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A Note on 1-Edge Balance Index Set

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Abstract: Let G be a graph with vertex set V and edge set E, and $Z_2 = \{0, 1\}$. Let f be a labeling from E to Z_2, so that the labels of the edges are 0 or 1. The edges labelled 1 are called 1-edges and edges labelled 0 are called 0-edges. The edge labeling f induces a vertex labeling $f^* : V \longrightarrow Z_2$ defined by

$$f^*(v) = \begin{cases} 1 & \text{if the number of 1-edges incident on } v \text{ is odd,} \\ 0 & \text{if the number of 1-edges incident on } v \text{ is even.} \end{cases}$$

For $i \in Z_2$ let $e_f(i) = e(i) = |\{e \in E : f(e) = i\}|$ and $v_f(i) = v(i) = |\{v \in V : f^*(v) = i\}|$. A labeling f is said to be edge-friendly if $| e(0) - e(1) | \leq 1$. The 1- edge balance index set ($OEBI$) of a graph G is defined by $\{| v_f(0) - v_f(1) | :$ the edge labeling f is edge-friendly$\}$. The main purpose of this paper is to completely determine the 1-edge balance index set of wheel and Mycielskian graph of a path.

Key Words: Mycielskian graph, edge labeling, edge-friendly, 1-edge balance index set, Smarandachely induced vertex labeling, Smarandachely edge-friendly graph.

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§1. Introduction

A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. Varieties of graph labeling have been investigated by many authors [2], [3] [5] and they serve as useful models for broad range of applications.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$ and $Z_2 = \{0, 1\}$. Let f be a labeling from $E(G)$ to Z_2, so that the labels of the edges are 0 or 1. The edges labelled 1 are called 1-edges and edges labelled 0 are called 0-edges. The edge labeling f induces a vertex labeling $f^* : V(G) \longrightarrow Z_2$, defined by

$$f^*(v) = \begin{cases} 1 & \text{if the number of 1-edges incident on } v \text{ is odd,} \\ 0 & \text{if the number of 1-edges incident on } v \text{ is even.} \end{cases}$$

For $i \in Z_2$, let $e_f(i) = e(i) = |\{e \in E(G) : f(e) = i\}|$ and $v_f(i) = v(i) = |\{v \in V(G) : f^*(v) = i\}|$. Generally, let $f : E(G) \to Z_p$ be a labeling from $E(G)$ to Z_p for an integer

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$p \geq 2$. A *Smarandachely induced vertex labeling* on G is defined by $f^v = (l_1, l_2, \cdots, l_p)$ with $n_k(v) \equiv l_k(\mathrm{mod} p)$, where $n_k(v)$ is the number of k-edges, i.e., edges labeled with an integer k incident on v. Let

$$e_k(G) = \frac{1}{2} \sum_{e \in E(G)} n_k(v)$$

for an integer $1 \leq k \leq p$. Then a Smarandachely edge-friendly graph is defined as follows.

Definition 1.1 *A graph G is said to be Smarandachely edge-friendly if $| e_k(G) - e_{k+1}(G) | \leq 1$ for integers $1 \leq k \leq p$. Particularly, if $p = 2$, such a Smarandachely edge-friendly graph is abbreviated to an edge-friendly graph.*

Definition 1.2 *The 1-edge balance index set of a graph G, denoted by $OEBI(G)$, is defined as $\{| v_f(1) - v_f(0) | : f$ is edge-friendly$\}$.*

For convenience, a vertex is called 0-vertex if its induced vertex label is 0 and 1-vertex, if its induced vertex label is 1.

In the mid 20^{th} century there was a question regarding the construction of triangle-free k-chromatic graphs, where $k \leq 3$. In this search Mycielski [4] developed an interesting graph transformation known as the Mycielskian which is defined as follows:

Definition 1.3 *For a graph $G = (V, E)$, the Mycielskian of G is the graph $\mu(G)$ with vertex set consisting of the disjoint union $V \cup V' \cup \{v_0\}$, where $V' = \{x' : x \in V\}$ and edge set $E \cup \{x'y : xy \in E\} \cup \{x'v_0 : x' \in V'\}$.*

