Vol.7,No.1,2011

SCIENTIA MAGNA

An international journal

Edited by Department of Mathematics
Northwest University, P.R.China

Vol. 7, No. 1, 2011 **ISSN 1556-6706**

SCIENTIA MAGNA

An international journal

Edited by

Department of Mathematics
Northwest University
Xi'an, Shaanxi, P.R.China

Scientia Magna is published annually in 400-500 pages per volume and 1,000 copies.

It is also available in **microfilm format** and can be ordered (online too) from:

Books on Demand
ProQuest Information & Learning
300 North Zeeb Road
P.O. Box 1346
Ann Arbor, Michigan 48106-1346, USA
Tel.: 1-800-521-0600 (Customer Service)
URL: http://wwwlib.umi.com/bod/

Scientia Magna is a **referred journal**: reviewed, indexed, cited by the following journals: "Zentralblatt Für Mathematik" (Germany), "Referativnyi Zhurnal" and "Matematika" (Academia Nauk, Russia), "Mathematical Reviews" (USA), "Computing Review" (USA), Institute for Scientific Information (PA, USA), "Library of Congress Subject Headings" (USA).

Printed in the United States of America
Price: US$ 69.95

Information for Authors

Papers in electronic form are accepted. They can be e-mailed in Microsoft Word XP (or lower), WordPerfect 7.0 (or lower), LaTeX and PDF 6.0 or lower.

The submitted manuscripts may be in the format of remarks, conjectures, solved/unsolved or open new proposed problems, notes, articles, miscellaneous, etc. They must be original work and camera ready [typewritten/computerized, format: 8.5 x 11 inches (21,6 x 28 cm)]. They are not returned, hence we advise the authors to keep a copy.

The title of the paper should be writing with capital letters. The author's name has to apply in the middle of the line, near the title. References should be mentioned in the text by a number in square brackets and should be listed alphabetically. Current address followed by e-mail address should apply at the end of the paper, after the references.

The paper should have at the beginning an abstract, followed by the keywords.

All manuscripts are subject to anonymous review by three independent reviewers.

Every letter will be answered.

Each author will receive a free copy of the journal.

Contributing to Scientia Magna

Authors of papers in science (mathematics, physics, philosophy, psychology, sociology, linguistics) should submit manuscripts, by email, to the

Editor-in-Chief:

Or anyone of the members of

Editorial Board:

Contents

M. A. Gungor and M. Sarduvan : A note on dual quaternions and matrices of dual quaternions 1

J. Tian and W. He : Generalized Weyl's theorem for Class A operators 12

R. Poovazhaki and V. Swaminathan : Weakly convex domination in graphs 18

G. Ilango and R. Marudhachalam : New hybrid filtering techniques for removal of speckle noise from ultrasound medical images 38

R. S. Maragatham and A. V. Jeyakumar : Introduction of eigen values on relative character graphs 54

Nicuşor Minculete : A result about Young's inequality and several applications 61

S. Panayappan, etc. : Composition operators of k-paranormal operators 69

M. Bencze, etc. : Inequalities between the sides and angles of an acute triangle 74

Y. Shang : The Estrada index of random graphs 79

Tèmítọ́pẹ́ Gbọ́láhàn Jaíyéọlá : Smarandache isotopy of second Smarandache Bol loops 82

M. Mohamadhasani and M. Haveshki : Implicative filters in pocrims 94

A. N. Murugan and A. Nagarajan: Magic graphoidal on special type of unicyclic graphs 99

D. Saglam, etc. : Minimal translation lightlike hypersurfaces 107

S. Harmaitree and U. Leerawat : The generalized f-derivations of lattices 114

Scientia Magna
Vol. 7 (2011), No. 1, 1-11

A note on dual quaternions and matrices of dual quaternions

M. A. Gungor† and M. Sarduvan‡

Department of Mathematics, Sakarya University, Faculty of Arts and Science,
Sakarya 54187, Türkiye
E-mail: agungor@sakarya.edu.tr msarduvan@sakarya.edu.tr

Abstract In this paper, it is investigated eigenvalues and eigenvectors of the dual Hamilton operators. Moreover, it is examined a special type dual quaternion equation using these eigenvalues and eigenvectors. Finally, it is given the nth power of a dual quaternion.

