Schwarzschild and other coordinate-dependent solutions of gravitational field in General Relativity

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(Dated: 14 November 2023)

We study the relation between the Schwarzschild solution and other coordinate-dependent solutions of the gravitational field produced by a point mass in the free space. We demonstrate that, although an infinite number of different coordinate systems can be used for solving this problem in general relativity, the physical solution is unique and unambiguous. Importantly, this physical solution differs from the Schwarzschild solution because the true physical distance R is distinct from the Schwarzschild coordinate distance r. This fact is often ignored and leads to many confusions. For instance, the event horizon present in the Schwarzschild solution is actually missing in the true physical solution, being just an apparent phenomenon associated with the Schwarzschild coordinates. Similarly, the concepts of the trapped surface at r lower than the Schwarzschild radius and the intrinsic singularity at $r = 0$, discussed in many papers and textbooks, are misleading.

I. INTRODUCTION

As commonly known, General Relativity (GR) equations are invariant to coordinate transformations, hence the physical solution should be independent of the used coordinates. Nevertheless, the choice of coordinates is important to solve Einstein's equations for some specific problem. For example, the static gravitational field associated with a point mass can effectively be studied using spherically symmetric metric tensor g_{kl}

$$ds^2 = -g_{tt}(r)c^2dt^2 + g_{rr}(r)dr^2 + g_{\theta\theta}(r)d\theta^2 + g_{\phi\phi}(r)d\phi^2 \,, \quad (1)$$

which reads in the Schwarzschild coordinate system[6,26,33,34]

$$ds^2 = -\left(1-\frac{r_s}{r}\right)c^2dt^2 + \left(1-\frac{r_s}{r}\right)^{-1}dr^2 + r^2d\Omega^2 \,, \quad (2)$$

where $d\Omega^2 = d\theta^2 + d\phi^2$ defines the element of the solid angle, $r_s = 2GM/c^2$ is the Schwarzschild radius, G is the gravitational constant, M is the mass of the body, and r and t are the coordinate (contravariant) distance and time. Velocity c is the speed of light far from the source of gravity. This solution is called the Schwarzschild black hole solution and a sphere with the Schwarzschild radius r_s is called the event horizon of the black hole. The event horizon with $r = r_s$ is characterized by $g_{tt} = 0$ and defines a volume around the point mass, from which no particle or light can escape[13,18,31]. Consequently, events inside the horizon can never influence external observers[26]. In addition, the radial term of the Schwarzschild metric is also anomalous at $r = r_s$, being characterized by the coordinate singularity with $g_{rr} = \infty$.

However, a solution of the same problem can be found also for other coordinate systems, such as the isotropic or harmonic coordinates[37]. As shown by Painlevé[28], Fromholz et al.[14] or Crothers[3,5], the number of possible alternative coordinate systems satisfying the Einstein's equations is even infinite. A particularly simple solution is obtained, for instance, for the so-called Brillouin coordinates[1]

$$ds^2 = -\left(1+\frac{r_s}{r}\right)^{-1}c^2dt^2 + \left(1+\frac{r_s}{r}\right)dr^2 + \left(1+\frac{r_s}{r}\right)^2 r^2d\Omega^2 \,, \quad (3)$$

where $r_s = 2GM/c^2$ is the Schwarzschild radius similarly as in Eq. (2). Interestingly, properties of the Brillouin solution are completely different from the Schwarzschild solution: the metric tensor is regular with no zeros or singularities for all r except for the origin of coordinates with $r = 0$, where the point mass is situated.

The differences between the solutions in Eq. (2) and Eq. (3) point out the fact that the presented metrics are coordinate-dependent. Therefore, a true physical solution, which should be invariant of coordinates, is yet to be found. If this necessary step is ignored, and solutions calculated in various coordinates are interpreted directly in physical terms, confusions and misinterpretations about the gravitational field of the point mass can arise[7,21,24,25,28].

