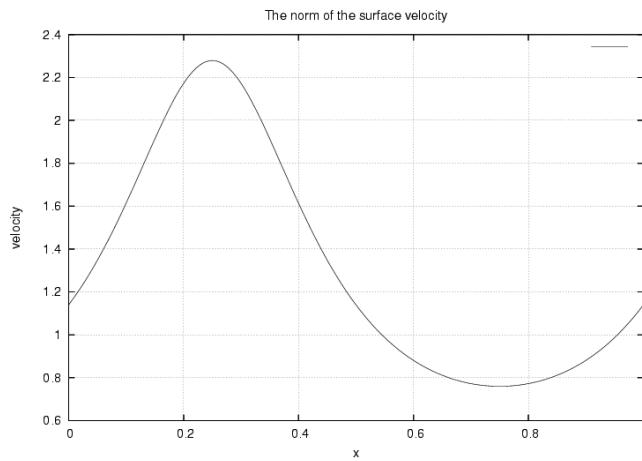
Figure 2: Velocities corresponding to a linear sloping surface and a sinusoidal bed. Sliding velocities at the bed. Left: horizontal velocity, right: vertical velocity.

(a) The norm of the velocities.

(b) The norm of the surface velocity.

Figure 3: The norm of the velocities corresponding to a linear sloping surface and a sinusoidal bed.

3 Analytical manufactured solutions of the 3-D isothermal steady-state full-Stokes ice-flow model

The dimensionless equation of the mass conservation and the surface and bed kinematic boundary conditions for three-dimensional case are as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{66}$$

$$u(x,y,s(x,y))\frac{\partial s}{\partial x} + v(x,y,s(x,y))\frac{\partial s}{\partial y} - w(x,y,s(x,y)) = \dot{a}, \tag{67}$$

$$u(x,y,b(x,y))\frac{\partial b}{\partial x} + v(x,y,b(x,y))\frac{\partial b}{\partial y} - w(x,y,b(x,y)) = 0, \tag{68}$$

If we assume as in the 2-D case that in the inside of the domain, $s(x,y) > b(x,y)$,

$$
\begin{aligned}
w(x,y,z) \;=\; & u(x,y,z)\left(\frac{\partial b}{\partial x}\frac{s-z}{s-b} + \frac{\partial s}{\partial x}\frac{z-b}{s-b}\right) \\
+\; & v(x,y,z)\left(\frac{\partial b}{\partial y}\frac{s-z}{s-b} + \frac{\partial s}{\partial y}\frac{z-b}{s-b}\right) - \dot{a}\frac{z-b}{s-b},
\end{aligned}
\tag{69}
$$

then the kinematic boundary conditions (67)-(68) are satisfied. From (69), it follows that

$$
\begin{aligned}
\frac{\partial w}{\partial z} \;=\; & \frac{\partial u}{\partial z}\left(\frac{\partial b}{\partial x}\frac{s-z}{s-b} + \frac{\partial s}{\partial x}\frac{z-b}{s-b}\right) + u\frac{\frac{\partial s}{\partial x} - \frac{\partial b}{\partial x}}{s-b} \\
+\; & \frac{\partial v}{\partial z}\left(\frac{\partial b}{\partial y}\frac{s-z}{s-b} + \frac{\partial s}{\partial y}\frac{z-b}{s-b}\right) + v\frac{\frac{\partial s}{\partial y} - \frac{\partial b}{\partial y}}{s-b} - \frac{\dot{a}}{s-b}.
\end{aligned}
\tag{70}
$$

If we substitute (70) into the incompressibility equation (66), we get the following equation containing only variables u, v and their derivatives:

$$
\begin{aligned}
& \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\left(\frac{\partial b}{\partial x}\frac{s-z}{s-b} + \frac{\partial s}{\partial x}\frac{z-b}{s-b}\right) + u\frac{\frac{\partial s}{\partial x} - \frac{\partial b}{\partial x}}{s-b} \\
& + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\left(\frac{\partial b}{\partial y}\frac{s-z}{s-b} + \frac{\partial s}{\partial y}\frac{z-b}{s-b}\right) + v\frac{\frac{\partial s}{\partial y} - \frac{\partial b}{\partial y}}{s-b} \\
& \hspace{8cm} - \frac{\dot{a}}{s-b} = 0.
\end{aligned}
\tag{71}
$$

