Everyone dealing with ice-shelf flow should derive Morland-MacAyal equations at least once in their live to get appreciation what physics is there, feelings about their limitations and basic understanding what is possible and more important what is not possible to model with them. Let this “once” happen today. First, few reminders.

Reminders

Tensor is an element of a tensor product of vector spaces, and could be represented as a multi-dimensional array relative to a choice of basis of the particular space on which it is defined (`http://en.wikipedia.org/wiki/Tensor`). Tensor that we'll be dealing with to describe stresses in ice is a second order tensor known as Cauchy stress tensor. It could have up to nine independent components, but luckily (due to conservation of the angular momentum), it's symmetric (*i.e.* components above and below its diagonal are the same) so we have to deal just with six independent components (`http://en.wikipedia.org/wiki/Stress_(mechanics)`).

The stress tensor looks like

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \tag{1}$$

Alternative notation for components of the stress tensor is σ_{ij}, where $i, j = x, y, z$.
A “dot” product of a tensor and a vector

$$\boldsymbol{\sigma} \cdot \vec{n} = \sigma_{ij} n_j = \vec{T} \tag{2}$$

A tensor of the second order has three invariants which are are coefficients of its characteristic polynomial

$$\det\left(\boldsymbol{\sigma} - \lambda \boldsymbol{I}\right) = 0 \tag{3}$$

where λ is eigenvalues and $\boldsymbol{I}$ is a unit tensor (that has ones on its diagonal and zeros everywhere else) The first invariant is a trace - a sum of diagonal elements

$$I_\sigma = \text{tr}\ \boldsymbol{\sigma} = \sigma_{ii} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}. \tag{4}$$

The second invariant is

$$II_\sigma = \frac{1}{2}\left[(\text{tr}\ \boldsymbol{\sigma})^2 - \text{tr}\ \boldsymbol{\sigma}\boldsymbol{\sigma}\right] = \frac{1}{2}\sum_{i,j}\sigma_{ij}\sigma_{ij}. \tag{5}$$

The third invariant is determinant of a tensor is its determinant

$$III_\sigma = \det \boldsymbol{\sigma} \tag{6}$$

Deviatoric stress tensor is defined as

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} + \boldsymbol{I}p = \sigma_{ij} + \delta_{ij}p, \tag{7}$$

where p is pressure in ice ($p = -\frac{1}{3}\sigma_{ii}$), δ_{ij} is the Kronecker delta.

From Nina's and Kees' lectures we know about a constitutive relationship between deviatoric stresses and strain rates in ice. It is so called Glen's flow low (http://websrv.cs.umt.edu/isis/index.php/Ice_Rheology)

$$\boldsymbol{\sigma'} = 2\nu\dot{\boldsymbol{\epsilon}}, \tag{8}$$

where $\dot{\boldsymbol{\epsilon}}$ is the strain rate tensor which elements are

$$\dot{\epsilon}_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{9}$$

where $u_{i,j}$ stands for $\frac{\partial u_i}{\partial x_j}$, and u_i are components of the velocity vector $\vec{u} = \{u, v, w\}$, and x_i are components of the coordinate vector $\{x, y, z\}$. For those who remembers the Hook's law, it relates the stress tensor to the strain tensor, *i.e.* $\boldsymbol{\sigma} = \boldsymbol{E}\ \boldsymbol{\epsilon}$.

Momentum balance equation

Of three conservation equations (see Christina Hulbe's notes `http://websrv.cs.umt.edu/isis/index.php/Introduction_to_Ice_Sheet_Modeling`) we focus on a last one - conservation of momentum. As we already know, we can neglect acceleration and advection terms and arrive to a following form of this equation

$$\frac{\partial \sigma_{ij}}{\partial x_j} - \rho g_i = 0 \tag{10}$$

In terms of the deviatoric stress tensor (7) these equations are

$$\frac{\partial}{\partial x_j}\left(\sigma'_{ij} - p\right) - \rho g_i = 0 \tag{11}$$

