SOME ISSUES OF ECONOMETRIC ASSESSMENT OF CES FUNCTIONS AND THEIR PRACTICAL APPLICATION IN ECONOMIC PROBLEMS

[1]Usmanova Dilfuza Ilhomovna, [2]Usmanov Farzod Shokhrukhovich
[1]Associate Professor of Samarkand Economics and Service Institute
[2]Master's student studying in the specialty, Samarkand Institute of Economics and Service
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Abstract. *This article is devoted to some issues of approximate estimation of functions with constant elasticity of substitution. The constant elasticity of substitution (CES) function is becoming popular in many areas of economics, but it is rarely used in econometric analysis because it cannot be estimated by standard linear regression methods. In this article, the author will reveal existing approaches to econometric assessment of this function and consider several examples of empirical assessment of the function using R software (micEconCES package).*
Keywords: *CES – function, loglinearization of the model, Taylor series, Kmenta method, running the model in R language.*

Introduction:

In modern economic research, it is customary to use the Cobb-Douglas production function for econometric assessments of macroeconomic models, models of the theory of Consumer Behavior, and assessment of the productivity of firms or sectors of the economy. Although this function is a widely used functional form in economic research, it has several disadvantages such as constant returns to scale, which may be unrealistic in some situations. However, it imposes strong restrictions on the basic functional constraints, in particular on the elasticity of substitution of factors, which are always equal to 1. Based on the above reasons, to describe different degrees of substitution between economic variables, scientists from Stanford University in 1961 [Arrow KJ, Chenery BH, Minhas BS, Solow RM (1961)][1]was proposed CES function – (function with constant elasticity of substitution) as a generalization of the Cobb–Douglas function, which allows the use of any (non-negative constant) elasticity of substitution.

Function with constant elasticity substitution shows how much the ratio of two variables or factors (micro- or macroeconomic) changes in percentage terms when the marginal rate of substitution of two factors changes.

The constancy of the elasticity of substitution of two variables is given by a differential equation in the form:

$$\frac{\partial\left(\frac{X_1}{X_2}\right)}{\frac{X_1}{X_2}} * \frac{\frac{\partial X_1}{\partial X_2}}{\partial\left(\frac{\partial X_1}{\partial X_2}\right)} = \frac{\partial\left(\ln\left(\frac{X_1}{X_2}\right)\right)}{\partial\left(\ln\left(\frac{\partial X_1}{\partial X_2}\right)\right)} = \delta \;\; or$$

[1] Arrow KJ, Chenery BH, Minhas BS, Solow RM (1961). \Capital-labor substitution and economic efficiency." The Review of Economics and Statistics, 43(3), 225 - 250. URLhttp://www.jstor.org/stable/1927286.

$$\partial\left(\ln\left(\frac{X_1}{X_2}\right)\right) = \delta * \ \partial\left(\ln\left(\frac{\partial X_1}{\partial X_2}\right)\right) \qquad (1.1)$$

Integrating equation 1.1: $\int \partial\left(\ln\left(\frac{X_1}{X_2}\right)\right) = \int \delta * \ \partial\left(\ln\left(\frac{\partial X_1}{\partial X_2}\right)\right) + \delta\ln(c)$mwe get$\ln\left(\frac{X_1}{X_2}\right) = \delta * \ \left(\ln\left(\frac{\partial X_1}{\partial X_2}\right)\right) + \delta\ln(c)$or Dividing the variables in the equation we get or . Integrating the resulting equation for the second time, we obtain the general form of the function:$\left(\frac{X_1}{X_2}\right)^{1/\delta} = c\left(\frac{\partial X_1}{\partial X_2}\right).\left(\frac{\partial X_2}{{X_2}^{1/\delta}}\right) = c\left(\frac{\partial X_1}{{X_1}^{1/\delta}}\right)$ $\quad \partial {X_2}^{(\delta-1)/\delta} = c\partial {X_1}^{(\delta-1)/\delta}$

