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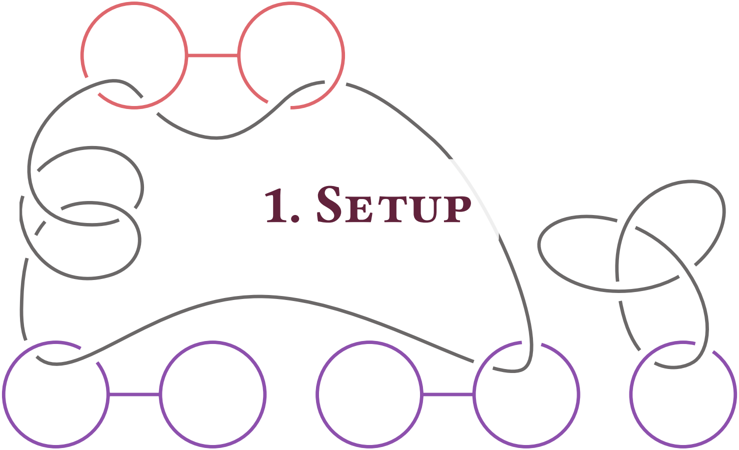
PG SEMINAR:

TOUR OF KNOTS & THETA FUNCTIONS

Introduction to abelian Chern-Simons theory

📍 Stellenbosch University
📅 October 2023

Talk Outline



1. SETUP

Motivation & goal

Goal

Establish isomorphism:

$$\left\{\begin{matrix}\text{space of THETA FUNCTIONS}\\ \text{associated with SURFACE}\end{matrix}\right\} \cong \left\{\begin{matrix}\text{space of SKEINS in}\\ \text{enclosed HANDLEBODY}\end{matrix}\right\}.$$

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Motivation

- These spaces are central to Chern-Simons theory
- My PhD is about improving this isomorphism

Notation

Notation (Manifolds)

- $\Sigma_g \quad \longrightarrow \quad$ genus-g Riemann surface

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- $\Sigma_{g,n} \quad \longrightarrow \quad$ genus-g Riemann surface with n boundary elements

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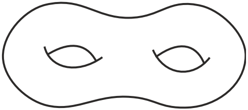
- $\Sigma_{g,n} \quad \longrightarrow \quad$ genus-g RIEMANN SURFACE with n BOUNDARY ELEMENTS
- $H_g \quad \longrightarrow \quad$ genus-g HANDLEBODY



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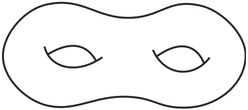

Notation (Symplectic mfds)

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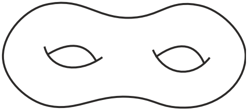

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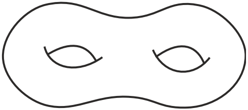

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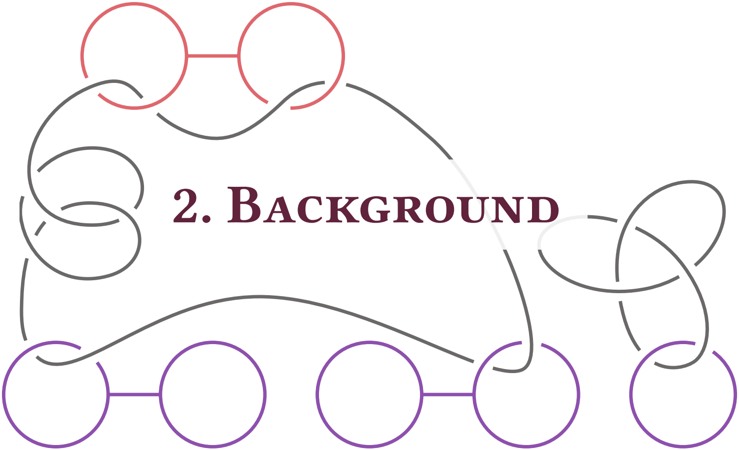
- X_f $\longrightarrow$ HAMILTONIAN vec field: $\omega(X_f, \cdot\,) = -df(\,\cdot\,)$
- $\{f, g\}$ $\longrightarrow$ POISSON BRACKET: $\{f, g\} := \omega(X_f, X_g)$



