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Smarandache's Orthic Theorem

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Abstract In this paper We present the Smarandache's Orthic Theorem in the geometry of the triangle.

Keywords Smarandache's Orthic Theorem, triangle.

§1. The main result

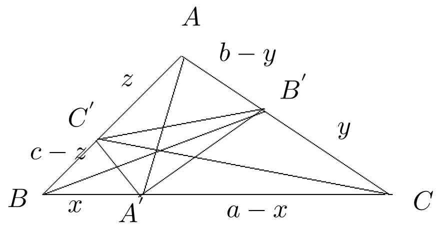
Smarandache's Orthic Theorem

Given a triangle ABC whose angles are all acute (acute triangle), we consider $A'B'C'$, the triangle formed by the legs of its altitudes.

In which conditions the expression:

$$\|A'B'\| \cdot \|B'C'\| + \|B'C'\| \cdot \|C'A'\| + \|C'A'\| \cdot \|A'B'\|$$

is maximum?



Proof. We have

$$\triangle ABC \sim \triangle A'B'C' \triangle AB'C \sim \triangle A'BC'. \tag{1}$$

We note

$$\|BA'\| = x, \|CB'\| = y, \|AC'\| = z.$$

It results that

$$\|A'C\| = a - x, \|B'A\| = b - y, \|C'B\| = c - z.$$

$$\widehat{BAC} = \widehat{B'A'C} = \widehat{BA'C'}; \widehat{ABC} = \widehat{AB'C'} = \widehat{A'B'C'}; \widehat{BCA} = \widehat{BC'A'} = \widehat{B'C'A}.$$

From these equalities it results the relation (1)

$$\triangle A'BC' \sim \triangle A'B'C \Rightarrow \frac{A'C'}{a-x} = \frac{x}{\|A'B'\|}, \tag{2}$$

$$\triangle A'B'C \sim \triangle AB'C' \Rightarrow \frac{A'C'}{z} = \frac{c-z}{\|B'c'\|}, \tag{3}$$

$$\triangle AB'C \sim \triangle A'B'C \Rightarrow \frac{B'C'}{y} = \frac{b-y}{\|A'B'\|}. \tag{4}$$

From (2), (3) and (4) we observe that the sum of the products from the problem is equal to:

$$x(a-x) + y(b-y) + z(c-z) = \frac{1}{4}(a^2+b^2+c^2) - (x-\frac{a}{2})^2 - (y-\frac{b}{2})^2 - (z-\frac{c}{2})^2,$$

which will reach its maximum as long as $x = \frac{a}{2}, y = \frac{b}{2}, z = \frac{c}{2}$, that is when the altitudes' legs are in the middle of the sides, therefore when the $\triangle ABC$ is equilateral. The maximum of the expression is $\frac{1}{4}(a^2+b^2+c^2)$.

§2. Conclusion (Smarandache's Orthic Theorem)

If we note the lengths of the sides of the triangle $\triangle ABC$ by $\|AB\| = c, \|BC\| = a, \|CA\| = b$, and the lengths of the sides of its orthic triangle $\triangle A^*B^*C^*$ by $\|A^*B^*\| = c^*, \|B^*C^*\| = a^*, \|C^*A^*\| = b^*$, then we proved that:

$$4(a^*b^* + b^*c^* + c^*a^*) \leq a^2+b^2+c^2.$$

§3. Open problems related to Smarandache's Orthic Theorem

1. Generalize this problem to polygons. Let $A_1A_2\cdots A_m$ be a polygon and P a point inside it. From P we draw perpendiculars on each side A_iA_{i+1} of the polygon and we note by $A_{i'}$ the intersection between the perpendicular and the side A_iA_{i+1}. A pedal polygon $A_{1'}A_{2'}\cdots A_{m'}$ is formed. What properties does this pedal polygon have?

2. Generalize this problem to polyhedrons. Let $A_1A_2\cdots A_n$ be a poliyhedron and P a point inside it. From P we draw perpendiculars on each polyhedron face F_i and we note by $A_{i'}$ the intersection between the perpendicular and the side F_i. A pedal polyhedron $A_{1'}A'_2\cdots A_{p'}$ is formed, where p is the number of polyhedron's faces. What properties does this pedal polyhedron have?

References

[1] Cătălin Barbu, Teorema lui Smarandache, Teoreme fundamentale din geometria triunghiului, Chapter II, Teoreme fundamentale din geometria triunghiului, Section II. 57, Editura Unique, Bacău, 2008, p. 337.

[2] József Sándor, On Smarandache's Pedal Theorem, Geometric Theorems, Diophantine Equations, and Arithmetic Functions, AR Press, Rehoboth, 2002, 9-10.

[3] Ion Pătraşcu, Smarandache's Orthic Theorem, http://www.scribd.com/doc/28311593/Smarandache-s-Orthic-Theorem.

[4] F. Smarandache, Eight Solved and Eight Open Problems in Elementary Geometry, in arXiv.org, Cornell University, NY, USA.

[5] F. Smarandache, Problèmes avec et san... problèmes!, Somipress, Fés, Morocco, 1983, Problem 5. 41, p. 59.