$${X_2}^{(\delta-1)/\delta} = c{X_1}^{(\delta-1)/\delta} + c_1. \ (1.2)$$

Assuming that we obtain the CES production function.$c = -\frac{\beta}{\alpha}, \quad c_1 = \left(\frac{1}{\alpha a}\right) * Y^{-\frac{1}{v}}, \ v = \frac{\gamma}{\rho}, \ \rho = (1-\delta)/\delta$

$$Y(X_1, X_2) = a\left(\alpha X_2^{-\rho} + \beta {X_1^{-\rho}}\right)^{-\frac{\gamma}{\rho}} \quad \rho \geq -1, \rho \neq 0 \qquad (1.3)$$

Below we show how the function CES may be a generalization of the Cobb–Douglas and Leontief functions when the function reaches its limiting cases. To achieve this, we first take the logarithm of the CES function:

$$\ln\left(Y(X_1, X_2)\right) = lna - \frac{\gamma}{\rho}\ln \ (1.4)\left(\alpha X_2^{-\rho} + \beta X_1^{-\rho}\right)$$

Let's pretend that. Aiming towards zero, when revealing the emerging uncertainty 0/0, we apply L'Hopital's rule:$\beta = 1 - \alpha\rho$

$$\lim_{\rho\to 0}\ln\left(Y(X_1, X_2)\right) = lna + \lim_{\rho\to 0}\frac{\gamma\left(\alpha {X_2}^{-\rho}\ln(X_2) + (1-\alpha){X_1}^{-\rho}\ln(X_1)\right)}{\left(\alpha {X_2}^{-\rho} + (1-\alpha){X_1}^{-\rho}\right)}$$
$$= lna + \gamma(\alpha\ln(X_2) + (1-\alpha)\ln(X_1)) \qquad (1.5)$$

Then, getting rid of logarithms, we get the Cobb–Douglas production function:

$$Y(X_1, X_2) = \ a * \ {X_2}^{\gamma\alpha}{X_1}^{\gamma(1-\alpha)} \qquad (1.6)$$

To obtain the Leontief function, we assume that, and setting the limit to, we again obtain uncertainty, where we again use L'Hopital's rule:$X_2 = \min(X_1, X_2)\rho$ к $+\infty$

$$\lim_{\rho\to +\infty}\ln\left(Y(X_1, X_2)\right) = lna + \lim_{\rho\to+\infty}\frac{\gamma\left(\alpha {X_2}^{-\rho}\ln(X_2) + (1-\alpha){X_1}^{-\rho}\ln(X_1)\right)}{\left(\alpha {X_2}^{-\rho} + (1-\alpha){X_1}^{-\rho}\right)} = lna +$$
$$\lim_{\rho\to+\infty}\frac{\gamma\left(\alpha\ln(X_2) + (1-\alpha)\left(\frac{X_1}{X_2}\right)^{-\rho}\ln(X_1)\right)}{\left(\alpha + (1-\alpha)\left(\frac{X_1}{X_2}\right)^{-\rho}\right)} = lna + \gamma\ln(X_2) \qquad (1.7)$$

Potentiating the resulting equality, we obtain at, the equation. Below, the author will consider asymptotic approaches to the econometric estimation of the equation.$\gamma = 1 Y(X_1, X_2) = a \min(X_1, X_2)$

Literature Review: Much work has been devoted to the empirical evaluation of CES functions. The author of the article reviewed some of them: For example, in the work [**Fragiadakis et al. (2012)]** was the coefficients of elasticity of substitution between capital and labor were assessed using the CES model. The WIOD SEA database was used for the period 1995–2009, collected across six economic sectors. The authors concluded that in most cases the values of short-term elasticities were less than one (i.e., consistent with the Cobb-Douglas specification) and sometimes even approached zero (i.e., Leontief specification), while long-term elasticities were above one.