2. BACKGROUND

Quantization

Definition (Quantization)

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1. $$\underset{\text{symp mfd }(\mathcal{M},\omega)}{\underset{\|}{\left\{\text{classical PHASE SPACE}\right\}}} \rightsquigarrow \underset{\text{Hilbert space }\mathcal{H}}{\underset{\|}{\left\{\text{QUANTUM STATE space}\right\}}}$$

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2. $$\begin{tikzcd} \left\{ \text{classical OBSERVABLES} \right\} \arrow[r, squiggly] \arrow[d, equal] & \left\{ \text{quantum OBSERVABLES} \right\} \arrow[d, equal] \\ \text{funcs } f \in C^{\infty}(M,\mathbb{R}) & \text{Hermitian ops } \mathsf{op}(f) \end{tikzcd}$$

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4. $\mathrm{op}(\{f,g\}) = \frac{i}{\hbar}[\mathrm{op}(f), \mathrm{op}(g)] + \mathcal{O}(h)$ ⟵ "correspondence principle"

Polarizations

on a symplectic manifold

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Given $2n$-DIM symp mfd (M, ω),

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   - $\boldsymbol{F} = \bar{\boldsymbol{F}}$ ⟶ "real" polarization
   - $\boldsymbol{F} \cap \bar{\boldsymbol{F}} = 0$ ⟶ "KÄHLER" polarization

Polarizations

on a symplectic manifold

Example (Kähler polarization)

Take $M = \mathbb{R}^n \times \mathbb{R}^n$, as for a ***n* particles in 1D**.

Writing $z_j = x_j + iy_j$, consider

$$\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) \quad \in \; T(\mathbb{R}^n \times \mathbb{R}^n) \otimes \mathbb{C}.$$

The polarization

$$\boldsymbol{F} := \operatorname{span}\left\{\frac{\partial}{\partial \bar{z}_1}, \frac{\partial}{\partial \bar{z}_2}, \dots, \frac{\partial}{\partial \bar{z}_n}\right\}$$

is Kähler.

Geometric quantization

The Hilbert space

Fix 2n-dim SYMP MFD (M, ω) & POLARIZATION $\boldsymbol{F}$ of M.

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"covariantly constant in direction of $\boldsymbol{F}$"

Define $\mathcal{H} := \left\{ M \overset{s}{\rightarrowtail} \mathcal{L} \;\middle|\; \forall \boldsymbol{v} \in \boldsymbol{F} : \nabla_{\boldsymbol{v}} s = 0 \right\}$, where

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Define INNER PRODUCT $\langle s, t\rangle := \displaystyle\int_{M/(\boldsymbol{F}\cap\bar{\boldsymbol{F}})} \langle s(p), t(p)\rangle \; d\,\mathrm{vol}_{M/(\boldsymbol{F}\cap\bar{\boldsymbol{F}})}$.

explain

Geometric quantization

The observables

Theorem (Weil's integrality condition)

$\mathcal{L}$ *exists iff* $\omega/(2\pi\hbar) \in H^2(M, \mathbb{Z})$.

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For $s \in \mathcal{H}$, $f \in C^\infty(M, \mathbb{R})$, define

$$\mathsf{op}(f)\, s := -i\hbar\nabla_{X_f} s + f \cdot s,$$

which satisfies DIRAC'S CONDITIONS.

First integral homology group

…of a surface

Definition (Intersection form)

Define the “intersection form” $\cdot\colon H_1(\Sigma_g, \mathbb{Z}) \times H_1(\Sigma_g, \mathbb{Z}) \to \mathbb{Z}$ as follows

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A "canonical basis" comprises ORIENTED SMOOTH SIMPLE CLOSED curves $a_1, \ldots, a_g, b_1, \ldots, b_g$ of $H_1(\Sigma_g, \mathbb{Z})$ with $a_j \cdot a_k = b_j \cdot b_k = 0, a_j \cdot b_k = \delta_{j,k}$.

