Four problems related to the Pseudo-Smarandache-Squarefree function

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Abstract For any positive integer n, the Pseudo-Smarandache-Squarefree function $Zw(n)$ is defined as the smallest positive integer m such that m^n is divisible by n. That is, $Zw(n) = \min\{m : m \in N, \ n \mid m^n\}$. In reference [2], Felice Russo proposed many problems and conjectures related to the Pseudo-Smarandache-Squarefree function $Zw(n)$. The main purpose of this paper is using the elementary methods to study several problems in [2], and four of them are solved.

Keywords Pseudo-Smarandache-Squarefree function, equation, positive integer solution.

§1. Introduction and results

For any positive integer n, the famous Pseudo-Smarandache-Squarefree function $Zw(n)$ is defined as the smallest positive integer m such that m^n is divisible by n. That is,

$$Zw(n) = \min\{m : m \in N, \ n \mid m^n\}.$$

It is easy to see that if $n > 1$, then

$$Zw(n) = \prod_{p \mid n} p,$$

where $\prod_{p \mid n}$ denotes the product over all different prime divisors of n (see reference [1]). From this formula, we can easily get the value of $Zw(n)$. For example, $Zw(1) = 1$, $Zw(2) = 2$, $Zw(3) = 3$, $Zw(4) = 2$, $Zw(5) = 5$, $Zw(6) = 6$, $Zw(7) = 7$, $Zw(8) = 2$, $Zw(9) = 3$, $Zw(10) = 10$, $\cdots$. In fact if n is a square-free number, then $Zw(n) = n$. In reference [2], Felice Russo studied the properties of $Zw(n)$, and proposed the following four problems:

Problem 1. Find all the values of n such that $Zw(n) = Zw(n+1) \cdot Zw(n+2)$.

Problem 2. Solve the equation $Zw(n) \cdot Zw(n+1) = Zw(n+2)$.

Problem 3. Solve the equation $Zw(n) \cdot Zw(n+1) = Zw(n+2) \cdot Zw(n+3)$.

Problem 4. Find all the values of n such that $S(n) = Zw(n)$, where $S(n)$ is the Smarandache function.

The main purpose of this paper is using the elementary methods to study these four problems, and solved them completely. That is, we shall prove the following conclusions:

Theorem 1. The following three equations have no positive integer solution.

$$Zw(n) = Zw(n+1) \cdot Zw(n+2); \tag{1}$$

$$Zw(n) \cdot Zw(n+1) = Zw(n+2); \tag{2}$$

$$Zw(n) \cdot Zw(n+1) = Zw(n+2) \cdot Zw(n+3). \tag{3}$$

Theorem 2. There exist infinite positive integers n such that the equation $S(n) = Zw(n)$, where $S(n)$ is the Smarandache function defined by $S(n) = \min\{k: \ k \in N, \ n \mid k!\}$.

§2. Proof of the theorems

In this section, we shall complete the proof of our theorems. First we prove that the equation $Zw(n) = Zw(n+1) \cdot Zw(n+2)$ has no positive integer solution. It is clear that $n = 1$ is not a solution of this equation. In fact if $n = 1$, then

$$1 = Zw(1) \neq 2 \cdot 3 = Zw(2) \cdot Zw(3).$$

If $n > 1$, suppose that the equation (1) has one positive integer solution $n = n_0$, then

$$Zw(n_0) = Zw(n_0 + 1) \cdot Zw(n_0 + 2).$$

For any prime divisor p of $n = n_0$, it is clear that $p \mid Zw(n_0)$.

From $\qquad\qquad Zw(n_0) = Zw(n_0 + 1) \cdot Zw(n_0 + 2)$

we deduce that $\qquad p \mid Zw(n_0 + 1) \cdot Zw(n_0 + 2).$

That is, $\qquad\qquad p \mid Zw(n_0 + 1)$ or $p \mid Zw(n_0 + 2).$

(a) If $p \mid Zw(n_0 + 1)$, then $p \mid (n_0 + 1)$, combining $p \mid n_0$ and $p \mid (n_0 + 1)$ we get $p \mid n_0 + 1 - n_0 = 1$, this is a contradiction.

(b) If $p \mid Zw(n_0 + 2)$, then $p \mid (n_0 + 2)$, combining $p \mid n_0$ and $p \mid (n_0 + 2)$ we deduce that $p \mid n_0 + 2 - n_0 = 2$, then we get $n_0 = p = 2$.

From equation (1) we have

$$2 = Zw(2) \neq 3 \cdot 2 = Zw(3) \cdot Zw(4),$$

It is not possible. So the equation (1) has no positive integer solution.

Using the similar method as in proving problem 1, we find that the equation (2) and equation (3) also have no positive integer solution. This completes the proof of Theorem 1.

In order to prove Theorem 2, we need some important properties of the Smarandache function $S(n)$, which we mentioned as the following:

Lemma 1. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime powers factorization of n, then

$$S(n) = \max_{1 \leq i \leq k}\{S(p_i^{\alpha_i})\}.$$

Proof. See reference [3].

Lemma 2. If p be a prime, then $S(p^k) \leq kp$; If $k < p$, then $S(p^k) = kp$, where k is any positive integer.

Proof.   See reference [4].

Now we use these two simple lemmas to prove our Theorem 2. It's clear that all prime p are the solutions of the equation $S(n) = Zw(n)$. So there are infinite positive integers n satisfying the equation $S(n) = Zw(n)$.

Now we construct infinite composite numbers n satisfying the equation $S(n) = Zw(n)$, let $n = p_1 \cdot p_2 \cdots p_{k-1} \cdot p_k^{\alpha_k}$, where p_i are the different primes, and $p_k > \alpha_k = p_1 p_2 \cdots \cdot p_{k-1}$. This time, from the definition of $S(n)$ and $Zw(n)$ we have $S(n) = p_1 \cdot p_2 \cdots p_{k-1} \cdot p_k$ and $Zw(n) = p_1 \cdot p_2 \cdots p_{k-1} \cdot p_k$. So all composite numbers $n = p_1 \cdot p_2 \cdots p_{k-1} \cdot p_k^{\alpha_k}$ (where p_i are the different primes, and $p_k > \alpha_k = p_1 p_2 \cdots \cdot p_{k-1}$) satisfying the equation $S(n) = Zw(n)$.

Note that k be any positive integer and there are infinite primes, so there are infinite composite numbers n satisfying the equation $S(n) = Zw(n)$.

This complete the proof of Theorem 2.

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