Equation (71) is a first-order quasi-linear partial differential equation with three independent variables (x, y, and z) and two dependent variables (u and v) of type:

$$F\left(x, y, z, u(x, y, z), v(x, y, z), \frac{\partial u}{\partial x}, \frac{\partial u}{\partial z}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z}\right) = 0. \tag{72}$$

Similar to the 2-D flowline manufactured solutions, let's choose velocity $u(x, y, z)$ as following function:

$$u(x, y, z) = (s - b)^{\gamma_1}\left(\frac{z - b}{s - b}\right)^{\lambda_1}, \tag{73}$$

where γ_1 and λ_1 are some constants.

Then the derivatives of function $u(x, z)$ are

$$\frac{\partial u}{\partial x} = \gamma_1 \frac{s'_x - b'_x}{s - b} u + \lambda_1 \frac{bs'_x - sb'_x - z(s'_x - b'_x)}{(z - b)(s - b)} u, \tag{74}$$

$$\frac{\partial u}{\partial z} = \lambda_1 \frac{1}{z - b} u,$$

where $s'_x = \frac{\partial s}{\partial x}$ and $b'_x = \frac{\partial b}{\partial x}$.

Substituting (74) into (71) generates a first-order quasi-linear partial differential equation:

$$u\left[\gamma_1 \frac{s'_x - b'_x}{s - b} + \lambda_1 \frac{bs'_x - sb'_x - z(s'_x - b'_x)}{(z - b)(s - b)} + \lambda_1 \frac{1}{z - b}\left(b'_x \frac{s - z}{s - b} + s'_x \frac{z - b}{s - b}\right) + \frac{s'_x - b'_x}{s - b}\right]$$

$$+ \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\left(b'_y \frac{s - z}{s - b} + s'_y \frac{z - b}{s - b}\right) + v\frac{s'_y - b'_y}{s - b} - \frac{\dot{a}}{s - b} = 0. \tag{75}$$

Finally, substitution of (73) into (75) generates a first-order quasi-linear partial differential equation with three independent variable (x, y, and z) and only one dependent variable (v):

$$\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\left(b'_y \frac{s - z}{s - b} + s'_y \frac{z - b}{s - b}\right) + v\frac{s'_y - b'_y}{s - b} \tag{76}$$

$$+ (\gamma_1 + 1)(s'_x - b'_x)(s - b)^{\gamma - 1}\left(\frac{z - b}{s - b}\right)^{\lambda_1} - \frac{\dot{a}}{s - b} = 0.$$

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The *characteristic system* of equation (76) is as follows:

$$\frac{d\,y}{1} = \frac{d\,z}{b'_y \frac{s-z}{s-b} + s'_y \frac{z-b}{s-b}} = -\frac{d\,v}{v \frac{s'_y - b'_y}{s-b} + (\gamma_1 + 1)(s'_x - b'_x)(s-b)^{\gamma_1 - 1} \left(\frac{z-b}{s-b}\right)^{\lambda_1} - \frac{\dot{a}}{s-b}} \quad (77)$$

Let's find two independent particular solutions of this system, solving the equations:

$$\frac{dy}{1} = \frac{dz}{b'_y \frac{s-z}{s-b} + s'_y \frac{z-b}{s-b}}, \quad (78)$$

$$\frac{dy}{1} = -\frac{dv}{v \frac{s'_y - b'_y}{s-b} + (\gamma_1 + 1)(s'_x - b'_x)(s-b)^{\gamma_1 - 1} \left(\frac{z-b}{s-b}\right)^{\lambda_1} - \frac{\dot{a}}{s-b}}. \quad (79)$$

Equation (78) has a solution

$$\frac{z}{s-b} = \frac{b}{s-b} + c_1, \quad \text{or} \quad c_1 = \frac{z-b}{s-b}, \quad (80)$$

where c_1 is a constant.