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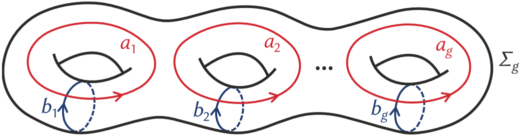

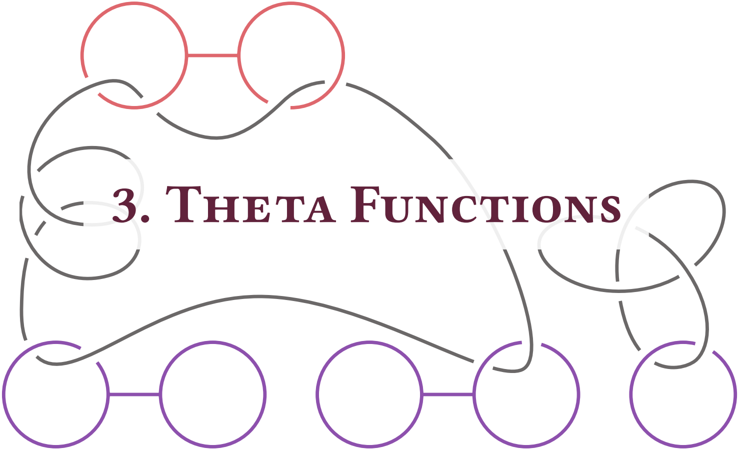

3. THETA FUNCTIONS

Overview

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\{\ \text{theta funcs}\ \}
\end{tikzcd}$$

Jacobian variety

Definition of the variety

Fix a CANONICAL BASIS, $a_1, \dots, a_g, b_1, \dots, b_g$ of $H_1(\Sigma_g, \mathbb{Z})$.

Theorem (Holomorphic basis)

$\exists!$ *basis* $\zeta_1, \dots, \zeta_g$ *of the space* $\mathscr{H}(\Sigma_g)$ *of* *HOLOMORPHIC 1-FORMS* *st*

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3. "JACOBIAN VARIETY" $\longrightarrow$ $\mathcal{J}(\Sigma_g) := \mathbb{C}^g / \boldsymbol{\Lambda}(\mathbf{1}, \Pi)$.

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… as a classical phase space

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2. $\mathscr{H}(\Sigma_g)^* \cong \mathbb{R}^{2g} \quad \longleftarrow \quad$ basis $\mathcal{B}$: $\zeta \mapsto \int_{a_j} \zeta, \, \zeta \mapsto \int_{b_j} \zeta$

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Factor each by $\boldsymbol{\Lambda}(\mathcal{B})$ to view $\mathcal{J}(\Sigma_g)$ as a complex (resp real) mfd.
The coords are related by $\boldsymbol{z} = \boldsymbol{x} + \Pi \boldsymbol{y}$.

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1. $\mathscr{H}(\Sigma_g)^* \cong \mathbb{C}^g \quad \longleftarrow \quad$ dual basis ζ_j^*
2. $\mathscr{H}(\Sigma_g)^* \cong \mathbb{R}^{2g} \quad \longleftarrow \quad$ basis $\mathcal{B}$: $\zeta \mapsto \int_{a_j} \zeta, \zeta \mapsto \int_{b_j} \zeta$

Factor each by $\mathbf{\Lambda}(\mathcal{B})$ to view $\mathcal{J}(\Sigma_g)$ as a complex (resp real) mfd.
The coords are related by $\boldsymbol{z} = \boldsymbol{x} + \Pi \boldsymbol{y}$.