The issue about the physical meaning of the Schwarzschild and other coordinate-dependent solutions of gravitational field in GR involves two key difficulties:

First, there is confusion regarding whether the Schwarzschild coordinates are preferable to other coordinates for certain reasons or not. For example, Misner et al.[26] advocate using the Schwarzschild coordinates as a particularly simple coordinate system with exceptional intrinsic geometric properties. In contrast, Hawking & Ellis[18] consider the Schwarzschild coordinates a poor choice. They argue that these coordinates produce the coordinate singularity $g_{rr} = \infty$ at $r = r_s$, which is not a real physical singularity. They claim that this singularity is apparent because it can be removed by using other coordinates. Obviously, this argument can be applied not only to the coordinate singularity but also to the existence of the event horizon in the Schwarzschild metric and its physical interpretation, as it also disappears in some coordinate systems.

Second, it is often ignored that the coordinate distance r in the Schwarzschild metric has not a direct physical meaning. For example, Landau & Lifshitz[23], Misner et al.[26], Hartle[16] or Lambourne[22] emphasize that we should be careful in interpretations of the Schwarzschild coordinates, because distance r in the Schwarzschild metric is not a real physical distance R of an observer from the point mass. Instead, it is a quantity related the area of a sphere in the Schwarzschild coordinates defined by

$$ds^2 = r^2 d\Omega^2 \,, \tag{4}$$

when r and t are assumed to be constant in the Schwarzschild metric. The physical distance R must be further calculated through an integral over r (see Landau & Lifshitz[23], their eq. 97.16; Lambourne[22], his eq. 5.20). However, the difference between r and R in the physical interpretations of the properties of the Schwarzschild solution is commonly ignored, and formulas are expressed in terms of r instead of R in the literature. For instance, the Schwarzschild radius r_s is often evaluated for various objects (e.g., Earth, sun, or other stars) and it is believed that this radius reflects a physical size and volume of a hypothetical black hole[18,20,22,26]. Similarly, orbits in the gravitational field are exclusively described as a function of the Schwarzschild coordinate r and misinterpreted as defining geometry of real physical orbits[2,22,26]. The same applies to studies of the collapse to a black hole when behaviour of particles near the event horizon is studied and implicitly assumed that its physical size is directly described by the Schwarzschild radius[18,20,26,29]. In this way, it is omitted that the properties of the solution can be essentially different when expressed in physical coordinates.

In this paper, we address the above-mentioned difficulties in the interpretation of the gravitational field in GR using Schwarzschild and other coordinate systems. We illustrate the correct method for manipulating various coordinate systems through the example of the radial propagation of photons in the gravitational field of a point mass. We investigate the properties of several metrics used to solve this problem and discuss their mutual relation. We highlight that many confusions in the interpretation of these solutions arise from mixing invariant quantities with coordinate-dependent quantities, as well as free-falling with non-inertial static frames. We emphasize that once the coordinate dependence of the solutions is properly eliminated by converting the used coordinate-dependent quantities into physical (proper) quantities, the problem becomes unique, and the interpretation of the physical solution becomes straightforward, even in non-inertial frames.

II. THEORY

A. Coordinate-dependent and physical quantities in GR

It is well-known that coordinates are irrelevant for physical quantities and they can, in principle, be assigned completely arbitrarily. According to the principle of general covariance, they are merely labels for spacetime events. The only physically meaningful quantities, as measured in experiments, are those invariant under coordinate transformations. Nevertheless, a choice of coordinates must be made, and it is often critical for finding solutions to Einstein's equations[14].

As a good example of a smartly selected coordinate system for solving the gravitational field around a point mass is the metric in Eq. (2) derived by Schwarzschild[33,34]. This metric belongs to the most famous solutions in GR, and its properties are thoroughly discussed in all relativistic textbooks[23,26,36,37]. The metric is spherically symmetric and is expressed using the Schwarzschild (contravariant) coordinates $(x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$. Although this coordinate system is straightforward, we must be careful in interpreting quantities in this system because its base vectors are not normalized. Consequently, the quantities evaluated in these coordinates (or any other curvilinear coordinates) may not represent true physical quantities[16,27].