Using Glen's flow low (8) we finally arrive to following three momentum balance equations

$$\frac{\partial}{\partial x}\left(2\nu\dot{\epsilon}_{xx} - p\right) + \frac{\partial}{\partial y}\left(2\nu\dot{\epsilon}_{xy}\right) + \frac{\partial}{\partial z}\left(2\nu\dot{\epsilon}_{xz}\right) = 0 \tag{12}$$

$$\frac{\partial}{\partial x}\left(2\nu\dot{\epsilon}_{xy}\right) + \frac{\partial}{\partial y}\left(2\nu\dot{\epsilon}_{yy} - p\right) + \frac{\partial}{\partial z}\left(2\nu\dot{\epsilon}_{yz}\right) = 0 \tag{13}$$

$$\frac{\partial}{\partial x}\left(2\nu\dot{\epsilon}_{xz}\right) + \frac{\partial}{\partial y}\left(2\nu\dot{\epsilon}_{yz}\right) + \frac{\partial}{\partial z}\left(2\nu\dot{\epsilon}_{zz} - p\right) = \rho g \tag{14}$$

As we already know, ice is treated as incompressible, therefore divergence of its flow is zero, *i.e.*

$$\dot{\epsilon}_{xx} + \dot{\epsilon}_{yy} + \dot{\epsilon}_{zz} = 0 \tag{15}$$

Equations (12)-(15) represent a field (valid through the whole domain) equations required to solve for unknowns velocity components u, v, w and the ice pressure p. To complete this system of equations we need to specify boundary conditions at the top and bottom boundaries of the ice shelf.

Boundary conditions

At the surface all surface forces are zero, *i.e.*

$$\boldsymbol{\sigma} \cdot \vec{n}_s = 0, \tag{16}$$

where $\vec{n}_s$ is the outward unit normal vector to the top surface Z_s at point (x, y)(`http://mathworld.wolfram.com/NormalVector.html`)

$$\vec{n}_s = \frac{1}{\sqrt{1 + \left(\frac{\partial Z_s}{\partial x}\right)^2 + \left(\frac{\partial Z_s}{\partial y}\right)^2}} \left\{ -\frac{\partial Z_s}{\partial x}, -\frac{\partial Z_s}{\partial y}, 1 \right\} \tag{17}$$

Applying "dot" product rule (2), we get following three equations that should be satisfied at the top surface

$$-\left(2\nu\dot{\epsilon}_{xx} - p\right)\frac{\partial Z_s}{\partial x} - 2\nu\dot{\epsilon}_{xy}\frac{\partial Z_s}{\partial y} + 2\nu\dot{\epsilon}_{xz} = 0 \tag{18}$$

$$-2\nu\dot{\epsilon}_{xy}\frac{\partial Z_s}{\partial x} - \left(2\nu\dot{\epsilon}_{yy} - p\right)\frac{\partial Z_s}{\partial y} + 2\nu\dot{\epsilon}_{yz} = 0 \tag{19}$$

$$-2\nu\dot{\epsilon}_{xz}\frac{\partial Z_s}{\partial x} - 2\nu\dot{\epsilon}_{yz}\frac{\partial Z_s}{\partial y} + \left(2\nu\dot{\epsilon}_{zz} - p\right) = 0 \tag{20}$$

At the bottom surface Z_b we have to satisfy continuity of the normal stress, *i.e.*

$$\boldsymbol{\sigma} \cdot \vec{n}_z = \rho_w g Z_b \vec{n}_b, \tag{21}$$

where ρ_w is the density of the water $\vec{n}_b$ is an outward normal vector to the bottom surface of the ice shelf

$$\vec{n}_s = \frac{1}{\sqrt{1 + \left(\frac{\partial Z_b}{\partial x}\right)^2 + \left(\frac{\partial Z_b}{\partial y}\right)^2}} \left\{ \frac{\partial Z_b}{\partial x}, \frac{\partial Z_b}{\partial y}, -1 \right\} \tag{22}$$