Classical mechanics on Jacobian variety

1. SYMPLECTIC FORM $\quad \longrightarrow \quad \omega = (d\boldsymbol{x})^T \wedge d\boldsymbol{y}$

Jacobian variety

… as a classical phase space

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1. SYMPLECTIC FORM $\quad \longrightarrow \quad \omega = (d\boldsymbol{x})^T \wedge d\boldsymbol{y}$
2. classical OBSERVABLES $\quad \longrightarrow \quad$ generated by $\exp(2\pi i(\boldsymbol{p}^T\boldsymbol{x} + \boldsymbol{q}^T\boldsymbol{y}))$,
   for $(\boldsymbol{p}, \boldsymbol{q}) \in \mathbb{Z}^{2g} \cong H_1(\Sigma_g, \mathbb{Z})$

Hilbert space of quantization of Jacobian variety

Unpacking the definition

Fix an EVEN $N \in \mathbb{N}$; set $\hbar = \dfrac{1}{2\pi N}$. ⟵ (to meet *Weil's integrality condition*)

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Quantizing $\mathcal{J}(\Sigma_g)$

Recall $\mathcal{H} := \left\{ M \overset{s}{\rightarrowtail} \mathcal{L} \;\middle|\; \forall \boldsymbol{v} \in \boldsymbol{F} : \nabla_{\boldsymbol{v}} s = 0 \right\}$;

1. $\mathcal{L}$ is a HOLOMORPHIC line bundle with curvature

$$\omega/\hbar = \pi i N (d\boldsymbol{z})^T \wedge Y^{-1} d\bar{\boldsymbol{z}}, \quad \text{where} \quad \Pi = X + iY.$$

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3. A COCYCLE $\Lambda\colon \mathbb{C}^g \times \boldsymbol{\Lambda}(\mathbf{1}, \Pi) \to \mathbb{C}$ encodes $\mathbb{C}^g \times \mathbb{C}$ by

$$(\boldsymbol{z}, \zeta) \sim (\boldsymbol{z}', \zeta') \quad \Longleftrightarrow \quad \begin{array}{l} \boldsymbol{z}' = \boldsymbol{z} + \boldsymbol{\lambda},\ \zeta' = \Lambda(\boldsymbol{z}, \boldsymbol{\lambda})\zeta \\ \text{for some } \boldsymbol{\lambda} \in \boldsymbol{\Lambda}(\mathbf{1}, \Pi). \end{array}$$

Hilbert space of quantization of Jacobian variety

Unpacking the definition

$\vdots$

4. Λ is HOLOMORPHIC in $\boldsymbol{z}$ (for $\mathcal{L}$ to be), & satisfies COCYCLE CONDITION:

$$\Lambda(\boldsymbol{z},\boldsymbol{\lambda})\Lambda(\boldsymbol{z}+\boldsymbol{\lambda},\boldsymbol{\mu})=\Lambda(\boldsymbol{z},\boldsymbol{\mu}+\boldsymbol{\lambda})\quad \text{for all } \boldsymbol{z}\in\mathbb{C}^g,\ \boldsymbol{\lambda},\boldsymbol{\mu}\in\boldsymbol{\Lambda}(\mathbf{1},\Pi).$$

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5 SIMPLEST SOLUTION: ⟵ other solutions just tensor $\mathcal{L}$ with a flat line bundle

$$\Lambda(\boldsymbol{z},\boldsymbol{\lambda}_j) = 1, \quad \Lambda(\boldsymbol{z},\boldsymbol{\lambda}_{g+j}) = e^{N\pi(2iz_j-\pi_{jj})}$$

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6 With $\boldsymbol{F} = \operatorname{span}\{\frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n}\}$, $\mathcal{H}$ is just HOLOMORPHIC SXNS. Pulled back to $\mathbb{C}^g$, they satisfy:

$$\begin{aligned} f(\boldsymbol{z} + \boldsymbol{\lambda}_j) &= f(\boldsymbol{z}) \\ f(\boldsymbol{z} + \boldsymbol{\lambda}_{g+j}) &= e^{N\pi(2iz_j - \pi_{jj})} f(\boldsymbol{z}). \end{aligned}$$

This is the set $\boldsymbol{\Theta}_N^{\Pi}(\Sigma_g)$ of “THETA FUNCTIONS”.

