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On the square-free number sequence

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Abstract The main purpose of this paper is to study the number of the square-free number sequence, and give two interesting asymptotic formulas for it. At last, give another asymptotic formula and a corollary.
Keywords Square-free number sequence; Asymptotic formula.

§1. Introduction

A number is called a square-free number if its digits don't contain the numbers: 0, 1, 4, 9. Let $\mathcal{A}$ denote the set of all square-free numbers. In reference [1], Professor F. Smarandache asked us to study the properties of the square-free number sequence. About this problem, it seems that none had studied it, at least we have not seen such a paper before. In this paper, we use the elementary method to study the number of the square-free number sequence, and obtain two interesting asymptotic formulas for it. That is, let $S(x) = \sum_{n\leq x, n\in\mathcal{A}} 1$, we shall prove the followings:

Theorem 1. For any real number $x \geq 1$, we have the asymptotic formula

$$\ln S(x) = \frac{\ln 6}{\ln 10} \times \ln x + O(1).$$

Theorem 2. For any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n\leq x, n\in\mathcal{B}} 1 = x + O\left(x^{\frac{2\ln 2}{\ln 10}}\right),$$

where $\mathcal{B}$ denote the complementary set of those numbers whose all digits are square numbers.

Let $\mathcal{B}'$ denote the set of those numbers whose all digits are square numbers. Then we have the following:

Theorem 3. For any real number $x \geq 1$,we have the asymptotic formula

$$\sum_{n\leq x, n\in\mathcal{B}} \frac{1}{n} = \ln x + \gamma - C + O\left(x^{-\frac{\ln\frac{5}{2}}{\ln 10}}\right),$$

where C is a computable constant, γ denotes the Euler's constant.

Let $\mathcal{A}'$ denote the complementary set of $\mathcal{A}$, we have following:

Corollary. For any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n\le x, n\in\mathcal{A}'} \frac{1}{n} = \ln x + \gamma - D + O\left(x^{-\frac{\ln\frac{5}{3}}{\ln 10}}\right),$$

where D is a computable constant.

§2. Proof of Theorems

In this section, we shall complete the proof of Theorems. First we need the following one simple lemma.

Lemma. For any real number $x \ge 1$, we have the asymptotic formula

$$\sum_{n\le x, n\in\mathcal{B}'} \frac{1}{n} = C + O\left(x^{-\frac{\ln\frac{5}{2}}{\ln 10}}\right).$$

Proof. In the interval $[10^{r-1}, 10^r), (r \ge 2)$, there are $3\times 4^{r-1}$ numbers belong to $\mathcal{B}'$, and every number's reciprocal isn't greater than $\frac{1}{10^{r-1}}$; when $r = 1$, there are 4 numbers belong to $\mathcal{B}'$ and their reciprocals aren't greater than 1. Then we have

$$\sum_{n\in\mathcal{B}'} \frac{1}{n} < 3 + \sum_{r=1}^{\infty} 3\times\frac{4^r}{10^r},$$

then $\sum_{n\in\mathcal{B}'} 1$ is convergent to a constant C. So

$$\sum_{n\le x, n\in\mathcal{B}'} \frac{1}{n} = \sum_{n\in\mathcal{B}'} \frac{1}{n} - \sum_{n>x, n\in\mathcal{B}'} \frac{1}{n} = C + O\left(\sum_{r=k}^{\infty} \frac{3\times 4^r}{10^r}\right) = C + O\left(x^{-\frac{\ln\frac{5}{2}}{\ln 10}}\right).$$

Now we come to prove Theorem 1. First for any real number $x \ge 1$, there exists a non-negative integer k, such that $10^k \le x < 10^{k+1} (k \ge 1)$ therefore $k \le \log x < k+1$. If a number belongs to $\mathcal{A}$, then its digits only contain these six numbers: 2, 3, 5, 6, 7, 8.

So in the interval $[10^{r-1}, 10^r) (r \ge 1)$, there are 6^r numbers belong to $\mathcal{A}$. Then we have

$$\sum_{n\le x, n\in\mathcal{A}} 1 \le \sum_{r=1}^{k+1} 6^r = \frac{6}{5}\times(6^{k+1}-1) < \frac{6^{k+2}}{5} < \frac{6^2}{5}\times x^{\frac{\ln 6}{\ln 10}},$$

and

$$\sum_{n\le x, n\in\mathcal{A}} 1 \ge \sum_{r=1}^{k} 6^r = \frac{6}{5}\times(6^k-1) \ge 6^k > \frac{1}{6}\times x^{\frac{\ln 6}{\ln 10}}.$$

So we have

$$\frac{1}{6}\times x^{\frac{\ln 6}{\ln 10}} < \sum_{n\le x, n\in\mathcal{A}} 1 < \frac{6^2}{5}\times x^{\frac{\ln 6}{\ln 10}}.$$

Taking the logarithm computation on both sides of the above, we get

$$\ln(x^{\frac{\ln 6}{\ln 10}}) + (-\ln 6) < \sum_{n\le x, n\in\mathcal{A}} 1 < \ln\left(x^{\frac{\ln 6}{\ln 10}}\right) + (2\times\ln 6 - \ln 5).$$

So

$$\ln S(x) = \ln\left(\sum_{n\le x, n\in\mathcal{A}} 1\right) = \ln\left(x^{\frac{\ln 6}{\ln 10}}\right) + O(1) = \frac{\ln 6}{\ln 10}\times\ln x + O(1).$$

This proves the Theorem 1.

Now we prove Theorem 2. It is clear that if a number doesn't belong to $\mathcal{B}$, then all of its digits are square numbers. So in the interval $[10^{r-1}, 10^r), (r\ge 2)$, there are $3\times 4^{r-1}$ numbers don't belong to $\mathcal{B}$; when $r=1$, there are 4 numbers don't belong to $\mathcal{B}$. Then we have

$$\begin{aligned}
&\sum_{n\le x, n\in\mathcal{B}} 1 = \sum_{n\le x} 1 - \sum_{n\le x, n\in\mathcal{B}'} 1\\
&= x + O\left(4 + 3\times 4 + 3\times 4^2 + \cdots + 3\times 4^k\right)\\
&= x + O\left(4^{k+1}\right) = x + O\left(x^{\frac{2\times\ln 2}{\ln 10}}\right).
\end{aligned}$$

This completes the proof of the Theorem 2. Now we prove the Theorem 3. In reference [2], we know the asymptotic formula:

$$\sum_{n\le x}\frac{1}{n} = \ln x + \gamma + O\left(\frac{1}{x}\right),$$

where γ is the Euler's constant.

Then from this asymptotic formula and the above Lemma, we have

$$\begin{aligned}
&\sum_{n\le x, n\in\mathcal{B}}\frac{1}{n} = \sum_{n\le x}\frac{1}{n} - \sum_{n\le x, n\in\mathcal{B}'}\frac{1}{n}\\
&= \ln x + \gamma + O\left(\frac{1}{x}\right) - C + O\left(x^{-\frac{\ln\frac{5}{2}}{\ln 10}}\right)\\
&= \ln x + \gamma - C + O\left(x^{-\frac{\ln\frac{5}{2}}{\ln 10}}\right).
\end{aligned}$$

This completes the proof of the Theorem 3. Now the Corollary immediately follows from the Lemma and Theorem 3.

Reference

[1] F.Smarandache, Only problems, Not Solutions, Xiquan Publ. House, Chicago, 1993.

[2] Tom M.Apostol, Introduction to Analytic Number Theory, Springer-Verlag, New York, 1976.