Other scientists [Koesler and Schymura (2015)] have conducted estimating CES functions in the form (KL)E using nonlinear econometric methods for panel data. The assessment was made using data from WIOD Socio-Economic Accounts and WIOD Energy Use, forming a balanced panel of 40 regions from 1995 to 2006 for each of the 35 sectors. The authors concluded that the common practice of using Cobb-Douglas or Leontief production functions in applied general equilibrium analyzes should be rejected in most cases, given the complexity and heterogeneity of the resulting estimates across sectors.

In [Németh et al. (2011)]provided elasticity estimates for two-level nested CES functions with the choice between domestic and imported goods at the top level and the choice between importing countries at the bottom level. The econometric estimation was based on panel data methods with fixed and random effects at the top and bottom levels, respectively. Eurostat's COMEXT and National Accounts databases for the period 1995–2005 were used as data sources. The authors concluded that relative changes in demand in response to relative price changes were less sensitive between domestic and import baskets (top level) than within the import basket (bottom level), with higher elasticities obtained in the latter case . Moreover, short-term elasticities tend to be lower than their long-term values in most cases.

[Saito (2004)] used the CES model to estimate substitution between domestic and imported goods (intergroup elasticities), as well as between import baskets from different countries (intragroup elasticities), using panel data analysis methods with fixed effects. His dataset included information from the International Sectoral Data Base and International Trade by Commodities Statistics, covering 14 regions over the period 1970 to 1990. In addition, the OECD Input-Output database was used for supporting calculations. In fact, between-group elasticities were treated as country-specific (estimated from each country's time series), while within-group elasticities were treated as uniform across all countries (based on panel data for all 14 countries). The author concluded that intergroup elasticities were higher than intragroup elasticities in the intermediate goods sectors, but equal to or lower than them in the final consumption sectors.

Main part: Having considered the properties of the CES model, below we move on to the econometric assessment of this function, which is based on the approximate expansion of the function in a Taylor series (Kmenta approximation). Today, an approximate estimate of the function for which was put forward by the author is used [$\rho \to 0$ **Uebe (2000)]**.
Consider the traditional CES model:

$$Y(X_1, X_2) = a\left(\alpha X_2^{-\rho} + \beta X_1^{-\rho}\right)^{-\frac{\gamma}{\rho}} \quad \rho \geq -1, \rho \neq 0$$

Let us assume that the model has constant returns to scale,
$\beta = \delta, \alpha = 1 - \delta$.Taking logarithms of the model on both sides we get that

$$\ln\left(Y(X_1, X_2)\right) = lna - \frac{\gamma}{\rho} ln\left((1-\delta)X_2^{-\rho} + (\delta)X_1^{-\rho}\right) \quad (1.8)$$

Let's introduce the following replacement:

$$f(\rho) = -\frac{\gamma}{\rho} ln\left((1-\delta)X_2^{-\rho} + (\delta)X_1^{-\rho}\right)(1.9)$$

And $g(\rho) = \left((1-\delta)X_2^{-\rho} + (\delta)X_1^{-\rho}\right)$(1.10)

We can further assume that the natural logarithm of the CES function is approximately equal to the expansion of the function into a Taylor series (1 - degree) with the value$\rho = 0$

$$\ln\left(Y(X_1, X_2)\right) \approx lna + f(0) + \rho f'(0) \quad (1.11)$$

Based on the above substitutions we have:

$$f(\rho) = -\frac{\gamma}{\rho} \ln\left(g(\rho)\right) \quad (1.12)$$

Let's find the first derivative of the function$f(\rho)$

$$f'(\rho) = \frac{\gamma}{\rho^2} \ln\left(g(\rho)\right) - \frac{\gamma}{\rho} * \frac{g'(p)}{g(\rho)} \quad (1.13)$$

and the first derivative of the function$g(\rho)$:

$$g'(p) = \left(-(1-\delta)X_2^{-\rho}\ln(X_2) - (\delta)X_1^{-\rho}\ln(X_1)\right) \quad (1.14)$$

At the point $\rho = 0$ we have:

$$g(0) = 1\,;\ g'(0) = \left(-(1-\delta)\ln(X_2) - (\delta)\ln(X_1)\right)\ (1.15)$$