Hilbert space of quantization of Jacobian variety

Explicit basis

Lemma (Basis for $\Theta_N^\Pi(\Sigma_g)$)

A BASIS *for* $\Theta_N^\Pi(\Sigma_g)$ *is given by the "theta series":*

$$\theta_{\boldsymbol{\mu}}^{\Pi}(\boldsymbol{z}) := \sum_{\boldsymbol{n}\in\mathbb{Z}^g} \exp\left(2\pi i N\left[\frac{1}{2}\left(\frac{\boldsymbol{\mu}}{N}+\boldsymbol{n}\right)^T \Pi\left(\frac{\boldsymbol{\mu}}{N}+\boldsymbol{n}\right)+\left(\frac{\boldsymbol{\mu}}{N}+\boldsymbol{n}\right)^T \boldsymbol{z}\right]\right),$$

for $\boldsymbol{\mu} \in \{0, \ldots, N-1\}^g \equiv \mathbb{Z}_N^g$.

Schrödinger representation

Action of quantized observables

Theorem (Weyl quantization)

QUANTIZED EXPONENTIALS act on $\Theta_N^{\Pi}(\Sigma_g)$ *as*

$$\mathsf{op}\left(e^{2\pi i(\boldsymbol{p}^T\boldsymbol{x}+\boldsymbol{q}^T\boldsymbol{y})}\right)\cdot\theta_{\boldsymbol{\mu}}^{\Pi}(\boldsymbol{z}) = e^{-\frac{i\pi}{N}(\boldsymbol{p}^T\boldsymbol{q}-2\boldsymbol{\mu}^T\boldsymbol{q})}\ \theta_{\boldsymbol{\mu}+\boldsymbol{p}}^{\Pi}(\boldsymbol{z}).$$

⟶ *"SCHRÖDINGER REP" of FINITE HEISENBERG GROUP on* $\Theta_N^{\Pi}(\Sigma_g)$

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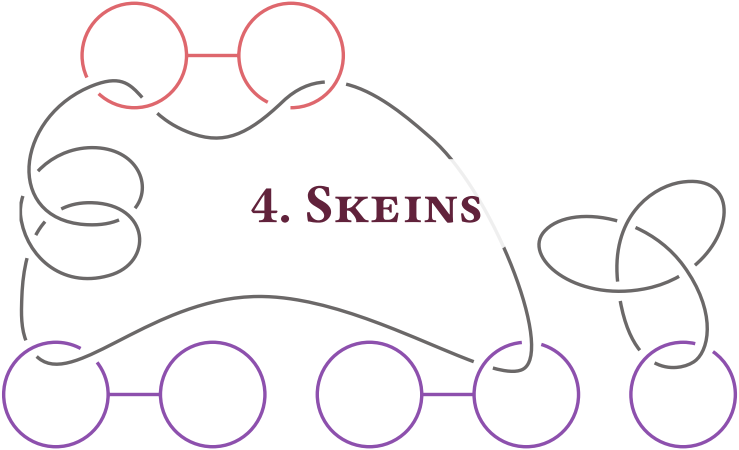
$\longrightarrow$ *"Schrödinger rep" of finite Heisenberg group on* $\Theta_N^{\Pi}(\Sigma_g)$

Theorem (Space of linear operators)

The space $L(\Theta_N^{\Pi}(\Sigma_g))$ *of linear operators on* $\Theta_N^{\Pi}(\Sigma_g)$ *has basis* $\mathsf{op}\left(e^{2\pi i(\boldsymbol{p}^T\boldsymbol{x}+\boldsymbol{q}^T\boldsymbol{y})}\right)$, *where* $\boldsymbol{p},\boldsymbol{q}\in\mathbb{Z}_N^g$.