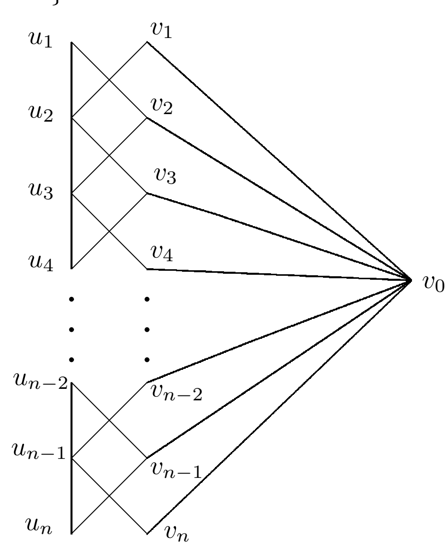

Figure 1 Mycielskian graph of the path P_n

Recently Chandrashekar Adiga et al. [1] have introduced and studied the 1-edge balance index set of several classes of graphs. In Section 2, we completely determine the 1-edge balance index set of the Mycielskian graph of path P_n. In Section 3, we establish that $OEBI(W_n) = \{0, 4, 8 \ldots, n\}$ if $n \equiv 0(mod\ 4)$, $OEBI(W_n) = \{2, 6, 10 \ldots, n\}$ if $n \equiv 2(mod\ 4)$ and $OEBI(W_n) = \{1, 2, 5 \ldots, n\}$ if n is odd.

§2. The 1-Edge Balance Index Set of $\mu(P_n)$

In this section we consider the Mycielskian graph of the path P_n $(n \geq 2)$, which consists of $2n+1$ vertices and $4n-3$ edges. To determine the $OEBI(\mu(P_n))$ we need the following theorem, whose proof is similar to the proof of the Theorem 1 in [6].

Theorem 2.1 *If the number of vertices in a graph G is even(odd) then the 1-edge balance index set contains only even(odd)numbers.*

Now we divide the problem of finding $OEBI(\mu(P_n))$ into two cases, viz,

$$n \equiv 0(mod\ \ 2) \qquad \text{and} \qquad n \equiv 1(mod\ \ 2),$$

Denoted by $max\{OEBI(\mu(P_n))\}$ the largest number in the 1-edge balance index set of $\mu(P_n)$. Then we get the follpowing result.

Theorem 2.2 *If* $n \equiv 0(mod\ \ 2)$ *i.e,* $n = 2k(k \in N)$, *then* $OEBI(\mu(P_n)) = \{1, 3, 5, \ldots, 2n+1\}$.

Proof Let f be an edge-friendly labeling on $\mu(P_n)$. Since the graph contains $2n+1 = 4k+1$ vertices, $4n-3 = 8k-3$ edges, we have two possibilities: i) $e(0) = 4k-1$, $e(1) = 4k-2$ ii) $e(0) = 4k-2$, $e(1) = 4k-1$. Now we consider the first case namely $e(0) = 4k-1$ and $e(1) = 4k-2$. Denote the vertices of $\mu(P_n)$ as in the Figure 1. Now we label the edges $u_{2q-1}v_{2q}$, $u_{2q+1}v_{2q}$ for $1 \leq q \leq k-1$, u_qu_{q+1} for $1 \leq q \leq 2k-3$, $u_{2k-2}v_{2k-1}$, $u_{2k}v_{2k-1}$ and $u_{2k-1}u_{2k}$ by 1 and label the remaining edges by 0. Then it is easy to observe that $v(0) = 4k+1$ and there is no 1-vertex in the graph. Thus $\mid v(1) - v(0) \mid = 4k+1 = 2n+1 = max\{OEBI(\mu(P_n))\}$.

Now we interchange the labels of the edges to get the remaining 1-edge balance index numbers. By interchanging the labels of edges $u_{2q}u_{2q+1}$ and $u_{2q}v_{2q+1}$ for $1 \leq q \leq k-2$, we get, $\mid v(0) - v(1) \mid = 4k+1-4q$. Further interchanging $u_{2k-1}u_{2k}$ and $u_{2k-1}v_{2k}$, we get $\mid v(0) - v(1) \mid = 5$.

