ENDPOINT FUNCTIONAL CONTINUOUS UNCERTAINTY PRINCIPLES

K. MAHESH KRISHNA
Post Doctoral Fellow
Statistics and Mathematics Unit
Indian Statistical Institute, Bangalore Centre
Karnataka 560 059, India
Email: kmaheshak@gmail.com
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Abstract: Let (Ω, μ), (Δ, ν) be measure spaces. Let $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ and $(\{g_\beta\}_{\beta\in\Delta}, \{\omega_\beta\}_{\beta\in\Delta})$ be continuous 1-Schauder (resp. ∞-Schauder) frames for a Banach space $\mathcal{X}$. Then for every $x \in \mathcal{X}\setminus\{0\}$, we show that

$$\mu(\operatorname{supp}(\theta_f x)) \geq \frac{1}{\sup\limits_{\alpha\in\Omega,\beta\in\Delta}|f_\alpha(\omega_\beta)|} \quad \left(\text{resp.} \quad \nu(\operatorname{supp}(\theta_g x)) \geq \frac{1}{\sup\limits_{\alpha\in\Omega,\beta\in\Delta}|f_\alpha(\omega_\beta)|}\right),$$

$$\nu(\operatorname{supp}(\theta_g x)) \geq \frac{1}{\sup\limits_{\alpha\in\Omega,\beta\in\Delta}|g_\beta(\tau_\alpha)|} \quad \left(\text{resp.} \quad \mu(\operatorname{supp}(\theta_f x)) \geq \frac{1}{\sup\limits_{\alpha\in\Omega,\beta\in\Delta}|g_\beta(\tau_\alpha)|}\right),$$

where

$$\theta_f : \mathcal{X} \ni x \mapsto \theta_f x \in \mathcal{L}^1(\Omega,\mu); \quad \theta_f x : \Omega \ni \alpha \mapsto (\theta_f x)(\alpha) := f_\alpha(x) \in \mathbb{K}$$
$$(\text{resp.} \quad \theta_f : \mathcal{X} \ni x \mapsto \theta_f x \in \mathcal{L}^\infty(\Omega,\mu); \quad \theta_f x : \Omega \ni \alpha \mapsto (\theta_f x)(\alpha) := f_\alpha(x) \in \mathbb{K}),$$
$$\theta_g : \mathcal{X} \ni x \mapsto \theta_g x \in \mathcal{L}^1(\Delta,\nu); \quad \theta_g x : \Delta \ni \beta \mapsto (\theta_g x)(\beta) := g_\beta(x) \in \mathbb{K}$$
$$(\text{resp.} \quad \theta_g : \mathcal{X} \ni x \mapsto \theta_g x \in \mathcal{L}^\infty(\Delta,\nu); \quad \theta_g x : \Delta \ni \beta \mapsto (\theta_g x)(\beta) := g_\beta(x) \in \mathbb{K}).$$

This solve two problems asked by K. M. Krishna in the paper 'Functional Continuous Uncertainty Principle' *[arXiv:2308.00312v1]*.
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1. Introduction

Given a collection $\{\tau_j\}_{j=1}^n$ in a finite dimensional Hilbert space $\mathcal{H}$ over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), let

$$\theta_\tau : \mathcal{H} \ni h \mapsto \theta_\tau h := (\langle h, \tau_j\rangle)_{j=1}^n \in \mathbb{K}^n.$$

Following is the most general form of discrete uncertainty principle for finite dimensional Hilbert spaces.

Theorem 1.1. ***(Donoho-Stark-Elad-Bruckstein-Ricaud-Torrésani Uncertainty Principle)*** *[2, 3, 16] Let $\{\tau_j\}_{j=1}^n$, $\{\omega_j\}_{j=1}^n$ be two Parseval frames for a finite dimensional Hilbert space $\mathcal{H}$. Then*

$$\left(\frac{\|\theta_\tau h\|_0 + \|\theta_\omega h\|_0}{2}\right)^2 \geq \|\theta_\tau h\|_0\|\theta_\omega h\|_0 \geq \frac{1}{\max\limits_{1\leq j,k\leq n}|\langle\tau_j, \omega_k\rangle|^2}, \quad \forall h \in \mathcal{H}\setminus\{0\}.$$

Recently, Theorem 1.1 has been greatly improved to continuous families in Banach spaces (even infinite dimensions). To state the result, we need a notion.