4. SKEINS

Overview

We turn to the other space from our goal:

$$\left\{\begin{array}{c}\text{space of THETA FUNCTIONS}\\ \text{associated with SURFACE}\end{array}\right\} \cong \left\{\begin{array}{c}\text{space of SKEINS in}\\ \text{enclosed HANDLEBODY}\end{array}\right\}$$

✓

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This construction is more direct:

$$\begin{tikzcd} H_g \arrow[d, "\text{\textcircled{1} construct skein module}"] \\ \mathcal{L}_N(H_g) \end{tikzcd}$$

Links & parallel powers

Fix smooth compact oriented 3-MFD M.

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The nth "parallel power" $K^{||N}$ of FRAMED KNOT $K \overset{\nu}{\hookrightarrow} M$ is obtained by restricting ν to $S^1 \times \bigsqcup_{k=1}^{n} \left[\frac{j}{n+1} - \frac{1}{2n}, \frac{j}{n+1} + \frac{1}{2n} \right]$.

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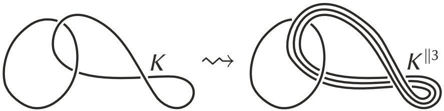

Linking number skein module

& the reduced version

Definition (Linking number skein module)

1. $\mathbb{C}[t, t^{-1}]\,\mathsf{Link}(M) \quad \longrightarrow \quad$ free $\mathbb{C}[t, t^{-1}]$-MODULE over $\mathsf{Link}(M)$

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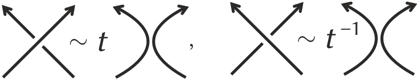
   - i 

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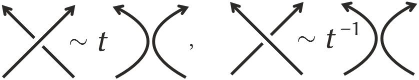
   i 

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$\mathcal{L}(M)$ is the "linking number SKEIN MODULE"; its elements are "skeins".

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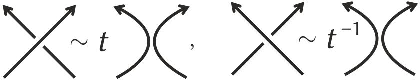
   - i 

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   $\mathcal{L}(M)$ is the "linking number SKEIN MODULE"; its elements are "skeins".
3. $\mathcal{L}_N(M) := \mathcal{L}(M)/\sim$ for further skein relations
   - iii $t\,\sigma \sim e^{\frac{i\pi}{N}}\,\sigma$
   - iv $L \sim L \cup K^{||n}$

   $\mathcal{L}_N(M)$ is the "REDUCED linking number skein module".

Skein algebra of a surface

= skein algebra of a cylinder over that surface

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We can define an ALGEBRA $\mathcal{L}(\Sigma)$ for smooth compact oriented SURFACE Σ:

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Write $\mathcal{L}(\Sigma) := \mathcal{L}(\Sigma \times [0, 1])$. A similar def applies to $\mathcal{L}_N(\Sigma)$.

Left action of skein algebra

… of a surface on the skein module of the enclosed handlebody

Fix a CANONICAL BASIS, $a_1, \dots, a_g, b_1, \dots, b_g$ of $H_1(\Sigma_g, \mathbb{Z})$.

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$\exists$ *orientation preserving* *DIFFEO* $f : \Sigma_g \to \partial H_g$*, depending only on* $b_1, \ldots, b_g$*, such that*

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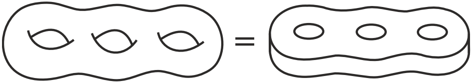

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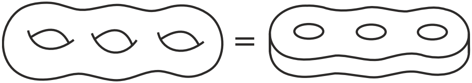
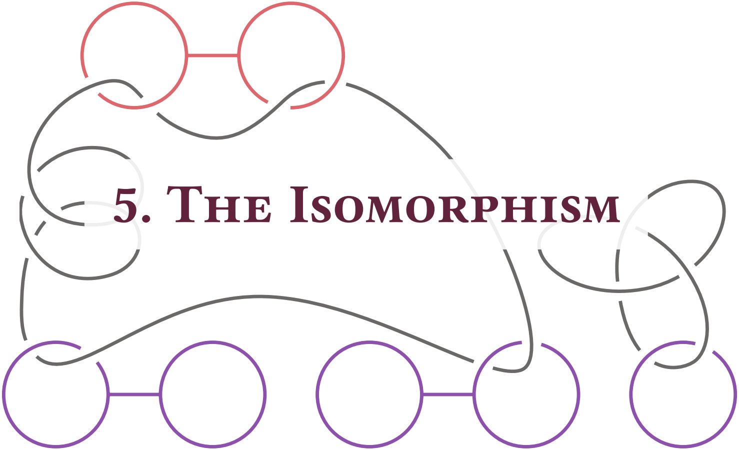