Three equations for the bottom boundary conditions are

$$(2\nu\dot{\epsilon}_{xx} - p)\frac{\partial Z_b}{\partial x} - 2\nu\dot{\epsilon}_{xy}\frac{\partial Z_b}{\partial y} + 2\nu\dot{\epsilon}_{xz} = -rho_w g Z_b \frac{\partial Z_b}{\partial x} \tag{23}$$

$$2\nu\dot{\epsilon}_{xy}\frac{\partial Z_b}{\partial x} - (2\nu\dot{\epsilon}_{yy} - p)\frac{\partial Z_b}{\partial y} + 2\nu\dot{\epsilon}_{yz} = \rho_w g Z_b \frac{\partial Z_b}{\partial y} \tag{24}$$

$$-2\nu\dot{\epsilon}_{xz}\frac{\partial Z_b}{\partial x} - 2\nu\dot{\epsilon}_{yz}\frac{\partial Z_b}{\partial y} + (2\nu\dot{\epsilon}_{zz} - p) = -\rho_w g Z_b \tag{25}$$

Solving equations (12)-(15) with boundary conditions (18)-(25) one gets three velocity components and pressure in the ice shelf.

Non-dimensionalization

Let us write all equations in non-dimensional variables. To do so we have to choose appropriate scales such that all terms in the above equations should be order of one. We start with choosing characteristic length scale L=10 km and ice thickness H_0=1000 m. A nondimensional parameter that describes the aspect ratio is $\delta = \frac{H}{L}$. A characteristic value for horizontal velocities u and v is U. A characteristic value for the vertical velocity should therefore be $w = \delta U$ (that comes from the continuity equation). All components of strain rates scale as $\frac{U}{L}$ except $\dot{\epsilon}_{xz}$ and $\dot{\epsilon}_{yz}$ that are $\delta^{-1}\frac{U}{L}$. A scale for Z_s and Z_b is H_0, and a scale for $p = \rho g H_0$. Scaled viscosity is

$$\nu = \frac{B(T)}{2II_{\dot{\epsilon}}^{\frac{n-1}{n}}} = \frac{B_0}{2\frac{U}{L}^{\frac{1-n}{n}}}\tilde{\nu} \tag{26}$$

where $\tilde{\nu}$ is nondimensional viscosity. Now, let make all our equations dimensionless. Notice, that $\frac{\partial}{\partial x,y} = \frac{1}{L}\frac{\partial}{\partial \tilde{x},\tilde{y}}$ and $\frac{\partial}{\partial z} = \frac{1}{H}\frac{\partial}{\partial \tilde{z}}$. If we substitute all dimensional variable using there nondimensional form into equations (12)-(14) we get following equations

$$\frac{1}{L}\frac{\partial}{\partial\tilde{x}}\left(\frac{B_0}{\left(\frac{U}{L}\right)^{\frac{2}{3}}}\frac{U}{L}\tilde{\nu}\dot{\epsilon}_{xx}-\rho g H_0\tilde{p}\right)+\frac{1}{L}\frac{\partial}{\partial\tilde{y}}\left(\frac{B_0}{\left(\frac{U}{L}\right)^{\frac{2}{3}}}\tilde{\nu}\frac{U}{L}\dot{\epsilon}_{xy}\right)+\frac{1}{H_0}\frac{\partial}{\partial\tilde{z}}\left(\frac{B_0}{\left(\frac{U}{L}\right)^{\frac{2}{3}}}\tilde{\nu}\delta^{-1}\frac{U}{L}\dot{\epsilon}_{xz}\right)=0 \tag{27}$$

$$\frac{1}{L}\frac{\partial}{\partial\tilde{x}}\left(\frac{B_0}{\left(\frac{U}{L}\right)^{\frac{2}{3}}}\frac{U}{L}\tilde{\nu}\dot{\epsilon}_{xy}\right)+\frac{1}{L}\frac{\partial}{\partial\tilde{y}}\left(\frac{B_0}{\left(\frac{U}{L}\right)^{\frac{2}{3}}}\tilde{\nu}\frac{U}{L}\dot{\epsilon}_{yy}-\rho g H_0\tilde{p}\right)+\frac{1}{H_0}\frac{\partial}{\partial\tilde{z}}\left(\frac{B_0}{\left(\frac{U}{L}\right)^{\frac{2}{3}}}\tilde{\nu}\delta^{-1}\frac{U}{L}\dot{\epsilon}_{yz}\right)=0 \tag{28}$$