Equation (79) can be re-written as follows:

$$\frac{dv}{dy} + \frac{s'_y - b'_y}{s-b} v = -(\gamma_1 + 1)(s'_x - b'_x)(s-b)^{\gamma_1 - 1} \left(\frac{z-b}{s-b}\right)^{\lambda_1} + \frac{\dot{a}}{s-b}. \quad (81)$$

This is a first-order ordinary differential equation of the form (44). The solution of the homogeneous equation is

$$v = \frac{a(y)}{s-b}. \quad (82)$$

where $a(y)$ is an unknown function.

Substituting (82) into (81), we get an equation for a:

$$a'(y) = -(\gamma_1 + 1)(s'_x - b'_x)(s-b)^{\gamma_1} \left(\frac{z-b}{s-b}\right)^{\lambda_1} + \dot{a}. \quad (83)$$

Equation (83) has a solution:

$$a(y) = -\int \left[(\gamma_1 + 1)(s'_x - b'_x)(s-b)^{\gamma_1} \left(\frac{z-b}{s-b}\right)^{\lambda_1} - \dot{a}\right] dy + c_2. \quad (84)$$

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Substituting (84) into equation (82), we get

$$v = \frac{-\int \left[(\gamma_1 + 1)(s'_x - b'_x)(s - b)^{\gamma_1} \left(\frac{z-b}{s-b}\right)^{\lambda_1} - \dot{a}\right] dy + c_2}{s - b} \qquad (85)$$

or

$$c_2 = v(s - b) + \int \left[(\gamma_1 + 1)(s'_x - b'_x)(s - b)^{\gamma_1} \left(\frac{z-b}{s-b}\right)^{\lambda_1} - \dot{a}\right] dy \qquad (86)$$

Then, the general solution of equation (76) can be written as

$$\theta \left(v(s - b) + \int \left[(\gamma_1 + 1)(s'_x - b'_x)(s - b)^{\gamma_1} \left(\frac{z-b}{s-b}\right)^{\lambda_1} - \dot{a}\right] dy, \frac{z-b}{s-b}\right) = 0, \quad (87)$$

where θ is an arbitrary function of two variables. With equation (87) solved for v, the general solution can be written in the form

$$\begin{aligned} v(x,y,z) &= \frac{1}{s-b}\vartheta\left(\frac{z-b}{s-b}\right) \\ &\quad - \frac{\int (\gamma_1 + 1)(s'_x - b'_x)(s - b)^{\gamma_1} \left(\frac{z-b}{s-b}\right)^{\lambda_1} dy - \dot{a}y}{s - b}, \end{aligned} \qquad (88)$$

where ϑ is an arbitrary function of one variable.

If we assume again that function ϑ in (88) is of the form

$$\vartheta(x) = x^{\lambda_2}, \qquad (89)$$

where λ_2 is a constant, then functions (73), (69), and (88) satisfying the mass balance equation and the kinematic boundary conditions are as follows:

$$u(x,y,z) = (s - b)^{\gamma_1} \left(\frac{z-b}{s-b}\right)^{\lambda_1}, \qquad (90)$$

$$\begin{aligned} v(x,y,z) &= \frac{1}{s-b}\left(\frac{z-b}{s-b}\right)^{\lambda_2} \\ &\quad - \frac{\int (\gamma_1 + 1)(s'_x - b'_x)(s - b)^{\gamma_1} \left(\frac{z-b}{s-b}\right)^{\lambda_1} dy - \dot{a}y}{s - b}, \end{aligned} \qquad (91)$$

$$\begin{aligned} w(x,y,z) &= u(x,y,z)\left(\frac{\partial b}{\partial x}\frac{s - z}{s - b} + \frac{\partial s}{\partial x}\frac{z - b}{s - b}\right) \\ &\quad + v(x,y,z)\left(\frac{\partial b}{\partial y}\frac{s - z}{s - b} + \frac{\partial s}{\partial y}\frac{z - b}{s - b}\right) - \dot{a}\frac{z - b}{s - b}, \end{aligned} \qquad (92)$$