Then we get that avna:$f(0)$ p

$$f(0) = \lim_{\rho\to 0} f(\rho) = \lim_{\rho\to 0} -\frac{\gamma}{\rho}\ln\left(g(\rho)\right) = \lim_{\rho\to 0} -\gamma\frac{g'(p)}{g(\rho)} = -\gamma\frac{g'(0)}{g(0)}$$

$$= \gamma(\delta\ln(X_1) + (1-\delta)\ln(X_2))$$

And the limit$f'(\rho)$when equal to:$\rho \to 0$

$$f'(0) = \lim_{\rho\to 0} f'(\rho) = \lim_{\rho\to 0}\left(\frac{\gamma}{\rho^2}\ln\left(g(\rho)\right) - \frac{\gamma}{\rho} * \frac{g'(p)}{g(\rho)}\right) = \lim_{\rho\to 0}\frac{\gamma\ln\left(g(\rho)\right) - \gamma\rho\frac{g'(\rho)}{g(\rho)}}{\rho^2} =$$

$$= \lim_{\rho\to 0}\frac{\gamma\frac{g'(\rho)}{g(\rho)} - \gamma\frac{g'(p)}{g(\rho)} - \gamma\rho\frac{g''(p)g(p) - (g'(p))^2}{\left(g(p)\right)^2}}{2\rho} = -\frac{\gamma}{2}\frac{g''(\rho)g(\rho) - (g'(\rho))^2}{\left(g(p)\right)^2} =$$

$$-\frac{\gamma}{2}\frac{g''(0)g(0) - (g'(0))^2}{\left(g(0)\right)^2} = -\frac{\gamma}{2}\left(\delta(lnx_1)^2 + (1-\delta)(lnx_2)^2 - (-\delta lnx_1 - (1-\delta)lnx_2)\right)^2$$

$$= -\frac{\gamma}{2}(\delta(lnx_1)^2 + (1-\delta)(lnx_2)^2 - \delta^2(lnx_1)^2 - 2\delta(1-\delta)lnx_1lnx_2 - (1-\delta)^2(lnx_2)^2)$$

$$= -\frac{\gamma\delta(1-\delta)}{2}((lnx_1)^2 - 2lnx_1lnx_2 + (lnx_2)^2) = -\frac{\gamma\delta(1-\delta)}{2}(lnx_1 - lnx_2)^2(1.16)$$

And so below we get a loglinearized form of the CES function expanded into a Taylor series (1 - degree).

$$\ln\left(Y(X_1, X_2)\right) \approx \ln a + \gamma(\delta\ln(X_1) + (1-\delta)\ln(X_2)) - \frac{1}{2}\gamma\rho\delta(1-\delta)(\ln(X_1) - \ln(X_2))^2(1.17)$$

In practice, to estimate CES functions today it is customary to use a loglinearized model in the form of:

$$\ln(y) = a_0 + a_1 \ln(x_1) + a_2 \ln(x_2) + \frac{1}{2} * \beta_{11}(\ln(x_1))^2 + \frac{1}{2} * \beta_{22}(\ln(x_2))^2 + \beta_{12} \ln(x_1) \ln(x_2) + \varepsilon \; (1.18)$$

where the validity of the hypothesis is tested:

$$\beta_{12} = -\beta_{11} = -\beta_{22}$$

If it is necessary to assess the consistency of the constant returns to scale, then the hypothesis is evaluated:

$$a_1 + a_2 = 1$$

The parameters of the CES function can be calculated based on the regressors of the model (1.18)

$$\alpha = \exp(a_0) \;; \gamma = a_1 + a_2; \; \delta = \frac{a_1}{a_1 + a_2}; \; \rho = \frac{\beta_{12}(a_1 + a_2)}{a_1 a_2}$$