Definition 1.2. *[8] Let (Ω, μ) be a measure space. Let $\{\tau_\alpha\}_{\alpha\in\Omega}$ be a collection in a Banach space $\mathcal{X}$ and $\{f_\alpha\}_{\alpha\in\Omega}$ be a collection in $\mathcal{X}^*$. The pair $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ is said to be a* ***continuous p-Schauder frame*** *for $\mathcal{X}$ $(1<p<\infty)$ if the following holds.*

(i) *For every $x \in \mathcal{X}$, the map $\Omega \ni \alpha \mapsto f_\alpha(x) \in \mathbb{K}$ is measurable.*
(ii) *For every $x \in \mathcal{X}$,*

$$\|x\|^p = \int\limits_{\Omega} |f_\alpha(x)|^p \, d\mu(\alpha).$$

(iii) *For every $x \in \mathcal{X}$, the map $\Omega \ni \alpha \mapsto f_\alpha(x)\tau_\alpha \in \mathcal{X}$ is weakly measurable.*
(iv) *For every $x \in \mathcal{X}$,*

$$x = \int\limits_{\Omega} f_\alpha(x)\tau_\alpha \, d\mu(\alpha),$$

where the integral is weak integral.

Note that condition (ii) in Definition 1.2 says that the map

$$\theta_f : \mathcal{X} \ni x \mapsto \theta_f x \in \mathcal{L}^p(\Omega, \mu); \quad \theta_f x : \Omega \ni \alpha \mapsto (\theta_f x)(\alpha) \coloneqq f_\alpha(x) \in \mathbb{K}$$

is a linear isometry.

Theorem 1.3. ***(Functional Continuous Uncertainty Principle)*** *[8] Let (Ω, μ), (Δ, ν) be measure spaces. Let $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ and $(\{g_\beta\}_{\beta\in\Delta}, \{\omega_\beta\}_{\beta\in\Delta})$ be continuous p-Schauder frames for a Banach space $\mathcal{X}$. Then for every $x \in \mathcal{X}\setminus\{0\}$, we have*

$$\mu(\operatorname{supp}(\theta_f x))^{\frac{1}{p}} \nu(\operatorname{supp}(\theta_g x))^{\frac{1}{q}} \geq \frac{1}{\sup\limits_{\alpha\in\Omega, \beta\in\Delta} |f_\alpha(\omega_\beta)|}, \quad \nu(\operatorname{supp}(\theta_g x))^{\frac{1}{p}} \mu(\operatorname{supp}(\theta_f x))^{\frac{1}{q}} \geq \frac{1}{\sup\limits_{\alpha\in\Omega, \beta\in\Delta} |g_\beta(\tau_\alpha)|},$$

where q is the conjugate index of p.

Corollary 1.4. ***(Functional Donoho-Stark-Elad-Bruckstein-Ricaud-Torrésani Uncertainty Principle)*** *[11] Let $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ and $(\{g_k\}_{k=1}^m, \{\omega_k\}_{k=1}^m)$ be p-Schauder frames for a finite dimensional Banach space $\mathcal{X}$. Then for every $x \in \mathcal{X}\setminus\{0\}$, we have*

$$\|\theta_f x\|_0^{\frac{1}{p}} \|\theta_g x\|_0^{\frac{1}{q}} \geq \frac{1}{\max\limits_{1\leq j\leq n, 1\leq k\leq m} |f_j(\omega_k)|} \quad \text{and} \quad \|\theta_g x\|_0^{\frac{1}{p}} \|\theta_f x\|_0^{\frac{1}{q}} \geq \frac{1}{\max\limits_{1\leq j\leq n, 1\leq k\leq m} |g_k(\tau_j)|},$$

where q is the conjugate index of p.

In paper [8], it is asked that whether we have version of Theorem 1.3 for $p = 1$ and $p = \infty$. In this paper, we solve these problems.

2. Functional Continuous Uncertainty Principle for Continuous 1-Schauder Frames

We clearly have the following definition from Definition 1.2.