Action of $\mathcal{L}_N(\Sigma_g)$ on $\mathcal{L}_N(H_g)$

By GLUING $\Sigma_g \times [0, 1]$ to H_g under f, we get an AXN of $\mathcal{L}_N(\Sigma_g)$ on $\mathcal{L}_N(H_g)$.



5. THE ISOMORPHISM

Overview

We finally remark on the isomorphism:

$$\underset{\checkmark}{\left\{\begin{array}{c}\text{space of THETA FUNCTIONS}\\ \text{associated with SURFACE}\end{array}\right\}} \underset{\substack{\uparrow\\ \text{now focus here}}}{\cong} \underset{\checkmark}{\left\{\begin{array}{c}\text{space of SKEINS in}\\ \text{enclosed HANDLEBODY}\end{array}\right\}}$$

ie. $\mathcal{L}_N(H_g) \cong \boldsymbol{\Theta}_N^{\Pi}(\Sigma_g)$

Operator algebras

Fix a CANONICAL BASIS, $a_1, \dots, a_g, b_1, \dots, b_g$ of $H_1(\Sigma_g, \mathbb{Z})$.

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$\mathcal{L}_N(\Sigma_g)$ has BASIS $\langle\, (\boldsymbol{p}, \boldsymbol{q})\, \rangle$, for $(\boldsymbol{p}, \boldsymbol{q}) \in \mathbb{Z}^{2g} \cong H_1(\Sigma_g, \mathbb{Z})$.

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Theorem (Operator algebras)

$\mathcal{L}_N(\Sigma_g) \;\cong\; L(\boldsymbol{\Theta}_N^{\sqcap}(\Sigma_g))$, as algebras.

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"Proof".

By above lemma & prior basis for $L(\Theta_N^{\sqcap}(\Sigma_g))$, isomorphism is

$$\langle\, (\boldsymbol{p}, \boldsymbol{q})\, \rangle \mapsto \mathsf{op}\left(e^{2\pi i (\boldsymbol{p}^T \boldsymbol{x} + \boldsymbol{q}^T \boldsymbol{y})} \right).$$

☐

Hilbert spaces

Lemma (Basis for $\mathcal{L}_N(H_g)$)

$\mathcal{L}_N(H_g) = \mathcal{L}_N(\Sigma_{0,g+1} \times [0,1])$ *has* ***BASIS*** $\langle\gamma\rangle$, *where* γ *ranges over multicurves representing homology classes of* $H_1(\Sigma_{0,g+1}, \mathbb{Z}_N)$

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Theorem (Main result)

$\mathcal{L}_N(H_g)$ $\qquad\qquad$ $\Theta_N^{\Pi}(\Sigma_g)$

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Theorem (Main result)

$$\begin{tikzcd} \mathcal{L}_N(\Sigma_g) & L(\boldsymbol{\Theta}_N^{\sqcap}(\Sigma_g)) \\ \mathcal{L}_N(H_g) \arrow[loop above] & \boldsymbol{\Theta}_N^{\sqcap}(\Sigma_g) \arrow[loop above] \end{tikzcd}$$

Hilbert spaces

Lemma (Basis for $\mathcal{L}_N(H_g)$)

$\mathcal{L}_N(H_g) = \mathcal{L}_N(\Sigma_{0,g+1} \times [0,1])$ *has* BASIS $\langle\gamma\rangle$, *where* γ *ranges over multicurves representing homology classes of* $H_1(\Sigma_{0,g+1}, \mathbb{Z}_N)$

Theorem (Main result)

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where γ ranges over MULTICURVES *in $\Sigma_{0,g+1} \cong H_g$, $[\gamma] \in H_1(H_g, \mathbb{Z}_N) = \mathbb{Z}_N^g$. This iso* INTERTWINES *the resp actions.*

Overview of the proof

Proof outline.