Today, for Econometric evaluation of a model, you can use the statistical programming language R, namely the built-in package micEconCES. The micEconCES package also implements a method for calculating model parameters using the Kmenta method. Let's look at a practical example of calculation and interpret the coefficients of the resulting model below. To do this, we will artificially create a database, because when evaluating a model with real data, researchers often encounter inconsistent estimates.

```
cesData <- data.frame (x1 = rchisq(200, 10), x2 = rchisq(200, 10))

cesData$y2 <- cesCalc(xNames = c("x1", "x2"), data = cesData, coef = c(gamma = 1, delta
= 0.6, rho = 0.5, nu = 1.1))
```

having a chi-square distribution with 10 degrees of freedom. The second line of code creates the dependent variable yc with parameters; $\alpha = 1$ (***gamma***), $\delta = 0.6$(***delta***), $\rho = 0.5$ (***rho***), $\gamma = 1.1$ (***nu***).

Below we will try to implement the model using the Kmenta method and obtain conclusions about the model.

```
cesKmenta <- cesEst(yName = "y2", xNames = c("x1", "x2"), data = cesData, method =
"Kmenta", vrs = TRUE)
```

where vrs (variable return to scale) is the variable return to scale.

```
Summary(cesKmenta)
```

Estimated CES function with variable returns to scale
Call:
cesEst(yName = "y2", xNames = c("x1", "x2"), data = cesData,
vrs = TRUE, method = "Kmenta")
Estimation by the linear Kmenta approximation
Test of the null hypothesis that the restrictions of the Translog
function required by the Kmenta approximation are true:
P-value = 0.4313435
Coefficients:
Estimate Std. Error t value Pr(>|t|)

gamma 0.92582 0.11766 7.869 3.59e-15 ***
delta 0.54626 0.02346 23.282 < 2e-16 ***
rho 0.51234 0.21924 2.337 0.0194 *
nu 1.12628 0.05738 19.628 < 2e-16 ***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 2.353881
Multiple R-squared: 0.7330527

Elasticity of Substitution:
Estimate Std. Error t value Pr(>|t|)
E_1_2 (all) 0.66123 0.09585 6.898 5.26e-12 ***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Let's draw conclusions from the model: We see that in the data there is a constant elasticity of the rate of substitution that is different from zero and consistent at the 1% significance level, which tells us that the Cob-Douglas dependence is rejected.

$$\frac{\partial\left(\frac{X_1}{X_2}\right)}{\frac{X_1}{X_2}} * \frac{\frac{\partial X_1}{\partial X_2}}{\partial\left(\frac{\partial X_1}{\partial X_2}\right)} = 0.66$$

In order to satisfy the exact dependence of the CES - function in the model, we assumed that the hypothesis must be satisfied in view of:

$$\beta_{12} = -\beta_{11} = -\beta_{22}$$

Let's check it:

```
summary(cesKmenta$kmenta)
```

OLS estimates for 'eq1' (equation 1)
Model Formula: y ~ 1 + a_1 + a_2 + b_1_1 + b_1_2 + b_2_2
<environment: 0x000001c738ba62e0>

Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.0770768 0.1270885 -0.60648 0.544897
a_1 0.6152361 0.0407933 15.08178 < 2e-16 ***
a_2 0.5110423 0.0372831 13.70706 < 2e-16 ***
b_1_1 -0.1430243 0.0598938 -2.38797 0.017892 *
b_1_2 0.1430243 0.0598938 2.38797 0.017892 *
b_2_2 -0.1430243 0.0598938 -2.38797 0.017892 *

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.249698 on 194 degrees of freedom
Number of observations: 200 Degrees of Freedom: 194
SSR: 12.220419 MSE: 0.062349 Root MSE: 0.249698

Multiple R-Squared: 0.673706 Adjusted R-Squared: 0.668712

We see that the coefficientsb_1_1, b_1_2 and b_2_2 in the model satisfy the above relationship.

Conclusion: In recent years, the CES function has become increasingly popular in macroeconomics, especially in economic growth theory. Based on global experience, the CES function does not always give consistent results due to problems such as model specification, endogeneity of variables and errors, missing variables. In our opinion, this article gives an idea for assessing the CES function, which may be useful to readers for further scientific research.

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