In order to calculate true physical (proper) quantities in curvilinear coordinate systems, we have to apply a system of orthonormal base vectors[16,27]. Consequently, the physical components of vectors that are independent of coordinates can be expressed as

$$v^{(i)} = v^i \sqrt{g_{ii}} \,, v_{(i)} = v_i \sqrt{g^{ii}} \,, \quad \text{(no summation over } i) \,, \tag{5}$$

where v^i and v_i are the contravariant and covariant components of vector $\mathbf{v}$, g^{kl} and g_{kl} are the contravariant and covariant metric tensors, and $v^{(i)}$ and $v_{(i)}$ are the contravariant and covariant physical components of vector $\mathbf{v}$. In orthogonal coordinate systems, both types of the physical components are equal:

$$v^{(i)} = v_{(i)} \,. \tag{6}$$

Hence, using Eqs (1) and (5) we get the following equations[16,22,23]

$$dT = \sqrt{g_{tt}} \, dt \,, \tag{7}$$

$$dR = \sqrt{g_{rr}} \, dr \,, \tag{8}$$

where dT is the element of the proper time, and dR is the element of the proper radial distance dR. Subsequently, the proper time T and the radial distance R read in a time-independent (static) coordinate system

$$T = \sqrt{g_{tt}} \, t \,, \tag{9}$$

$$R = \int \sqrt{g_{rr}} \, dr \,. \tag{10}$$

B. Coordinate-dependent and physical speeds of light

The physical speed of light in free-falling frames in the gravitational field is exactly equal to c. However, special attention should be paid to calculating the physical speed of light in the gravitational field in non-inertial frames that are at rest with respect to the mass producing the gravitational field. This velocity is of particular interest because it describes the scenario in which the observer experiences a static gravitational field, and his position is fixed with respect to this field. In the following calculations and discussions regarding the physical speed of light, we will refer to this non-trivial case.

Let us assume a spherically symmetric metric tensor $g_{\alpha\beta}$ defined in Eq. (1) and light propagating in a radial direction ($d\theta = 0, d\phi = 0$)

$$ds^2 = -g_{tt}c^2dt^2 + g_{rr}dr^2 \,. \tag{11}$$

The propagation of the electromagnetic waves obeys the equation of the null spacetime distance, $ds^2 = 0$. Hence,

$$g_{tt}c^2dt^2 = g_{rr}dr^2 \,, \tag{12}$$

and the contravariant (coordinate-dependent) speed of light c_g^r along the r - axis reads

$$c_g^r = \frac{dr}{dt} = \sqrt{\frac{g_{tt}}{g_{rr}}}\, c \,, \tag{13}$$

where subscript g means that the speed is affected by gravity in a static non-inertial frame.

In order to express the physical (proper) speed of light, which is coordinate invariant and evaluated in the fixed frame, we have to express the speed of light in the orthonormal coordinate basis[16,27]. Hence, the i-th component of the proper speed of light is

$$c_{g(r)} = \sqrt{g_{rr}}\, c_g^r = \sqrt{g_{tt}}\, c \,. \tag{14}$$

Calculating tangential components of the speed of light in an analogous way, we will find that the proper speed of light has the same magnitude in all directions. Hence, we can simply write

$$c_g = \sqrt{g_{tt}}\, c \,. \tag{15}$$

Emphasize that the proper speed of light c_g is the quantity measured in the frame at rest with respect to the sources of the static gravitational field. The system is not inertial or free-falling; hence, it is affected by gravity. This causes the speed of light not constant but varying, dependent on the distance from the observer to the source of gravity. Clearly, the physical speed of light in a free-falling system is simply c, defined as the ratio of elements of proper distance and proper time: $c = dR/dT$.