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The constructed velocities satisfy the surface and bed kinematic boundary conditions (24) - (25) and the mass conservation equation (20). They do not necessarily satisfy the conservation of momentum equations and the basal and surface boundary conditions (not given in this paper). To make the chosen velocity functions into exact solutions of these equations, we substitute them into those equations and calculate the right-hand side functions which accommodate the solutions. This can be done when specific surface $s(x)$ and bed $b(x)$ are chosen.

3.1 An analytical solution for a flow with a linear sloping surface and a sinusoidal bed

To generate a particular solution, assume that the steady-state flow's linear sloping surface and a sinusoidal bed are defined similar to the ones of experiment A in [8]. In the expression below we used the following notation: $a = z + x \cdot \tan(\alpha) + 1$.

$$s(x,y) = -x \cdot \tan(\alpha), \tag{93}$$

$$b(x,y) = s(x,y) - 1 + \frac{1}{2}\sin(2\pi x)\sin(2\pi y). \tag{94}$$

3.1.1 Parameters $\gamma_1 = 2, \lambda_1 = 2, \lambda_2 = 2$

Let's first calculate integral in (91) by substituting functions (93)-(94) for bed and surface into the integral in (91):

$$
\begin{aligned}
I &= -3\int \pi\cos(2\pi x)\sin(2\pi y)\left(a - \frac{1}{2}\sin(2\pi x)\sin(2\pi y)\right)^2 dy \tag{95}\\
&= -3\pi\cos(2\pi x)\int \sin(2\pi y)\left(a^2 - a\sin(2\pi x)\sin(2\pi y) + \frac{1}{4}\sin^2(2\pi x)\sin^2(2\pi y)\right)dy\\
&= 3\pi\cos(2\pi x)\left[\frac{a^2\cos(2\pi y)}{2\pi} + \frac{a\sin(2\pi x)}{2}\int(1 - \cos(4\pi y))dy + \frac{\sin^2(2\pi x)}{8\pi}\int_{t=\cos(2\pi y)}(1-t^2)dt\right]\\
&= \frac{3}{2}\cos(2\pi x)\left[a^2\cos(2\pi y) + a\pi\sin(2\pi x)\left(y - \frac{\sin(4\pi y)}{4\pi}\right) + \frac{\sin^2(2\pi x)}{4}\left(\cos(2\pi y) - \frac{\cos^3(2\pi y)}{3}\right)\right]
\end{aligned}
$$

If we substitute the calculated integral and functions (93)-(94) for bed and sur-

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face into (90) - (92), we get the following formulas for velocities:

$$u(x,y,z) \;=\; (z-b)^2, \tag{96}$$

$$v(x,y,z) \;=\; \frac{1}{s-b}\left(\frac{z-b}{s-b}\right)^2 - \frac{I-\dot{a}y}{s-b}, \tag{97}$$

$$
\begin{aligned}
w(x,y,z) \;=\;& u(x,y,z)\left(\frac{\partial b}{\partial x}\frac{s-z}{s-b} + \frac{\partial s}{\partial x}\frac{z-b}{s-b}\right) \\
+\;& v(x,y,z)\left(\frac{\partial b}{\partial y}\frac{s-z}{s-b} + \frac{\partial s}{\partial y}\frac{z-b}{s-b}\right) - \dot{a}\frac{z-b}{s-b}.
\end{aligned}
\tag{98}
$$

