On Tangents and Secants of Infinite Sums

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Abstract. We prove some identities involving tangents, secants, and cosecants of infinite sums.

For $k = 0, 1, 2, \ldots$ let e_k be the kth-degree elementary symmetric function of $\tan\theta_j$, $j = 1, 2, 3, \ldots$, i.e. the sum of all products of k of the tangents. It is routine to prove by induction on the number of terms, when that number is finite, that

$$\begin{aligned} \tan\sum_j \theta_j &= \frac{e_1 - e_3 + e_5 - \cdots}{e_0 - e_2 + e_4 - \cdots}, \\ \sec\sum_j \theta_j &= \frac{\prod_j \sec\theta_j}{e_0 - e_2 + e_4 - \cdots}, \\ \csc\sum_j \theta_j &= \frac{\prod_j \sec\theta_j}{e_1 - e_3 + e_5 - \cdots}. \end{aligned} \tag{1}$$

As far as I know, the last two identities do not appear in any refereed source. The case of the first one in which only finitely many terms appear on the left appears in [**1**, page 47].

We will prove that the last two identities hold when the sum on the left converges absolutely. The first identity in that case follows as a corollary. I added a quick sketch of these proofs to Wikipedia's "List of trigonometric identities" [**2**] in 2012.

1. ELEMENTARY SYMMETRIC FUNCTIONS AND CONVERGENCE.

- The kth-degree elementary symmetric function $e_{n,k}$ in finitely many variables x_i, $i \in \{1, \ldots, n\}$ is the sum of all products of k of those variables:

$$e_{n,k} = \sum_{\substack{A\subseteq\{1,\ldots,n\}\\ |A|=k}} \prod_{j\in A} x_j.$$

  In particular, $e_{n,0} = 1$ and if $k > n$ then $e_{n,k} = 0$. For example, $e_{4,2} = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$.

- The kth-degree elementary symmetric function e_k in variables x_i, $i \in \mathbb{N} = \{1, 2, 3, \ldots\}$ is the sum of all products of k of those variables:

$$e_k = \sum_{\substack{A\subseteq\mathbb{N}\\ |A|=k}} \prod_{j\in A} x_j. \tag{2}$$

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- In the two infinite series

$$e_0 - e_2 + e_4 - \cdots$$
$$e_1 - e_3 + e_5 - \cdots$$

each term shown is itself ± 1 times one of the infinite series (2). As long as convergence is absolute, the order of summation will not affect the value of the sum, but our proofs will involve limits of partial sums. Hence, we will evaluate the sums in the following order:

$$\begin{aligned} e_0 - e_2 + e_4 - \cdots &= \lim_{n\to\infty} \sum_{\text{even } k \le n} (-1)^{k/2} e_{n,k} \\ e_1 - e_3 + e_5 - \cdots &= \lim_{n\to\infty} \sum_{\text{odd } k \le n} (-1)^{(k-1)/2} e_{n,k}. \end{aligned} \tag{3}$$

2. EXAMPLES OF TRIGONOMETRIC IDENTITIES. In all that follows, we will have $x_j = \tan\theta_j$ for $j = 1, 2, 3, \ldots$. Here are our identities in the case where only three θs are not 0:

$$\tan(\theta_1 + \theta_2 + \theta_3) = \frac{\tan\theta_1 + \tan\theta_2 + \tan\theta_3 - \tan\theta_1 \tan\theta_2 \tan\theta_3}{1 - \tan\theta_1 \tan\theta_2 - \tan\theta_1 \tan\theta_3 - \tan\theta_2 \tan\theta_3},$$

$$\sec(\theta_1 + \theta_2 + \theta_3) = \frac{\sec\theta_1 \sec\theta_2 \sec\theta_3}{1 - \tan\theta_1 \tan\theta_2 - \tan\theta_1 \tan\theta_3 - \tan\theta_2 \tan\theta_3},$$

$$\csc(\theta_1 + \theta_2 + \theta_3) = \frac{\sec\theta_1 \sec\theta_2 \sec\theta_3}{\tan\theta_1 + \tan\theta_2 + \tan\theta_3 - \tan\theta_1 \tan\theta_2 \tan\theta_3}.$$

3. RESULTS IN THE INFINITE CASE.

Theorem 1. *Suppose $\sum_{j=1}^{\infty} \theta_j$ converges absolutely. Then the three identities (1) hold, with the infinite sums involving symmetric functions defined as in (3). Convergence in (3) is absolute.*

Proof. We take the codomain of trigonometric functions to be the one-point compactification $\mathbb{R} \cup \{\infty\}$. Thus, they are continuous everywhere, and as $n \to \infty$, the secant, cosecant, and tangent of $\sum_{j=1}^{n} \theta_j$ approach the values of those functions at $\sum_{j=1}^{\infty} \theta_j$, including the case in which a pole occurs at the value of that sum.

Our central tactic is to rearrange the last two identities in (1) in the case involving only finitely many terms as follows:

$$\begin{aligned} e_{n,0} - e_{n,2} + e_{n,4} - \cdots &= \frac{\prod_{j=1}^{n} \sec\theta_j}{\sec \sum_{j=1}^{n} \theta_j} \\ e_{n,1} - e_{n,3} + e_{n,5} - \cdots &= \frac{\prod_{j=1}^{n} \sec\theta_j}{\csc \sum_{j=1}^{n} \theta_j} \end{aligned} \tag{4}$$

(the sums on the left have only finitely many nonzero terms). The identities (4) are proved by a routine induction on n.

For large enough N, for all $j \geq N$, we have θ_j so close to 0 that $1 \leq \sec\theta_j \leq 1 + \theta_j^2 \leq \exp(\theta_j^2)$. Hence,

$$1 \leq \prod_{j=N}^{n} \sec\theta_j \leq \prod_{j=N}^{n} (1 + \theta_j^2) \leq \prod_{j=N}^{n} \exp(\theta_j^2) = \exp\left(\sum_{j=N}^{n} \theta_j^2\right),$$

and that converges as $n \to \infty$ since $\sum_j \theta_j$ converges absolutely. Thus, the right sides of (4) converge and, therefore, so do the left sides, and to the same limit.

To show that convergence on the left side is absolute, we let $f_{n,k}$ be the kth-degree elementary symmetric function in the absolute values $|x_1|, \ldots, |x_n|$. It is enough to prove that

$$\lim_{n\to\infty} \sum_{k=0}^{n} f_{n,k} < \infty.$$

We have

$$\lim_{n\to\infty} \sum_{k=0}^{n} f_{n,k} = \lim_{n\to\infty} \prod_{i=1}^{n} (1 + |x_i|) \leq \lim_{n\to\infty} \prod_{i=1}^{n} \exp|x_i| = \lim_{n\to\infty} \exp \sum_{i=1}^{n} |x_i| < \infty.$$

Finally, since $\tan = \sec/\csc$, the first identity in (1) follows. ■

REFERENCES

1. E.W. Hobson, *A Treatise on Plane Trigonometry*, Cambridge Univ. Press, Cambridge, 1891.
2. https://en.wikipedia.org/wiki/List_of_trigonometric_identities

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