$$\frac{1}{L}\frac{\partial}{\partial\tilde{x}}\left(\frac{B_0}{\left(\frac{U}{L}\right)^{\frac{2}{3}}}\delta^{-1}\frac{U}{L}\tilde{\nu}\dot{\epsilon}_{xz}\right)+\frac{1}{L}\frac{\partial}{\partial\tilde{y}}\left(\frac{B_0}{\left(\frac{U}{L}\right)^{\frac{2}{3}}}\tilde{\nu}\delta^{-1}\frac{U}{L}\dot{\epsilon}_{yz}\right)+\frac{1}{H_0}\frac{\partial}{\partial\tilde{z}}\left(\frac{B_0}{\left(\frac{U}{L}\right)^{\frac{2}{3}}}\tilde{\nu}\frac{U}{L}\dot{\epsilon}_{zz}-\rho g H_0\tilde{p}\right)=\rho g \tag{29}$$

Re-arranging terms and dropping "tilde" sign we get a following equations

$$\frac{\partial}{\partial x}\left(\nu\dot{\epsilon}_{xx}-\Gamma p\right)+\frac{\partial}{\partial y}\left(\nu\dot{\epsilon}_{xy}\right)+\delta^{-2}\frac{\partial}{\partial z}\left(\nu\dot{\epsilon}_{xz}\right)=0 \tag{30}$$

$$\frac{\partial}{\partial x}\left(\nu\dot{\epsilon}_{xy}\right)+\frac{\partial}{\partial y}\left(\nu\dot{\epsilon}_{yy}-\Gamma p\right)+\delta^{-2}\frac{\partial}{\partial z}\left(\nu\dot{\epsilon}_{yz}\right)=0 \tag{31}$$

$$\frac{\partial}{\partial x}\left(\nu\dot{\epsilon}_{xz}\right)+\frac{\partial}{\partial y}\left(\nu\dot{\epsilon}_{yz}\right)+\frac{\partial}{\partial z}\left(\nu\dot{\epsilon}_{yz}-\Gamma p\right)=\Gamma \tag{32}$$

where $\Gamma=\frac{\rho g H_0}{B_0\left(\frac{U}{L}\right)^{\frac{1}{3}}}$.
Repeating the same steps for the boundary conditions we get following equations. For the top surface

$$-\frac{\partial Z_s}{\partial x}\left(\nu\dot{\epsilon}_{xx}-\Gamma p\right)-\frac{\partial Z_s}{\partial y}\left(\nu\dot{\epsilon}_{xy}\right)+\delta^{-2}\nu\epsilon_{xz}=0 \tag{33}$$

$$-\frac{\partial Z_s}{\partial x}\left(\nu\dot{\epsilon}_{xy}\right)-\frac{\partial Z_s}{\partial y}\left(\nu\dot{\epsilon}_{yy}-\Gamma p\right)+\delta^{-2}\nu\epsilon_{yz}=0 \tag{34}$$

$$-\frac{\partial Z_s}{\partial x}\left(\nu\dot{\epsilon}_{xz}\right)-\frac{\partial Z_s}{\partial y}\left(\nu\dot{\epsilon}_{yz}\right)+\left(\nu\epsilon_{zz}-\Gamma p\right)=0 \tag{35}$$

For the bottom surface

$$\frac{\partial Z_b}{\partial x}\left(\nu\dot{\epsilon}_{xx}-\Gamma p\right)+\frac{\partial Z_b}{\partial y}\left(\nu\dot{\epsilon}_{xy}\right)-\delta^{-2}\nu\dot{\epsilon}_{xz}=\frac{\rho_w}{\rho}\Gamma Z_b\frac{\partial Z_b}{\partial x} \tag{36}$$