Figures (4) show the surface horizontal velocities corresponding to the linear sloping surface with a slope $\alpha = 0.5\,^\circ$ and a sloping sinusoidal bed. Unfortunately,
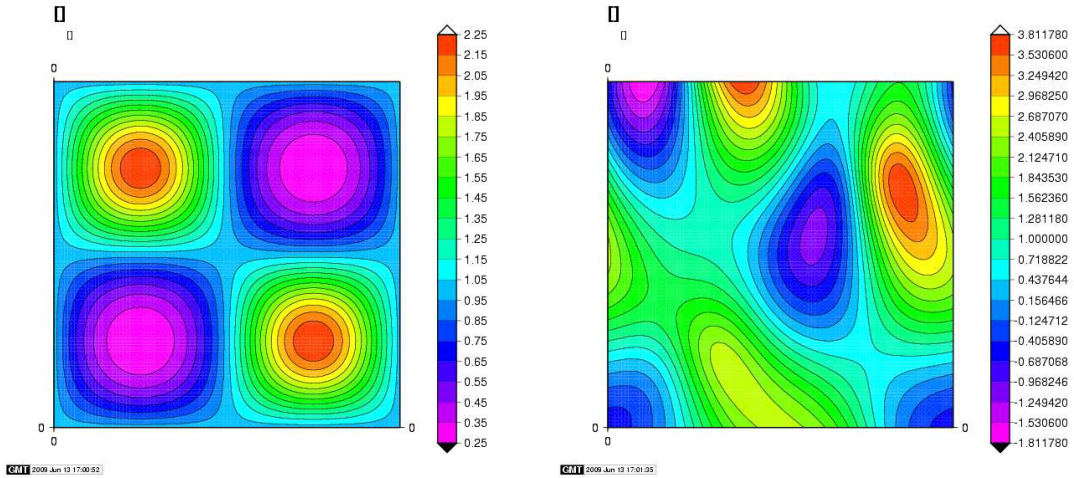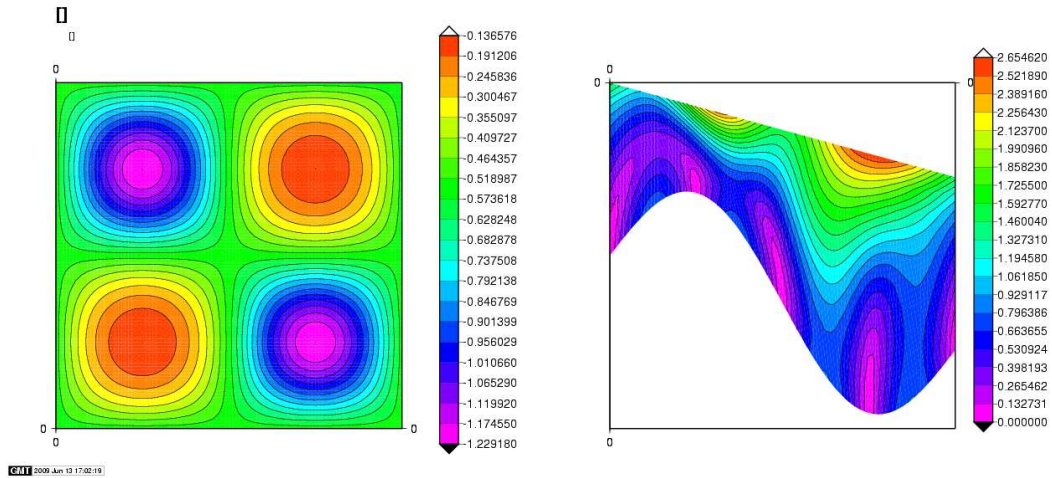
(a) Surface u_x velocity field.

(b) Surface u_y velocity field.

Figure 4: Surface horizontal velocity fields.

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the constructed velocities satisfy periodic boundary conditions only in horizontal direction x and do not satisfy the periodic boundary conditions in horizontal direction y.

Figures (5) show the surface vertical velocity and the norm of velocities at the vertical transect at $y = 1/4$.