Definition 2.1. *Let (Ω, μ) be a measure space. Let $\{\tau_\alpha\}_{\alpha\in\Omega}$ be a collection in a Banach space $\mathcal{X}$ and $\{f_\alpha\}_{\alpha\in\Omega}$ be a collection in $\mathcal{X}^*$. The pair $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ is said to be a* ***continuous 1-Schauder frame*** *for $\mathcal{X}$ if the following holds.*

(i) *For every* $x \in \mathcal{X}$, *the map* $\Omega \ni \alpha \mapsto f_\alpha(x) \in \mathbb{K}$ *is measurable.*

(ii) *For every* $x \in \mathcal{X}$,

$$\|x\| = \int\limits_{\Omega} |f_\alpha(x)| \, d\mu(\alpha).$$

(iii) *For every* $x \in \mathcal{X}$, *the map* $\Omega \ni \alpha \mapsto f_\alpha(x)\tau_\alpha \in \mathcal{X}$ *is weakly measurable.*

(iv) *For every* $x \in \mathcal{X}$,

$$x = \int\limits_{\Omega} f_\alpha(x)\tau_\alpha \, d\mu(\alpha),$$

where the integral is weak integral.

We note that condition (ii) in Definition 2.1 says that the map

$$\theta_f : \mathcal{X} \ni x \mapsto \theta_f x \in \mathcal{L}^1(\Omega, \mu); \quad \theta_f x : \Omega \ni \alpha \mapsto (\theta_f x)(\alpha) \coloneqq f_\alpha(x) \in \mathbb{K}$$

is a linear isometry.

Theorem 2.2. ***(Functional Continuous Uncertainty Principle for Continuous 1-Schauder Frames)*** *Let* (Ω, μ), (Δ, ν) *be measure spaces. Let* $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ *and* $(\{g_\beta\}_{\beta\in\Delta}, \{\omega_\beta\}_{\beta\in\Delta})$ *be continuous 1-Schauder frames for a Banach space* $\mathcal{X}$. *Then for every* $x \in \mathcal{X} \setminus \{0\}$, *we have*

$$\mu(\operatorname{supp}(\theta_f x)) \geq \frac{1}{\sup\limits_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(\omega_\beta)|}, \quad \nu(\operatorname{supp}(\theta_g x)) \geq \frac{1}{\sup\limits_{\alpha\in\Omega,\beta\in\Delta} |g_\beta(\tau_\alpha)|}.$$

In particular,

$$\mu(\operatorname{supp}(\theta_f x))\nu(\operatorname{supp}(\theta_g x)) \geq \frac{1}{\left(\sup\limits_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(\omega_\beta)|\right)\left(\sup\limits_{\alpha\in\Omega,\beta\in\Delta} |g_\beta(\tau_\alpha)|\right)}$$

and

$$\mu(\operatorname{supp}(\theta_f x)) + \nu(\operatorname{supp}(\theta_g x)) \geq \frac{1}{\sup\limits_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(\omega_\beta)|} + \frac{1}{\sup\limits_{\alpha\in\Omega,\beta\in\Delta} |g_\beta(\tau_\alpha)|}.$$

Proof. Let $x \in \mathcal{X} \setminus \{0\}$. First using θ_f is an isometry and later using θ_g is an isometry, we get

$$
\begin{aligned}
\|x\| &= \|\theta_f x\| = \int\limits_{\Omega} |f_\alpha(x)|\, d\mu(\alpha) = \int\limits_{\operatorname{supp}(\theta_f x)} |f_\alpha(x)|\, d\mu(\alpha)\\
&= \int\limits_{\operatorname{supp}(\theta_f x)} \left| f_\alpha\left(\int\limits_{\Delta} g_\beta(x)\omega_\beta\, d\nu(\beta)\right)\right| d\mu(\alpha) = \int\limits_{\operatorname{supp}(\theta_f x)} \left|\int\limits_{\Delta} g_\beta(x) f_\alpha(\omega_\beta)\, d\nu(\beta)\right| d\mu(\alpha)\\
&= \int\limits_{\operatorname{supp}(\theta_f x)} \left|\int\limits_{\operatorname{supp}(\theta_g x)} g_\beta(x) f_\alpha(\omega_\beta)\, d\nu(\beta)\right| d\mu(\alpha) \leq \int\limits_{\operatorname{supp}(\theta_f x)} \int\limits_{\operatorname{supp}(\theta_g x)} |g_\beta(x) f_\alpha(\omega_\beta)|\, d\nu(\beta)\, d\mu(\alpha)\\
&\leq \left(\sup_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(\omega_\beta)|\right) \int\limits_{\operatorname{supp}(\theta_f x)} \int\limits_{\operatorname{supp}(\theta_g x)} |g_\beta(x)|\, d\nu(\beta)\, d\mu(\alpha)\\
&= \left(\sup_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(\omega_\beta)|\right) \mu(\operatorname{supp}(\theta_f x)) \int\limits_{\operatorname{supp}(\theta_g x)} |g_\beta(x)|\, d\nu(\beta)\\
&= \left(\sup_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(\omega_\beta)|\right) \mu(\operatorname{supp}(\theta_f x))\|\theta_g x\| = \left(\sup_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(\omega_\beta)|\right) \mu(\operatorname{supp}(\theta_f x))\|x\|.
\end{aligned}
$$

