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Smarandache mukti-squares[1]

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Abstract In [4] we have introduced Smarandache quasigroups which are Smarandache non-associative structures. A quasigroup is a groupoid whose composition table is a Latin square. There are squares in the Latin squares which seem to be of importance to study the structure of Latin Squares. We consider the particular type of squares properly contained in the Latin squares which themselves contain a Latin square. Such Latin squares are termed as Smarandache Mukti-Squares or SMS. Extension of some SMS to Latin squares is also considered.

§1. Introduction

Latin squares were first studied by Euler near the end of eighteenth century. A Latin square of order n is an n by n array containing symbols from some alphabet of size n, arranged so that each symbol appears exactly once in each row and exactly once in each column. Orthogonal Latin squares play an important role in the development of the theory of Latin squares. The best introduction of Latin Squares is in Bose and Manvel [1]. Today Latin squares have wide applications varying from 'Experimental Designs' in Agriculture to Cryptography and Computer science. There are some typical squares properly contained in some Latin squares. These squares themselves contain a Latin square. We have termed such squares as Smarandache Mukti-Squares. In this paper, we are initiating the study of such squares. We prove some properties and some important results.

Definition 1.1. An $n \times n$ array containing symbols from some alphabet of size m with $m \geq n$ is called a square of order n.

Definition 1.2. A Latin square of order n is an n by n array containing symbols from some alphabet of size n, arranged so that each symbol appears exactly once in each row and exactly once in each column.

Definition 1.3. If a Latin square L contains a Latin square S properly, then S is called a sub-Latin square.

Definition 1.4. A square in which ;

1. No element in the first row is repeated.
2. No element in the first column is repeated.

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3. Elements in first row and first column are same, is called a Mukti-Square.

Example 1.1. The following are Mukti-Squares of order 3 and 4 with alphabets $\{0,\ 1,\ 2,\ 3,\ 4\}$ and $\{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7\}$ respectively.

$$\begin{array}{ccc} 3 & 1 & 0 \\ 0 & 3 & 4 \\ 1 & 2 & 3 \end{array}$$

and

$$\begin{array}{cccc} 4 & 5 & 6 & 7 \\ 5 & 3 & 1 & 2 \\ 6 & 1 & 5 & 3 \\ 7 & 2 & 3 & 1 \end{array}$$

Definition 1.5. If a square contains a Latin Square properly then the square is called a Smarandache Mukti-Square or SMS.

Example 1.2. The following is a Smarandache Mukti-Square of order 4.

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 3 \\ 3 & 2 & 3 & 1 \\ 4 & 3 & 1 & 2 \end{array}$$

Clearly Mukti-Square on $\{1, 2, 3, 4\}$ contains a Latin square

$$\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{array}$$

on $\{1, 2, 3\}$.

Remark.

1. Any Latin square can be rotated about the axis through its center perpendicular to its plane.

2. The angles of rotation are $0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}$.

Theorem 1.1. Order of an SMS is greater than or equal to 3.

Proof. Follows from the definition of SMS.

§2. Orthogonal

Smarandache Mukti-squares Two Smarandache Mukti-Squares are said to be orthogonal if the Latin squares contained in them are orthogonal. The following SMSs are orthogonal to each other.

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 3 \\ 3 & 2 & 3 & 1 \\ 4 & 3 & 1 & 2 \end{matrix}$$

is orthogonal to

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 3 \\ 4 & 3 & 1 & 2 \\ 3 & 2 & 3 & 1 \end{matrix}$$

§3. Latin squares which contain SMS

Theorem 3.1. If a Latin square has no subLatin square properly contained in it then it has no SMS.

Proof. Follows from the definition of SMS.

Example 3.1. Consider the Latin squares;

$$\begin{matrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{matrix}$$

$$\longrightarrow$$

$$\begin{matrix} 3 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \\ 0 & 1 & 3 & 2 \\ 1 & 0 & 2 & 3 \end{matrix}$$

$$\longrightarrow$$

$$\begin{matrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{matrix}$$

$$\longrightarrow$$

3	2	0	1
2	3	1	0
1	0	3	2
0	1	2	3

Note that each rotation of the Latin square through 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ yields a new SMS in the Latin square.