(a) Surface u_z velocity field.

(b) The norm of velocity at the vertical transect $y = 1/4$.

Figure 5: The surface vertical velocity and the surface velocity profile at the vertical transect $y = 1/4$.

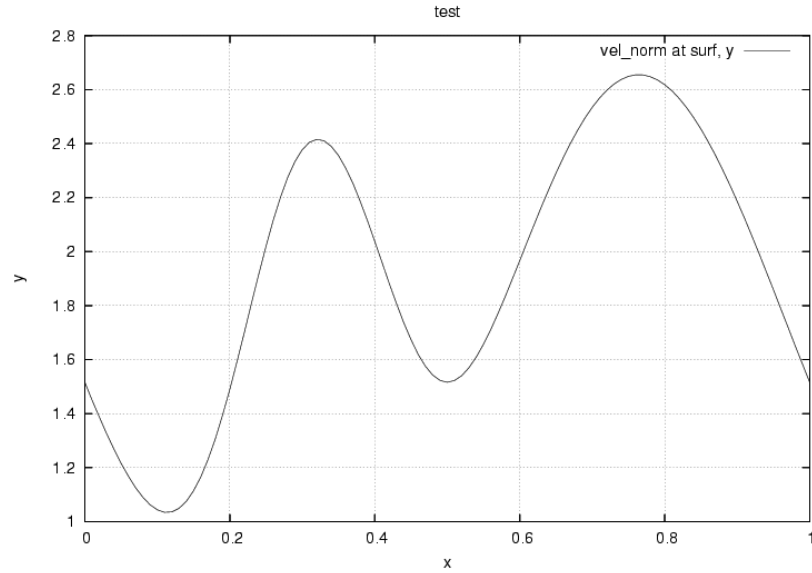Figure (6) shows the norm of the surface velocity along $y = 1/4$.

Figure 6: The norm of the surface velocity along $y = 1/4$.

4   Conclusion

In this paper, the manufactured analytical solutions are constructed for the steady-state isothermal 2-D flowline and 3-D full-Stokes ice flow models. The exact solutions allow the modelers to compute stress and velocity fields for a 2-D or a 3-D full-Stokes problem with variable viscosity for different bed and surface geometries. For a 2-D model, the boundary conditions can be specified as essential, Dirichlet, conditions or as the periodic boundary conditions similar to boundary conditions of experiment B (ice flow over a rippled bed) [8]. For a 3-D model, the boundary conditions are specified as Dirichlet conditions.

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5 Appendix A. Calculation of compensatory source functions in 2-D flowline diagnostic equations

5.1 Artificial terms in diagnostic equations and in the boundary conditions

The constructed velocities (64)-(65) satisfy the surface and bed kinematic boundary conditions (24) - (25) and the mass conservation equation (20). They do not necessarily satisfy the conservation of momentum equations (21) - (22) and its basal and surface boundary conditions (27) - (28) and (29) - (30). Following [3], let's introduce artificial stresses Σ_x and Σ_z in the conservation of momentum equations to make the chosen velocity functions into exact solutions of the equations.

$$\delta\frac{\partial\left(2\mu\frac{\partial u}{\partial x}+p\right)}{\partial x}+\frac{\partial\left(\mu\left(\frac{1}{\delta}\frac{\partial u}{\partial z}+\delta\frac{\partial w}{\partial x}\right)\right)}{\partial z}=\Sigma_x, \tag{99}$$

$$\delta\frac{\partial\left(\mu\left(\delta\frac{\partial w}{\partial x}+\frac{1}{\delta}\frac{\partial u}{\partial z}\right)\right)}{\partial x}+\frac{\partial\left(2\mu\frac{\partial w}{\partial z}+p\right)}{\partial z}-1=\Sigma_z, \tag{100}$$