Therefore

$$
\frac{1}{\sup\limits_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(\omega_\beta)|} \leq \mu(\operatorname{supp}(\theta_f x)).
$$

On the other way, first using θ_g is an isometry and θ_f is an isometry, we get

$$
\begin{aligned}
\|x\| &= \|\theta_g x\| = \int\limits_{\Delta} |g_\beta(x)|\, d\nu(\beta) = \int\limits_{\operatorname{supp}(\theta_g x)} |g_\beta(x)|\, d\nu(\beta)\\
&= \int\limits_{\operatorname{supp}(\theta_g x)} \left| g_\beta\left(\int\limits_{\Omega} f_\alpha(x)\tau_\alpha\, d\mu(\alpha)\right)\right| d\nu(\beta) = \int\limits_{\operatorname{supp}(\theta_g x)} \left|\int\limits_{\Omega} f_\alpha(x) g_\beta(\tau_\alpha)\, d\mu(\alpha)\right| d\nu(\beta)\\
&= \int\limits_{\operatorname{supp}(\theta_g x)} \left|\int\limits_{\operatorname{supp}(\theta_f x)} f_\alpha(x) g_\beta(\tau_\alpha)\, d\mu(\alpha)\right| d\nu(\beta) \leq \int\limits_{\operatorname{supp}(\theta_g x)} \int\limits_{\operatorname{supp}(\theta_f x)} |f_\alpha(x) g_\beta(\tau_\alpha)|\, d\mu(\alpha)\, d\nu(\beta)\\
&\leq \left(\sup_{\alpha\in\Omega,\beta\in\Delta} |g_\beta(\tau_\alpha)|\right) \int\limits_{\operatorname{supp}(\theta_g x)} \int\limits_{\operatorname{supp}(\theta_f x)} |f_\alpha(x)|\, d\mu(\alpha)\, d\nu(\beta)\\
&= \left(\sup_{\alpha\in\Omega,\beta\in\Delta} |g_\beta(\tau_\alpha)|\right) \nu(\operatorname{supp}(\theta_g x)) \int\limits_{\operatorname{supp}(\theta_f x)} |f_\alpha(x)|\, d\mu(\alpha)\\
&= \left(\sup_{\alpha\in\Omega,\beta\in\Delta} |g_\beta(\tau_\alpha)|\right) \nu(\operatorname{supp}(\theta_g x))\|\theta_f x\| = \left(\sup_{\alpha\in\Omega,\beta\in\Delta} |g_\beta(\tau_\alpha)|\right) \nu(\operatorname{supp}(\theta_g x))\|x\|.
\end{aligned}
$$

Therefore

$$
\frac{1}{\sup\limits_{\alpha\in\Omega,\beta\in\Delta} |g_\beta(\tau_\alpha)|} \leq \nu(\operatorname{supp}(\theta_g x)).
$$