In the next four steps we interchange two pairs of edges as follows to see that $1, 3, 7, 11 \in OEBI(\mu(P_n))$

$$u_1v_2 \text{ and } v_1v_0,\ u_2v_3 \text{ and } v_2v_0.$$
$$u_3v_2 \text{ and } v_3v_0,\ u_3v_4 \text{ and } v_4v_0.$$
$$u_4v_5 \text{ and } v_5v_0,\ u_5v_4 \text{ and } v_6v_0.$$
$$u_5v_6 \text{ and } v_7v_0,\ u_6v_7 \text{ and } v_8v_0.$$

Now we interchange $u_{2\lfloor\frac{q-1}{2}\rfloor+7}\ v_{2\lceil\frac{q-1}{2}\rceil+6}$ and $v_{2q+7}\ v_0$, $u_{2q+6}\ v_{2q+7}$ and $v_{2q+8}\ v_0$ for $1 \leq q \leq k-5$ to obtain $\mid v(0) - v(1) \mid = 4q+11$. Finally by interchanging the labels of the edges $u_{2\lfloor\frac{k-5}{2}\rfloor+7}\ v_{2\lceil\frac{k-5}{2}\rceil+6}$ and $u_{2k-2}\ u_{2k-1}$ we get $\mid v(0) - v(1) \mid = 4k-5$ and $u_{2\lfloor\frac{k-4}{2}\rfloor+7}\ v_{2\lceil\frac{k-4}{2}\rceil+6}$ and $u_{2k-1}\ v_0$ we get $\mid v(0) - v(1) \mid = 4k-1$.

Proof of the second case follows similarly. Thus

$$OEBI(\mu(P_n)) = \{1, 3, 5, \cdots, 2n+1\}. \qquad \square$$

Theorem 2.3 *If* $n \equiv 1(mod\ \ 2)$ *i.e,* $n = 2k+1(k \in N)$, *then* $OEBI(\mu(P_n)) = \{1, 3, 5, \ldots, 2n+1\}$.

Proof Let f be an edge-friendly labeling on $\mu(P_n)$. Since the graph contains $2n+1 = 4k+3$ vertices, $4n-3 = 8k+1$ edges, we have two possibilities: i) $e(0) = 4k+1$, $e(1) = 4k$ ii) $e(0) = 4k$, $e(1) = 4k+1$. Now we consider the first case namely $e(0) = 4k+1$ and $e(1) = 4k$. Denote the vertices of $\mu(P_n)$ as in the Figure 1. Now we label the edges $u_{2q-1}v_{2q}$, $u_{2q+1}v_{2q}$ for $1 \leq q \leq k$ and $u_q u_{q+1}$ for $1 \leq q \leq 2k$ by 1 and label the remaining edges by 0. Then it is easy to observe that $v(0) = 4k+3$ and there is no 1-vertex in the graph. Thus $| v(1) - v(0) |= 4k+3 = 2n+1 = max\{OEBI(\mu(P_n))\}$.

Now we interchange the labels of the edges to get the remaining 1-edge balance index numbers. By interchanging the labels of edges $u_{2q}u_{2q+1}$ and $u_{2q}v_{2q+1}$ for $1 \leq q \leq k$ we get $| v(0)-v(1) |= 4k+3-4q$. Further interchanging $u_{2k}v_{2k+1}$ and $v_{2k+1}v_0$ we get $| v(0)-v(1) |= 1$.

In the next four steps we interchange two pairs of edges as follows to see that $5, 9, 13.17 \in OEBI(\mu(P_n))$

$$u_1v_2 \text{ and } v_1v_0,\ u_2v_3 \text{ and } v_2v_0.$$
$$u_3v_2 \text{ and } v_3v_0,\ u_3v_4 \text{ and } v_4v_0.$$
$$u_4v_5 \text{ and } v_5v_0,\ u_5v_4 \text{ and } v_6v_0.$$
$$u_5v_6 \text{ and } v_7v_0,\ u_6v_7 \text{ and } v_8v_0.$$

And finally by interchanging the labels of edges $u_{2\lfloor \frac{q-1}{2} \rfloor+7}\ v_{2\lceil \frac{q-1}{2} \rceil+6}$ and $v_{2q+7}\ v_0$, $u_{2q+6}\ v_{2q+7}$ and $v_{2q+8}\ v_0$ for $1 \leq q \leq k-4$, we Obtain $| v(0) - v(1) |= 4q + 17$.