The ice pressure $\left(\tilde{p}=\sigma'_{\tilde{x}\tilde{x}}-\tilde{\rho}\tilde{g}(\tilde{s}-\tilde{z})=2\tilde{\mu}\frac{\partial\tilde{u}}{\partial\tilde{x}}-\tilde{\rho}\tilde{g}(\tilde{s}-\tilde{z})\right)$ in (99)-(100) in nondimensional form is as follows:

$$p=2\mu\frac{\partial u}{\partial x}-(s-z). \tag{101}$$

Let's also introduce an artificial terms $\upsilon_x, \upsilon_z, \tau_b$, and τ_z in the boundary conditions to make the chosen velocities satisfy the boundary conditions::

at the upper surface $s(x,t)$:

$$\frac{1}{\sqrt{1+\delta^2\left(\frac{ds}{dx}\right)^2}}\left[-\delta\frac{ds}{dx}\left(2\mu\frac{\partial u}{\partial x}+p\right)+\mu\left(\frac{1}{\delta}\frac{\partial u}{\partial z}+\delta\frac{\partial w}{\partial x}\right)\right]=\upsilon_x, \tag{102}$$

$$\frac{1}{\sqrt{1+\delta^2\left(\frac{ds}{dx}\right)^2}}\left[-\delta\frac{ds}{dx}\left(\mu\left(\delta\frac{\partial w}{\partial x}+\frac{1}{\delta}\frac{\partial u}{\partial z}\right)\right)+\left(2\mu\frac{\partial w}{\partial z}+p\right)\right]=\upsilon_z \tag{103}$$

at the lower surface $b(x, t)$:

$$\frac{1}{\sqrt{1 + \delta^2 \left(\frac{db}{dx}\right)^2}} \left[\delta \frac{db}{dx} \left(2\mu \frac{\partial u}{\partial x} + p \right) - \mu \left(\frac{1}{\delta} \frac{\partial u}{\partial z} + \delta \frac{\partial w}{\partial x} \right) \right] = \tau_x, \qquad (104)$$

$$\frac{1}{\sqrt{1 + \delta^2 \left(\frac{db}{dx}\right)^2}} \left[\delta \frac{db}{dx} \left(\mu \left(\delta \frac{\partial w}{\partial x} + \frac{1}{\delta} \frac{\partial u}{\partial z} \right) \right) - \left(2\mu \frac{\partial w}{\partial z} + p \right) \right] + 1 = 0 \qquad (105)$$

$$(106)$$

5.2   Calculation of derivatives

Calculation of the artificial stress terms require calculation of derivatives of the exact solutions (57) - (58). Let's re-write these functions as follows to simplify calculation of the derivatives:

$$u(x, z) = \frac{1}{s(x) - b(x)} \left(\frac{z - b(x)}{s(x) - b(x)} \right)^{\lambda}, \qquad (107)$$

$$w(x, z) = u(x, z) f(x), \qquad (108)$$

$$\text{where } f(x) = b' \frac{s - z}{s - b} + s' \frac{z - b}{s - b}. \qquad (109)$$

Then, the first derivatives of functions (57-58) are as follows:

$$\frac{\partial u}{\partial x} = -u \frac{s' - b'}{s - b} - \lambda u \frac{b'(s - b) + (z - b)(s' - b')}{(s - b)(z - b)} \qquad (110)$$

$$= -u \left[(1 + \lambda) \frac{s' - b'}{s - b} + \lambda \frac{b'}{z - b} \right] = -u \left[\frac{s' - b'}{s - b} + \lambda \frac{f}{z - b} \right],$$

$$\frac{\partial u}{\partial z} = u \frac{\lambda}{z - b},$$

$$\frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} f + u \frac{\partial f}{\partial x},$$

$$\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x},$$

$$\frac{\partial f}{\partial x} = \frac{b''(s - z) + s''(z - b)}{s - b} - f \frac{s' - b'}{s - b}.$$