□

Corollary 2.3. *(**Functional Donoho-Stark-Elad-Bruckstein-Ricaud-Torrésani Uncertainty Principle for 1-Schauder Frames**) Let $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ and $(\{g_k\}_{k=1}^m, \{\omega_k\}_{k=1}^m)$ be 1-Schauder frames for a finite dimensional Banach space $\mathcal{X}$. Then for every $x \in \mathcal{X} \setminus \{0\}$, we have*

$$\|\theta_f x\|_0 \geq \frac{1}{\displaystyle\max_{1\leq j\leq n, 1\leq k\leq m} |f_j(\omega_k)|} \quad \text{and} \quad \|\theta_g x\|_0 \geq \frac{1}{\displaystyle\max_{1\leq j\leq n, 1\leq k\leq m} |g_k(\tau_j)|}.$$

In particular,

$$\|\theta_f x\|_0 \|\theta_g x\|_0 \geq \frac{1}{\left(\displaystyle\max_{1\leq j\leq n, 1\leq k\leq m} |f_j(\omega_k)|\right)\left(\displaystyle\max_{1\leq j\leq n, 1\leq k\leq m} |g_k(\tau_j)|\right)}$$

and

$$\|\theta_f x\|_0 + \|\theta_g x\|_0 \geq \frac{1}{\displaystyle\max_{1\leq j\leq n, 1\leq k\leq m} |f_j(\omega_k)|} + \frac{1}{\displaystyle\max_{1\leq j\leq n, 1\leq k\leq m} |g_k(\tau_j)|}.$$

Remark 2.4. *We note that the $\mathcal{L}^1$-norm uncertainty principles derived in [1, 13, 17] differ from our result.*

3. Functional Continuous Uncertainty Principle for Continuous ∞-Schauder Frames

We first formulate the ∞-version of Definition 1.2.

Definition 3.1. *Let (Ω, μ) be a measure space. Let $\{\tau_\alpha\}_{\alpha\in\Omega}$ be a collection in a Banach space $\mathcal{X}$ and $\{f_\alpha\}_{\alpha\in\Omega}$ be a collection in $\mathcal{X}^*$. The pair $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ is said to be a **continuous ∞-Schauder frame** for $\mathcal{X}$ if the following holds.*

(i) *For every $x \in \mathcal{X}$, the map $\Omega \ni \alpha \mapsto f_\alpha(x) \in \mathbb{K}$ is measurable.*

(ii) *The map*

$$\theta_f : \mathcal{X} \ni x \mapsto \theta_f x \in \mathcal{L}^\infty(\Omega, \mu); \quad \theta_f x : \Omega \ni \alpha \mapsto (\theta_f x)(\alpha) \coloneqq f_\alpha(x) \in \mathbb{K}$$

is a well-defined linear isometry.

(iii) *For every $x \in \mathcal{X}$, the map $\Omega \ni \alpha \mapsto f_\alpha(x)\tau_\alpha \in \mathcal{X}$ is weakly measurable.*

(iv) *For every $x \in \mathcal{X}$,*

$$x = \int_\Omega f_\alpha(x)\tau_\alpha \, d\mu(\alpha),$$

where the integral is weak integral.

Theorem 3.2. *(**Functional Continuous Uncertainty Principle for Continuous ∞-Schauder Frames**) Let (Ω, μ), (Δ, ν) be measure spaces. Let $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ and $(\{g_\beta\}_{\beta\in\Delta}, \{\omega_\beta\}_{\beta\in\Delta})$ be continuous ∞-Schauder frames for a Banach space $\mathcal{X}$. Then for every $x \in \mathcal{X} \setminus \{0\}$, we have*

$$\mu(\operatorname{supp}(\theta_f x)) \geq \frac{1}{\displaystyle\sup_{\alpha\in\Omega, \beta\in\Delta} |g_\beta(\tau_\alpha)|}, \quad \nu(\operatorname{supp}(\theta_g x)) \geq \frac{1}{\displaystyle\sup_{\alpha\in\Omega, \beta\in\Delta} |f_\alpha(\omega_\beta)|}.$$

In particular,

$$\mu(\operatorname{supp}(\theta_f x))\nu(\operatorname{supp}(\theta_g x)) \geq \frac{1}{\left(\displaystyle\sup_{\alpha\in\Omega, \beta\in\Delta} |f_\alpha(\omega_\beta)|\right)\left(\displaystyle\sup_{\alpha\in\Omega, \beta\in\Delta} |g_\beta(\tau_\alpha)|\right)}$$

and

$$\mu(\operatorname{supp}(\theta_f x)) + \nu(\operatorname{supp}(\theta_g x)) \geq \frac{1}{\sup\limits_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(\omega_\beta)|} + \frac{1}{\sup\limits_{\alpha\in\Omega,\beta\in\Delta} |g_\beta(\tau_\alpha)|}.$$