Proof of the second case follows similarly. Thus

$$OEBI(\mu(P_n)) = \{1, 3, 5, \ldots, 2n+1\}. \qquad \square$$

§3. The 1-Edge Balance Index Set of Wheel

In this section we consider the wheel, denoted by W_n which consists of n vertices and $2n-2$ edges. To determine the $OEBI(W_n)$ we consider four cases, namely,

$$n \equiv 0(mod\ \ 4), \qquad n \equiv 1(mod\ \ 4),$$
$$n \equiv 2(mod\ \ 4), \qquad n \equiv 3(mod\ \ 4).$$

Theorem 3.1 *If $n \equiv 0(mod\ \ 4)$ i.e, $n = 4k(k \in N)$, then $OEBI(W_n) = \{0, 4, 8, \ldots, n\}$.*

Proof Let f be an edge-friendly labeling on W_n. Since the graph contains $n = 4k$ vertices, $2n-2 = 8k-2$ edges, we must have $e(0) = e(1) = 4k-1$. Denote the vertices on the rim of the wheel by $v_0, v_1, v_2, \cdots, v_{4k-1}$ and denote the center by v_0. Now we label the edges $v_q v_{q+1}$ for $1 \leq q \leq 4k-2$ and $v_{4k-1}v_1$ by 1 and label the remaining edges by 0. Then it is easy to observe that $v(0) = 4k$ and there is no 1-vertex in the graph. Thus $| v(1) - v(0) |= 4k = n = max\{OEBI(W_n)\}$.

Now we interchange the labels of the edges to get the remaining 1-edge balance index numbers. By interchanging the labels of edges $v_{2q-1}v_{2q}$ and $v_{2q-1}v_0$, $v_{2q}v_{2q+1}$ and $v_{2q}v_0$ for $1 \leq q \leq k$ we get $| v(0) - v(1) |= 4k - 4q$. Thus $0, 4, 8, \cdots, n$ are elements of $OEBI(W_n)$.

Let $a_i = card\{v \in V \mid$ number of 1-edges incident on v is equal to $i\}$, $i = 1, 2, 3, \ldots, 4k-1$. Then we have

$$\sum_{i=1}^{4k-1} ia_i = a_1 + 2a_2 + 3a_3 +, \ldots, +(4k-1)a_{4k-1} = 8k-2$$

implies that $a_1 + 3a_3 + 5a_5 +, \ldots, +(4k-1)a_{4k-1}$ is even, which is possible if and only if, $a_1 + a_3 + a_5 +, \ldots, +a_{4k-1}$ is even, that is, the number of 1-vertices is even and hence the number of 0-vertices is also even. Therefore, the numbers $2, 6, 10, \ldots, n-2$ are not elements of $OEBI(W_n)$. □

Theorem 3.2 *If* $n \equiv 1(mod\ \ 4)$ *i.e,* $n = 4k+1 (k \in N)$, *then* $OEBI(W_n) = \{1, 3, 5, \ldots, n\}$.

Proof Let f be an edge-friendly labeling on W_n. Since the graph contains $n = 4k+1$ vertices, $2n-2 = 8k$ edges, we must have $e(0) = e(1) = 4k$. Denote the vertices on the rim of the wheel by $v_0, v_1, v_2, \cdots, v_{4k}$ and denote the center by v_0. Now we label the edges $v_q v_{q+1}$ for $1 \leq q \leq 4k-1$ and $v_{4k}v_1$ by 1 and label the remaining edges by 0. Then it is easy to observe that $v(0) = 4k+1$ and there is no 1-vertex in the graph. Thus $\mid v(1) - v(0) \mid = 4k+1 = n = max\{OEBI(W_n)\}$.

Now we interchange the labels of the edges to get the remaining 1-edge balance index numbers. By interchanging the labels of edges $v_{2q-1}v_{2q}$ and $v_{2q-1}v_0$, $v_{2q}v_{2q+1}$ and $v_{2q}v_0$ for $1 \leq q \leq 2k-1$, we get $\mid v(0) - v(1) \mid = \mid 4k+1-4q \mid$ and by interchanging the labels of edges $v_{4k-1}v_{4k}$ and $v_{4k-1}v_0$, $v_{4k}v_1$ and $v_{4k}v_0$, we get $\mid v(0) - v(1) \mid = 4k-1$. Thus

$$OEBI(W_n) = \{1, 3, 5, \ldots, n\}.$$ □

Similarly one can prove the following results.

Theorem 3.3 *If* $n \equiv 2(mod\ \ 4)$ *i.e,* $n = 4k+2 (k \in N)$, *then* $OEBI(W_n) = \{2, 6, 10, \ldots, n\}$.

Theorem 3.4 *If* $n \equiv 3(mod\ \ 4)$ *i.e,* $n = 4k+3 (k \in N)$, *then* $OEBI(W_n) = \{1, 3, 5, \ldots, n\}$.

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