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The second derivatives are

$$
\frac{\partial^2 u}{\partial x^2} = u \left[\frac{s' - b'}{s - b} + \lambda \frac{f}{z - b} \right]^2 \tag{111}
$$
$$
-u \left[\frac{(s'' - b'')(s - b) - (s' - b')^2}{(s - b)^2} + \lambda \frac{\frac{\partial f}{\partial x}(z - b) + b'f}{(z - b)^2} \right],
$$
$$
\frac{\partial^2 u}{\partial x \partial z} = u \frac{\lambda}{(z - b)^2} \left[b' - \lambda f - \frac{s' - b'}{s - b}(z - b) \right],
$$
$$
\frac{\partial^2 u}{\partial z^2} = u \frac{\lambda(\lambda - 1)}{(z - b)^2},
$$
$$
\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} f + 2 \frac{\partial u}{\partial x} \frac{\partial f}{\partial x} + u \frac{\partial^2 f}{\partial x^2},
$$
$$
\frac{\partial^2 w}{\partial x \partial z} = -\frac{\partial^2 u}{\partial x^2},
$$
$$
\frac{\partial^2 w}{\partial z^2} = -\frac{\partial^2 u}{\partial x \partial z},
$$
$$
\frac{\partial^2 f}{\partial x^2} = \frac{b'''(s - z) + b''s' + s'''(z - b) - s''b'}{s - b} - 2 \frac{(s' - b')(b''(s - z) + s''(z - b))}{(s - b)^2}
$$
$$
+ 2f \frac{(s' - b')^2}{(s - b)^2} - f \frac{s'' - b''}{s - b}.
$$

where, for a linear sloping surface (62) and a sinusoidal bed (63),

$$
s' = -\tan\alpha, \quad s'' = 0, \tag{112}
$$
$$
b' = -\tan\alpha + \pi\cos(2\pi x), \quad b'' = -2\pi^2\sin(2\pi x), \quad b''' = -4\pi^3\cos(2\pi x). \tag{113}
$$

If we name the expression

$$
\nu = \frac{1}{2} \left(\frac{1}{\delta} \frac{\partial u}{\partial z} + \delta \frac{\partial w}{\partial x} \right)^2 - \frac{\partial u}{\partial x} \frac{\partial w}{\partial z}, \tag{114}
$$
$$
\text{then } \mu = \nu^{\frac{1-n}{2n}}.
$$

For further calculations we will need the following derivatives:

$$\frac{\partial \mu}{\partial x} = \frac{1-n}{2n}\frac{\mu}{\nu}\left[\left(\frac{1}{\delta}\frac{\partial u}{\partial z}+\delta\frac{\partial w}{\partial x}\right)\left(\frac{1}{\delta}\frac{\partial^2 u}{\partial x \partial z}+\delta\frac{\partial^2 w}{\partial x^2}\right)-\frac{\partial^2 u}{\partial x^2}\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x \partial z}\right] \quad (115)$$

$$\frac{\partial \mu}{\partial z} = \frac{1-n}{2n}\frac{\mu}{\nu}\left[\left(\frac{1}{\delta}\frac{\partial u}{\partial z}+\delta\frac{\partial w}{\partial x}\right)\left(\frac{1}{\delta}\frac{\partial^2 u}{\partial z^2}+\delta\frac{\partial^2 w}{\partial x \partial z}\right)-\frac{\partial^2 u}{\partial x \partial z}\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial z^2}\right], \quad (116)$$

Substituting (110)-(111) and (115)-(116) into (99)-(100), (102)-(103), and (104)-(105) generates formulas for artificial terms $\Sigma_x, \Sigma_z, \upsilon_x, \upsilon_z, \tau_x,$ and τ_z.

If constant λ in (56) is chosen so that $\lambda > 2$, then velocities $u(x, z)$ and $w(x, z)$ equal zero at the bed, that is, frozen bed case. In this case, artificial stress terms in the basal boundary conditions τ_x and τ_z both are zeros. Artificial vertical stress terms in the momentum equation Σ_z and in the surface boundary condition υ_z are small, while the horizontal stress terms Σ_x and υ_x are big.