Keywords Dual quaternion, dual matrix equation, normal matrix, eigenvalue, rank.

§1. Introduction and preliminaries

Quaternions were invented by Sir William Rowan Hamilton as an extension to the complex numbers. Until the middle of the 20th century, the practical use of quaternions was minimal in comparison with other methods. But, currently, this situation has changed. Today, quaternions play a significant role in several areas of the physical science; namely, in differential geometry, in analysis and synthesis of mechanism and machines, simulation of particle motion in molecular physics, and quaternionic formulation of equation of motion in theory of relativity. Moreover, quaternions are used especially in the area of computer vision, computer graphics, animation, and to solve optimization problems involving the estimation of rigid body transformations (see, for example, [1, 4, 6, 8, 19]).

Each element of the set

$$\mathbb{D} = \left\{ \tilde{a} = a + \varepsilon a^* \,:\, a, a^* \in \mathbb{R} \ \text{ and } \ \varepsilon \neq 0,\ \varepsilon^2 = 0 \right\} = \{ \tilde{a} = (a, a^*) \,:\, a, a^* \in \mathbb{R} \}$$

is called a dual number. A dual number $\tilde{a} = a + \varepsilon a^*$ can be expressed in the form $\tilde{a} = \mathrm{Re}\,(\tilde{a}) + \varepsilon \mathrm{Du}\,(\tilde{a})$, where $\mathrm{Re}\,(\tilde{a}) = a$ and $\mathrm{Du}\,(\tilde{a}) = a^*$. The conjugate of $\tilde{a} = a + \varepsilon a^*$ is defined as $\overline{\tilde{a}} = a - \varepsilon a^*$. Summation and multiplication of two dual numbers are defined as similar to the complex numbers. However, it will not be forgotten that $\varepsilon^2 = 0$. Thus, $\mathbb{D}$ is a commutative ring with a unit element [11]. Clifford introduced dual numbers to form bi-quaternions (called dual quaternions nowadays) for studying noneuclidean geometry [5]. First applications of dual numbers to mechanics was generalized by Kothelnikov [15] and Study [20] in their principle of transference. Recently, dual numbers have been applied to study the kinematics, dynamics, and calibration of open-chain robot manipulators. Moreover, dual numbers are useful for analytical treatment in kinematics and dynamics of spatial mechanisms (see, for example, [7, 16, 17, 18]).

Furthermore, each element of the set

$$\mathbb{C}_D = \left\{ \hat{z} = \tilde{a} + \tilde{b}i : \ \tilde{a}, \tilde{b} \in \mathbb{D} \text{ and } i^2 = -1 \right\}$$

is called a dual complex number. A dual complex number $\hat{z} = \tilde{a} + \tilde{b}i$ can be expressed in the form $\hat{z} = \mathrm{Du}\,(\hat{z}) + i\mathrm{Im}\,(\hat{z})$, where $\mathrm{Du}\,(\hat{z}) = \tilde{a}$ and $\mathrm{Im}\,(\hat{z}) = \tilde{b}$. The conjugate of $\hat{z} = \tilde{a} + \tilde{b}i$ is defined as $\bar{\hat{z}} = \tilde{a} - \tilde{b}i$. Summation and multiplication of any two dual complex numbers $\hat{z} = \tilde{a} + \tilde{b}i$ and $\hat{w} = \tilde{c} + \tilde{d}i$ are defined in the following ways,

$$\hat{z} + \hat{w} = (\tilde{a} + \tilde{c}) + \left(\tilde{b} + \tilde{d}\right) i$$

and

$$\hat{z}.\hat{w} = \left(\tilde{a} + \tilde{b}i\right)\left(\tilde{c} + \tilde{d}i\right) = \left(\tilde{a}\tilde{c} - \tilde{b}\tilde{d}\right) + \left(\tilde{a}\tilde{d} + \tilde{b}\tilde{c}\right) i.$$

The dual complex numbers defined as the dual quaternions were considered as a generalization of complex numbers by Ata and Yayli [3].

In this paper, it is assumed that the reader is already familiar regular quaternions, otherwise (see, for example, [10, 13, 22, 24]). The matrix representation of spatial displacements of rigid bodies has an important role in kinematics and the mathematical description of displacements. Veldkamp and Yang-Freudenstein investigated the use of dual numbers, dual numbers matrix, and dual quaternions in instantaneous spatial kinematics in [21] and [23], respectively. Agrawal [2] worked on Hamilton operators and dual quaternions in kinematics. In [2], the algebra of dual quaternions is developed by using two Hamilton operators. Properties of these operators are used to find some mathematical expressions for screw motion.