Proof. Let $x \in \mathcal{X} \setminus \{0\}$ and $\alpha \in \Omega$. Then

$$\begin{aligned}
|(\theta_f x)(\alpha)| = |f_\alpha(x)| &= \left| f_\alpha\left(\int\limits_\Delta g_\beta(x)\omega_\beta \, d\nu(\beta)\right)\right| = \left|\int\limits_\Delta g_\beta(x) f_\alpha(\omega_\beta)\, d\nu(\beta)\right| \\
&= \left|\int\limits_{\operatorname{supp}(\theta_g x)} g_\beta(x) f_\alpha(\omega_\beta)\, d\nu(\beta)\right| \leq \int\limits_{\operatorname{supp}(\theta_g x)} |g_\beta(x) f_\alpha(\omega_\beta)|\, d\nu(\beta) \\
&\leq \left(\sup_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(\omega_\beta)|\right) \int\limits_{\operatorname{supp}(\theta_g x)} |g_\beta(x)|\, d\nu(\beta) \\
&\leq \left(\sup_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(\omega_\beta)|\right) \nu(\operatorname{supp}(\theta_g x))\|\theta_g x\| \\
&= \left(\sup_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(\omega_\beta)|\right) \nu(\operatorname{supp}(\theta_g x))\|x\|.
\end{aligned}$$

Now using the fact that θ_f is an isometry, we get

$$\|x\| = \|\theta_f x\| = \operatorname{ess\,sup}(\theta_f x) \leq \left(\sup_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(\omega_\beta)|\right) \nu(\operatorname{supp}(\theta_g x))\|x\|.$$

Therefore

$$\frac{1}{\sup\limits_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(\omega_\beta)|} \leq \nu(\operatorname{supp}(\theta_g x)).$$

Now let $\beta \in \Delta$. Then

$$\begin{aligned}
|(\theta_g x)(\beta)| = |g_\beta(x)| &= \left| g_\beta\left(\int\limits_\Omega f_\alpha(x)\tau_\alpha \, d\mu(\alpha)\right)\right| = \left|\int\limits_\Omega f_\alpha(x) g_\beta(\tau_\alpha)\, d\mu(\alpha)\right| \\
&= \left|\int\limits_{\operatorname{supp}(\theta_f x)} f_\alpha(x) g_\beta(\tau_\alpha)\, d\mu(\alpha)\right| \leq \int\limits_{\operatorname{supp}(\theta_f x)} |f_\alpha(x) g_\beta(\tau_\alpha)|\, d\mu(\alpha) \\
&\leq \left(\sup_{\alpha\in\Omega,\beta\in\Delta} |g_\beta(\tau_\alpha)|\right) \int\limits_{\operatorname{supp}(\theta_f x)} |f_\alpha(x)|\, d\mu(\alpha) \\
&\leq \left(\sup_{\alpha\in\Omega,\beta\in\Delta} |g_\beta(\tau_\alpha)|\right) \mu(\operatorname{supp}(\theta_f x))\|\theta_f x\| \\
&= \left(\sup_{\alpha\in\Omega,\beta\in\Delta} |g_\beta(\tau_\alpha)|\right) \mu(\operatorname{supp}(\theta_f x))\|x\|.
\end{aligned}$$

Since θ_g is an isometry,

$$\|x\| = \|\theta_g x\| = \operatorname{ess\,sup}(\theta_g x) \leq \left(\sup_{\alpha\in\Omega,\beta\in\Delta} |g_\beta(\tau_\alpha)|\right) \mu(\operatorname{supp}(\theta_f x))\|x\|.$$

Therefore

$$\frac{1}{\sup\limits_{\alpha\in\Omega,\beta\in\Delta}|g_\beta(\tau_\alpha)|}\leq \mu(\operatorname{supp}(\theta_f x)).$$

□

Noncommutative version of Donoho-Stark and entropic uncertainty principles are recently derived [4–6, 14, 15]. Thus it is desirable to obtain noncommutative versions of uncertainty principles obtained in this paper and the paper [7–12].

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