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$$\langle(\boldsymbol{p}, \boldsymbol{q})\rangle \cdot \langle a_1^{||\mu_1} \cdots a_g^{||\mu_g}\rangle = t^{-\boldsymbol{p}^T\boldsymbol{q}-2\boldsymbol{\mu}^T\boldsymbol{q}}\langle a_1^{||\mu_1+p_1} \cdots a_g^{||\mu_g+p_g}\rangle$$

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    5. setting $t = e^{\frac{i\pi}{N}}$, we recognize the SCHRÖDINGER REP
$$\mathrm{op}\left(e^{2\pi i(\boldsymbol{p}^T\mathbf{x}+\boldsymbol{q}^T\mathbf{y})}\right) \cdot \theta^{\Pi}_{\boldsymbol{\mu}}(\boldsymbol{z}) = e^{-\frac{i\pi}{N}(\boldsymbol{p}^T\boldsymbol{q}-2\boldsymbol{\mu}^T\boldsymbol{q})}\ \theta^{\Pi}_{\boldsymbol{\mu}+\boldsymbol{p}}(\boldsymbol{z}).$$

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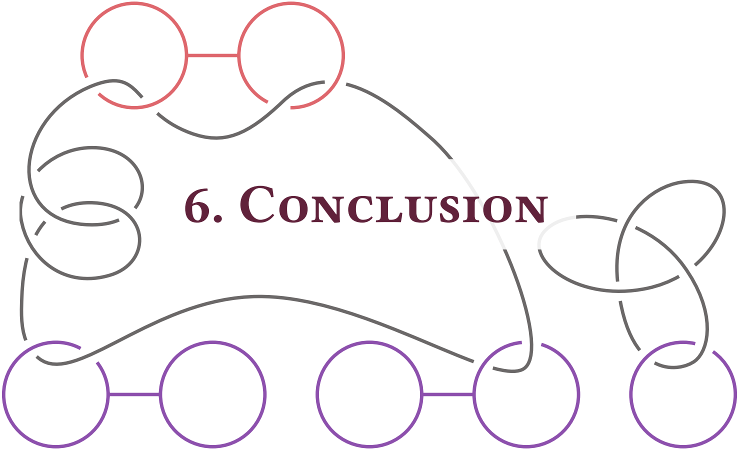
Overview of the proof

Observation / lamentation

Of course, we have relied on BASES for each of

$$\Theta_N^{\Pi}(\Sigma_g), \quad \mathcal{L}_N(H_g), \quad L(\Theta_N^{\Pi}(\Sigma_g)), \quad \mathcal{L}_N(\Sigma_g).$$



6. CONCLUSION

Concluding remarks

Why the result is interesting

1. It gives a much simpler TOPOLOGICAL VERSION of the SCHRÖDINGER REP on theta functions:

$$\mathsf{op}\left(e^{2\pi i(\boldsymbol{p}^T\boldsymbol{x}+\boldsymbol{q}^T\boldsymbol{y})}\right)\cdot\theta^{\Pi}_{\boldsymbol{\mu}}(\boldsymbol{z})=e^{-\frac{i\pi}{N}(\boldsymbol{p}^T\boldsymbol{q}-2\boldsymbol{\mu}^T\boldsymbol{q})}\ \theta^{\Pi}_{\boldsymbol{\mu}+\boldsymbol{p}}(\boldsymbol{z}).$$

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Connection to my work

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1. $\mathcal{J}(\Sigma_g) \quad \longrightarrow \quad$ moduli space of flat SU(2) connections on Σ_g;
2. skein relations are more complicated;

…otherwise, same result. My work involves making this iso basis-free.

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