C. Gravitational field of point mass in various coordinate systems

In this section, we investigate the relation between the coordinate-dependent and physical speed of light propagating radially in the gravitational field of a point mass in various coordinate systems. Specifically, we will examine the Schwarzschild and Brillouin metrics, as described by Eqs (2) and (3). Additionally, we will explore the solution of the gravitational field in the isotropic coordinates defined by[7,26,37]

$$\begin{aligned} ds^2 = &-\left(1 - \frac{r_s}{4r}\right)^2 / \left(1 + \frac{r_s}{4r}\right)^2 c^2dt^2 \\ &+ \left(1 + \frac{r_s}{4r}\right)^4 \left(dr^2 + r^2 d\Omega^2\right] \,, \end{aligned} \tag{16}$$

and in the harmonic coordinates defined by[37]

$$\begin{aligned} ds^2 = &-\left(1 - \frac{r_s}{2r}\right) / \left(1 + \frac{r_s}{2r}\right) c^2dt^2 \\ &+ \left(1 + \frac{r_s}{2r}\right) / \left(1 - \frac{r_s}{2r}\right) dr^2 + r^2 \left(1 + \frac{r_s}{2r}\right)^2 d\Omega^2 \,. \end{aligned} \tag{17}$$

Using Eq. (10) and Eq. (15), the physical distance and the physical speed of light read for the four coordinate systems as follows:

- the Schwarzschild coordinates

$$\begin{aligned} R &= \int_{r_s}^{r} \frac{1}{\sqrt{1 - \frac{r_s}{r}}}\, dr \,, \\ c_g &= c\sqrt{1 - \frac{r_s}{r}} \,,\; r \geq r_s \,, \end{aligned} \tag{18}$$

- the Brillouin coordinates

$$\begin{aligned} R &= \int_{0}^{r} \sqrt{1 + \frac{r_s}{r}}\, dr \,, \\ c_g &= c/\sqrt{1 + \frac{r_s}{r}} \,,\; r \geq 0 \,, \end{aligned} \tag{19}$$

- the isotropic coordinates

$$\begin{aligned} R &= \int_{r_s/4}^{r} \left(1 + \frac{r_s}{4r}\right)^2 dr \,, \\ c_g &= c\left(1 - \frac{r_s}{4r}\right) / \left(1 + \frac{r_s}{4r}\right) \,,\; r \geq \frac{1}{4r_s} \,, \end{aligned} \tag{20}$$

- the harmonic coordinates

$$\begin{aligned} R &= \int_{r_s/2}^{r} \sqrt{\left(1 + \frac{r_s}{2r}\right) / \left(1 - \frac{r_s}{2r}\right)}\, dr \,, \\ c_g &= c\sqrt{\left(1 - \frac{r_s}{2r}\right) / \left(1 + \frac{r_s}{2r}\right)} \,,\; r \geq \frac{1}{2r_s} \,. \end{aligned} \tag{21}$$

The range of the coordinate distance r is chosen the physical distance R and the physical speed of light c_g to be positive or zero. For other values of r, the solutions are not physical.

Obviously, the formulas for the physical distances far from the point mass ($r \to \infty$) coincide in all coordinate systems in the first-order approximation of $1/r$:

$$dR = \left(1 + \frac{r_s}{2r}\right) dr + O\left(r^{-2}\right) \,, \tag{22}$$

and similarly the physical speed of light c_g behaves in the same way

$$c_g = \left(1 - \frac{r_s}{2r}\right) c + O\left(r^{-2}\right) . \tag{23}$$

As expected, the physical distance R in Eqs. (18-21) is always positive and ranges from zero (when the observation point is at the origin of coordinates) to infinity. This confirms the fact that radius r is not a physical distance[16], and the Schwarzschild radius r_s is not a real physical quantity. Additionally, this implies that radius r does not need to cover the whole range of values from zero to infinity as commonly incorrectly assumed in the concept of black holes. This mistake leads to confusions about the physical interpretation of the Schwarzschild solution for r in the interval $0 \leq r \leq r_s$, including the speculations about the existence and visibility of the trapped surfaces at $r < r_s$ and the intrinsic singularity at $r = 0$[17,19,22,26,29,30].