$$\frac{\partial Z_b}{\partial x}\left(\nu\dot{\epsilon}_{xy}\right)-\frac{\partial Z_b}{\partial y}\left(\nu\dot{\epsilon}_{yy}-\Gamma p\right)+\delta^{-2}\nu\dot{\epsilon}_{yz}=\frac{\rho_w}{\rho}\Gamma Z_b\frac{\partial Z_b}{\partial y} \tag{37}$$

$$\frac{\partial Z_b}{\partial x}\left(\nu\dot{\epsilon}_{xz}\right)+\frac{\partial Z_s}{\partial y}\left(\nu\dot{\epsilon}_{yz}\right)-\left(\nu\dot{\epsilon}_{zz}-\Gamma p\right)=-\frac{\rho_w}{\rho}\Gamma Z_b \tag{38}$$

Equations (30)-(31) are expressed through two nondimensional parameters Γ and δ. Physically, Γ is the ratio of the horizontal stress induced by gravity to the stress required to deform the ice at the reference strain rate $\frac{U}{L}$. Also, it is a measure of the importance of ice thickness gradient to ice flow.

Perturbation Series Expansion

At this point we employ a perturbation theory (`http://en.wikipedia.org/wiki/Perturbation_theory`). A major idea is to express all non-dimensional variables in terms of a small parameter δ^{-2}. For convenience, let's denote it ε. Every variable is written in terms of an infinite power series of a small parameter ε, *i.e.*

$$\dot{\epsilon}_{xx}=\dot{\epsilon}_{xx}^{(0)}+\varepsilon\dot{\epsilon}_{xx}^{(1)}+\varepsilon^2\dot{\epsilon}_{xx}^{(2)}+\ldots \tag{39}$$

Substituting all variables in a form (39) to (30)-(32) and collecting terms with the same power of ε produces a set of approximations of the various orders

Zero order approximation

From equation (30)-(32) we get terms with ε^{-1}

$$\frac{\partial}{\partial z}\left(\nu^{(0)}\dot{\epsilon}_{xz}^{(0)}\right) = 0 \tag{40}$$

$$\frac{\partial}{\partial z}\left(\nu^{(0)}\dot{\epsilon}_{yz}^{(0)}\right) = 0 \tag{41}$$

$$\frac{\partial}{\partial x}\left(\nu^{(0)}\dot{\epsilon}_{xz}^{(0)}\right) + \frac{\partial}{\partial y}\left(\nu^{(0)}\dot{\epsilon}_{yz}\right) + \frac{\partial}{\partial z}\left(\nu^{(0)}\dot{\epsilon}_{zz}^{(0)}\right) - \Gamma\left(\frac{\partial p^{(0)}}{\partial z} + 1\right) = 0 \tag{42}$$

Boundary conditions at the top surface

$$\dot{\epsilon}_{xz}^{(0)} = 0 \tag{43}$$

$$\dot{\epsilon}_{yz}^{(0)} = 0 \tag{44}$$

$$-\frac{\partial Z_s}{\partial x}\left(\nu^{(0)}\dot{\epsilon}_{xz}^{(0)}\right) - \frac{\partial Z_s}{\partial y}\left(\nu^{(0)}\dot{\epsilon}_{yz}^{(0)}\right) + \left(\nu^{(0)}\dot{\epsilon}_{zz}^{(0)} - \Gamma p^{(0)}\right) = 0 \tag{45}$$

and at the bottom surface

$$\dot{\epsilon}_{xz}^{(0)} = 0 \tag{46}$$

$$\dot{\epsilon}_{yz}^{(0)} = 0 \tag{47}$$

$$\frac{\partial Z_b}{\partial x}\left(\nu^{(0)}\dot{\epsilon}_{xz}^{(0)}\right) + \frac{\partial Z_b}{\partial y}\left(\nu^{(0)\dot{\epsilon}_{yz}}\right) - \left(\nu^{(0)}\dot{\epsilon}_{zz}^{(0)} - \Gamma p^{(0)}\right) = \frac{\rho_w}{\rho}\Gamma Z_b \tag{48}$$