In this case we say that the given Latin square is Fully SMS- Symmetric.

Theorem 3.2. If a Latin square of order 4 is fully SMS-symmetric then its ortogonal Latin square can not be fully SMS-symmetric.

Proof. Consider a fully SMS-symmetric Latin square of order 4 as:

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

It can be practically verified that the another Latin square which is fully SMS-symmetric is not orthogonal to it.

There do exist Latin squares which contain no SMS. The example follows:

1	2	3	4
4	1	2	3
3	4	1	2
2	3	4	1

Theorem 3.3. A Latin square of order 3 does not posses an SMS.

Proof. Follows from the definition of SMS.

Theorem 3.4. A Latin square may have more than one SMS. They may be of different orders.

Proof. We prove the theorem by giving an example:

7	3	2	1	6	5	4
3	6	1	2	7	4	5
2	1	5	3	4	7	6
1	2	3	4	5	6	7
6	7	4	5	3	1	2
5	4	7	6	1	2	3
4	5	6	7	2	3	1

This is a partially SMS-symmetric Latin square. We can see the SMS on $\{4, 5, 6, 7\}$. Other SMSs can be seen by rotating the Latin square. There are two SMS on $\{4, 6, 7\}$ which are identical.

§4. Extension of Smarandache Mukti-Squares

In this section we have tried to find out which SMSs can be extended to form a Latin square.

Theorem 4.1. If a Latin square of order 2 is extended to form a Latin square then the minimum order of such a Latin square will be 4 and it must contain an SMS.

Proof. Actual extension of such a Latin square will verify the theorem. Consider an SMS;

$$\begin{matrix} d & a \\ a & d \end{matrix}$$

This can be extended to a Latin square as;

$$\begin{matrix} a & d & c & b \\ d & a & b & c \\ c & b & d & a \\ b & c & a & d \end{matrix}$$

Theorem 4.2. An SMS of order three can be extended to a Latin square of order 4 if the Latin square contained in the SMS has one element outside the alphabet of the SMS.

Proof. Again we prove the theorem by constructing an example. Consider an SMS;

$$\begin{matrix} a & b & c \\ b & d & a \\ c & a & d \end{matrix}$$

This can be extended to a Latin square as

$$\begin{matrix} a & d & c & b \\ d & a & b & c \\ c & b & d & a \\ b & c & a & d \end{matrix}$$

Theorem 4.3. It is not possible to extend an SMS of order 3 to a Latin square if the Latin square contained in the SMS has two elements outside the alphabet of the SMS. It has been practically tried out but could not construct the Latin square.

Theorem 4.4. An SMS of order 4 can be extended to a Latin square of order 7 if the Latin square contained in the SMS has order 3 and contains all the elements outside the alphabet of the SMS.

Proof. We construct an example. Consider an SMS as

$$\begin{matrix} 4 & 5 & 6 & 7 \\ 5 & 3 & 1 & 2 \\ 6 & 1 & 2 & 3 \\ 7 & 2 & 3 & 1 \end{matrix}$$

Then this SMS can be extended to

$$\begin{array}{ccccccc}
7 & 3 & 2 & 1 & 6 & 5 & 4 \\
3 & 6 & 1 & 2 & 7 & 4 & 5 \\
2 & 1 & 5 & 3 & 4 & 7 & 6 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
6 & 7 & 4 & 5 & 3 & 1 & 2 \\
5 & 4 & 7 & 6 & 1 & 2 & 3 \\
4 & 5 & 6 & 7 & 2 & 3 & 1
\end{array}$$

We report some of the observations in this study.

1. Every SMS can not be extended to form a Latin square.
2. Order of the SMS does not divide the order of the Latin square containing it.
3. Number of SMS do not change if the Latin square is rotated through $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

§5. Some open problems

1. Can we extend an SMS of order 3 when all the elements of the Latin square contained in the SMS are outside the alphabet of the SMS?

2. Can we extend an SMS of order 4 when one element of the Latin square contained in the SMS is outside the alphabet of the SMS?

3. Can we extend an SMS of order 4 when two element of the Latin square contained in the SMS are outside the alphabet of the SMS?

References

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