D. Properties of the physical solution at a non-inertial static frame

Here, we numerically examine the properties of the physical solution for the radial propagation of photons in the gravitational field of a point mass. Specifically, we investigate the physical speed of light measured by a static observer as a function of the physical distance from the point mass at rest. Since GR equations should yield a unique solution to this problem, we expect that the physical solutions obtained using different coordinate systems will coincide.

Figure 1 shows the physical radial speed of light c_g as a function of the physical radial distance R of an observation point from the point mass. The figure indicates that all four coordinate systems yield the same dependence, $c_g = c_g(R)$. This confirms our expectation that the physical solution should be unique and coordinate-independent. Interestingly, the speed of light in the non-inertial static frame varies depending on the position of the observation point. The speed of light c_g is zero directly at the point mass but rapidly increases with distance. The increase of c_g slows down with distance, and finally c_g converges to c at very large distance, which characterizes the speed of light in a medium with no gravity.

III. DISCUSSION

The gravitational field of a point mass can be studied in an infinite number of coordinates with different metric tensors that satisfy the Einstein field equations. Therefore, finding a suitable metric tensor is not the final goal; it represent just the first step in solving the problem. The essential step is to solve the geodesics equation and express its solution in physical (coordinate-independent) quantities. Evidently, in contrast to the ambiguity of the used coordinates, the physical solution must be unique and unambiguous.

This procedure is demonstrated in solving a simple problem of the radial propagation of photons in the gravitational field produced by a point mass situated at the origin of coordinates. Four completely different metric tensors that satisfy the Einstein equations are applied to solve the geodesics equation for the propagation of photons. Subsequently, the physical speed of light is expressed in terms of the physical distance from the point mass, $c_g = c_g(R)$. This dependence is identical for all coordinate systems and confirms that GR yields an unambiguous solution, and the presented procedure gives correct results.

As a consequence, we must be aware that specific properties of metrics related to individual coordinate systems cannot be interpreted directly in physical terms. This applies, for example, to an anomalous behaviour of the Schwarzschild metric at the Schwarzschild radius, $r = r_s$, when g_{tt} becomes zero (the event horizon) and g_{rr} goes to infinity (the coordinate singularity). The coordinate singularity is considered as apparent and produced by the choice of coordinates[13,18]. In contrast, this argument has not surprisingly been applied to the event horizon that is commonly assumed a real phenomenon attributed to a physical sphere with a radius r_s around the point mass. This opinion is misleading because it ignores the fact that the coordinate r in the Schwarzschild metric is not the physical distance and it need not cover the full range of values $0 \leq r \leq \infty$. Actually, the coordinate r covers only the interval of $r_s \leq r \leq \infty$, because $r = r_s$ corresponds to the physical distance $R = 0$ from the point mass (see Eq. (18)). Hence, the coordinate distance $r = r_s$ is not associated with any surface in the physical space but with the point singularity at the origin of the physical space. Consequently, the event horizon associated with the Schwarzschild black hole is an apparent phenomenon similarly as the coordinate singularity in the Schwarzschild coordinates[3–5]. Similarly, the concepts of the trapped surfaces at $r < r_s$ and the intrinsic singularity at $r = 0$ discussed in many papers and textbooks[13,17,19,21,22,26,29,30,37] are misleading.

Figure 1 indicates that the physical speed of light in a non-inertial frame, which is at rest with respect to the static point mass, behaves in an essentially different manner than in free-falling frames. The physical speed of light is not constant but varies depending on the distance from the point mass. It is zero at the origin of coordinates but rapidly increases with distance. At a large distance from the point mass, where the gravitational field is weak, the speed of light c_g converges to the speed of light c in vacuum, undistorted by gravity.