Integrating equations (40)-(42) with respect to z and using boundary conditions (43)-(48) we get following

$$\dot{\epsilon}_{xz}^{(0)} = 0 \tag{49}$$

$$\dot{\epsilon}_{yz}^{(0)} = 0 \tag{50}$$

$$\Gamma p^{(0)} = \Gamma(Z_s - z) + \nu^{(0)}\dot{\epsilon}_{zz}^{(0)} \tag{51}$$

$$Z_s = Z_b\left(1 - \frac{\rho_w}{\rho}\right) \tag{52}$$

These expressions imply that to the first order of the aspect ratio the vertical strain rates $\dot{\epsilon}_{xz}$ and $\dot{\epsilon}_{yz}$ are zero, the ice shelf is in hydrostatic equilibrium, *i.e.* flotation condition is satisfied, and pressure linearly depends on the vertical coordinate. Zero vertical shear implies that horizontal velocities are independent on the vertical coordinate.

First order approximation

To determine $\dot{\epsilon}_{xx}^{(0)}, \dot{\epsilon}_{yy}^{(0)}, \dot{\epsilon}_{zz}^{(0)}, \dot{\epsilon}_{xy}^{(0)}$ we need to consider the first order approximation of the stress balance equations

$$\frac{\partial}{\partial x}\left(\nu^{(0)}\dot{\epsilon}_{xx}^{(0)} - \Gamma p^{(0)}\right) + \frac{\partial}{\partial y}\left(\nu^{(0)}\dot{\epsilon}_{xy}^{(0)}\right) = -\frac{\partial}{\partial z}\left(\nu^{(0)}\dot{\epsilon}_{zz}^{(1)}\right) \tag{53}$$

$$\frac{\partial}{\partial x}\left(\nu^{(0)}\dot{\epsilon}_{xy}^{(0)}\right) + \frac{\partial}{\partial y}\left(\nu^{(0)}\dot{\epsilon}_{yy}^{(0)} - \Gamma p^{(0)}\right) = -\frac{\partial}{\partial z}\left(\nu^{(0)}\dot{\epsilon}_{xz}^{(1)}\right) \tag{54}$$

$$\tag{55}$$

Boundary conditions at the top surface

$$-\frac{\partial Z_s}{\partial x}\left(\nu^{(0)}\dot{\epsilon}_{xx}^{(0)} - \Gamma p^{(0)}\right) - \frac{\partial Z_s}{\partial y}\left(\nu^{(0)}\dot{\epsilon}_{xy}\right) = -\nu^{(0)}\dot{\epsilon}_{yz}^{(1)} \tag{56}$$

$$-\frac{\partial Z_s}{\partial x}\left(\nu^{(0)}\dot{\epsilon}_{xy}^{(0)}\right) - \frac{\partial Z_s}{\partial y}\left(\nu^{(0)}\dot{\epsilon}_{yy}^{(0)} - \Gamma p^{(0)}\right) = -\nu^{(0)}\dot{\epsilon}_{yz}^{(1)} \tag{57}$$

At the bottom surface

$$\frac{\partial Z_b}{\partial x}\left(\nu^{(0)}\dot{\epsilon}^{(0)} - \Gamma p^{(0)}\right) + \frac{\partial Z_b}{\partial y}\left(\nu^{(0)}\dot{\epsilon}_{xy}^{(0)}\right) = \nu^{(0)}\dot{\epsilon}_{xz}^{(1)} + \frac{\rho_w}{\rho}\Gamma Z_b \frac{\partial Z_b}{\partial x} \tag{58}$$

$$\frac{\partial Z_b}{\partial x}\left(\nu^{(0)}\dot{\epsilon}_{yx}^{(0)}\right) + \frac{\partial Z_b}{\partial y}\left(\nu^{(0)}\dot{\epsilon}_{yy}^{(0)} - \Gamma p^{(0)}\right) = \nu^{(0)}\dot{\epsilon}_{yz}^{(1)} + \frac{\rho_w}{\rho}\Gamma Z_b \frac{\partial Z_b}{\partial y} \tag{59}$$