Note that the concept of the varying speed of light seems apparently to be against the basic principles of theory of the Special and General Relativity. However, this is misleading, because we are not studying the speed of light in free-falling inertial frames, but in non-inertial

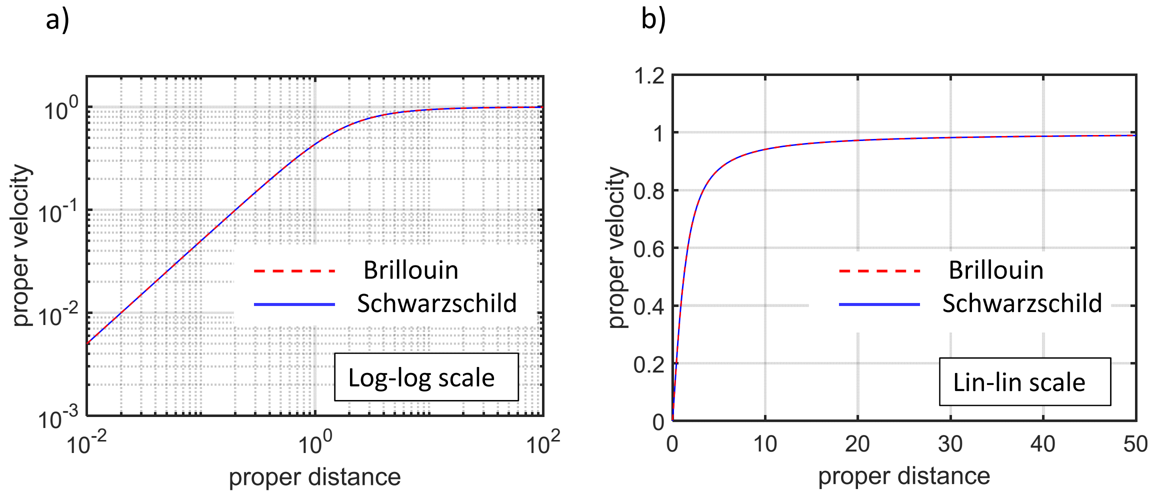

FIG. 1. The physical speed if light c_g as a function of the physical (proper) distance R of an observation point from the point mass. (a) The logarithmic scale, (b) the linear scale. The Schwarzschild solution - the blue full line, the Brillouin solution - the red dashed line. Note that the physical solution in the isotropic and harmonic coordinates is identical with the Schwarzschild and Brillouin solutions. The axes are in relative units: the speed of light c_g is normalized to c and the physical distance R is normalized to the Schwarzschild radius r_s.

frames. These frames are not equivalent because the acceleration due to gravity is not cancelled with the gravitational field in non-inertial frames. As emphasized by Einstein[8]: '*the law of constancy of the velocity of light in vacuo, which constitutes one of the fundamental assumptions in the special theory of relativity and to which we have already frequently referred, cannot clam an unlimited validity its results hold only so long as we are able to disregards the influences of gravitational fields on the phenomena (e.g. of light)*'. A typical example of an observation of the varying speed of light is gravitational lensing, since '*A curvature of rays of light can only take place when the velocity of propagation of light varies with position*'[8].

Considering the varying speed of light in the gravitational field, we can define the refractive index of the vacuum distorted by gravity as

$$n = \frac{c}{c_g} = \frac{1}{\sqrt{g_{tt}}}\,, \tag{24}$$

and simulate the propagation of photons in curved spacetimes using the Fermat's principle in a manner very analogous to that used for light in dielectric media. This approach is mentioned in several textbooks[23,27] and has been applied by many authors to astrophysical and cosmological problems[9–12,15,32,35,38]. The validity of this approach is, however, limited to weak gravitational fields, where the physical distance dR and the Schwarzschild covariant distance dr are close, as seen in Eq. (22). Nevertheless, this limitation can easily be removed by transforming dr to dR using Eq. (18). Consequently, the exact ray fields can be calculated even for strong gravitational fields in the close vicinity of massive objects.

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