Vertical integration of equations (53) and (54), application of the Leibnitz rule and substitution of the boundary conditions (56)-(59) for $\nu^{(0)}\dot{\epsilon}_{xz}^{(1)}$ and $\nu^{(0)}\epsilon_{yz}^{(1)}$ result in zero-order horizontal stress balance

$$\frac{\partial}{\partial x}\left(2\nu_{av}^{(0)}\dot{\epsilon}_{xx}^{(0)}+H\nu_{av}^{(0)}\dot{\epsilon}_{yy}^{(0)}\right)+\frac{\partial}{\partial y}\left(H\nu_{av}^{(0)}\dot{\epsilon}_{xy}\right)=\Gamma H\frac{\partial Z_s}{\partial x} \tag{60}$$

$$\frac{\partial}{\partial x}\left(H\nu_{av}^{(0)}\dot{\epsilon}_{xy}^{(0)}\right)+\frac{\partial}{\partial y}\left(2H\nu_{av}^{(0)}\dot{\epsilon}_{yy}+H\nu_{av}^{(0)}\dot{\epsilon}_{xx}\right)=\Gamma H\frac{\partial Z_s}{\partial y} \tag{61}$$

where H is nondimensional thickness $H = Z_s - Z_b$. Finally, using expressions for the strain rates through velocities, getting back to dimensional form we arrive to following equations

$$\frac{\partial}{\partial x}\left[2\bar{\nu}H\left(2\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left[\bar{\nu}H\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]=\rho gH\frac{\partial S}{\partial x}, \tag{62}$$

$$\frac{\partial}{\partial x}\left[\bar{\nu}H\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[2\bar{\nu}H\left(\frac{\partial u}{\partial x}+2\frac{\partial v}{\partial y}\right)\right]=\rho gH\frac{\partial S}{\partial y} \tag{63}$$

where $\bar{\nu}$ is the depth-averaged effective viscosity

$$\nu=\frac{1}{H}\int_{Z_b}^{Z_s}dz\frac{B(T(z))}{2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\frac{1}{4}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right]^{\frac{n-1}{2n}}}, \tag{64}$$

Boundary conditions

There are two types of boundary conditions: kinematic and dynamic. Kinematic conditions (specication of velocity) are usually applied where ice-shelves abut stagnant, zero slip coastlines (such as where the Ross Ice Shelf abuts the Transantarctic mountains), or where ice streams ow into the ice shelf (in which case the velocity is specified from an examination of ice-stream dynamics). Dynamic conditions (specification of stress) are usually applied at the seaward, iceberg-calving front.

Kinematic conditions

Depth-averaged ice velocity is specied at all junctions with grounded ice or coastlines where the ice shelf shears past stagnant rock. Typically, ice flow into the ice shelf at grounding lines of ice sheets can have z -dependence which is incompatible with the z -independent horizontal ow of the ice shelf. As described by Barcilon and MacAyeal [1988], the z -dependent structure of the input velocity is winnowed out of the net horizontal ow within a narrow transition zone between ice-sheet and ice-shelf ow regimes. This winnowing process is of little interest in most glaciological problems involving ice shelves. Thus, it is sufficient to ignore the winnowing process and simply specify the depth-averaged input velocity as the correct boundary condition.

Dynamic conditions

The balance of forces at the seaward ice front also introduces z -dependent structure in the ice-shelf ow which is winnowed away within a narrow transition zone extending inward from the ice front. As suggested by Morland [1987], this winnowing is of little interest, and may be safely ignored in the specfication of boundary conditions for the ice-shelf stress-equilibrium equations. The balance of forces at the ice front which is relevant as a boundary condition is the depth-integrated balance:

$$\int_{Z_b}^{Z_s} dz\, \boldsymbol{\sigma}' \cdot \vec{n} = -\frac{1}{2}\rho_w g \left(\frac{\rho}{\rho_w} H\right)^2 \vec{n} \tag{65}$$

where $\vec{n}$ is the outward-pointing normal to the portion of the boundary that represents the ice front ($\vec{n}$ is restricted to lie in the horizontal plain). Here we have made use of the assumption that the ice shelf oats in hydrostatic equilibrium with seawater, *i.e.* $Z_b = -\frac{\rho}{\rho_w} H$.