Each element of the set

$$\mathbb{H}_D = \left\{ \tilde{Q} = \tilde{a}_0 + \tilde{a}_1 i + \tilde{a}_2 j + \tilde{a}_3 k \ : \ \tilde{a}_0, \tilde{a}_1, \tilde{a}_2, \tilde{a}_3 \in \mathbb{D} \right\}$$

is called a dual quaternion, where i, j, and k are special elements of $\mathbb{H}_D$ satisfying

$$i^2 = j^2 = k^2 = ijk = -1$$

and

$$ij = k = -ji\,, \quad jk = i = -kj\,, \quad ki = j = -ik.$$

A dual quaternion $\tilde{Q} = \tilde{a}_0 + \tilde{a}_1 i + \tilde{a}_2 j + \tilde{a}_3 k$ is pieced into two parts with real part $\mathcal{R}\left(\tilde{Q}\right) := \tilde{a}_0$ and imaginary part $\Im\left(\tilde{Q}\right) := \tilde{a}_1 i + \tilde{a}_2 j + \tilde{a}_3 k$. Summation and multiplication of any two dual quaternions $\tilde{Q} = \tilde{a}_0 + \tilde{a}_1 i + \tilde{a}_2 j + \tilde{a}_3 k$ and $\tilde{P} = \tilde{b}_0 + \tilde{b}_1 i + \tilde{b}_2 j + \tilde{b}_3 k$ are defined as

$$\tilde{Q} + \tilde{P} = \left(\tilde{a}_0 + \tilde{b}_0\right) + \left(\tilde{a}_1 + \tilde{b}_1\right) i + \left(\tilde{a}_2 + \tilde{b}_2\right) j + \left(\tilde{a}_3 + \tilde{b}_3\right) k$$

and

$$\begin{aligned} \tilde{Q}\tilde{P} \quad = \quad & \left(\tilde{a}_0\tilde{b}_0 - \tilde{a}_1\tilde{b}_1 - \tilde{a}_2\tilde{b}_2 - \tilde{a}_3\tilde{b}_3\right) + \left(\tilde{a}_1\tilde{b}_0 + \tilde{a}_0\tilde{b}_1 - \tilde{a}_3\tilde{b}_2 + \tilde{a}_2\tilde{b}_3\right) i \\ & + \left(\tilde{a}_2\tilde{b}_0 + \tilde{a}_3\tilde{b}_1 + \tilde{a}_0\tilde{b}_2 - \tilde{a}_1\tilde{b}_3\right) j + \left(\tilde{a}_3\tilde{b}_0 - \tilde{a}_2\tilde{b}_1 + \tilde{a}_1\tilde{b}_2 + \tilde{a}_0\tilde{b}_3\right) k. \end{aligned}$$

Thus, with this multiplication operator, $\mathbb{H}_D$ is called dual quaternion algebra [12]. The conjugate of $\tilde{Q} = \tilde{a}_0 + \tilde{a}_1 i + \tilde{a}_2 j + \tilde{a}_3 k$ is defined as $\overline{\tilde{Q}} = \tilde{a}_0 - \tilde{a}_1 i - \tilde{a}_2 j - \tilde{a}_3 k$. For any two quaternions $\tilde{Q}$ and $\tilde{P}$ we have $\overline{\tilde{Q}\tilde{P}} = \overline{\tilde{P}}\,\overline{\tilde{Q}}$.

In this paper, it is employed a matrix oriented approach to the dual quaternions topic, by representing dual quaternions as four-dimensional vectors and the multiplication of dual quaternions as matrix-by-vector product, since this approach might be easier to grasp than the traditional axiomatic point of view.

The purpose of this paper is mainly three fold: first to investigate eigenvalues and eigenvectors of the dual Hamilton Operators, second to examine a special type dual quaternion equation using these eigenvalues and eigenvectors, and finally to give the nth power of a dual quaternion.

§2. Basic properties of the dual fundamental matrices

It is nearby to identify a dual quaternion $\tilde{Q} \in \mathbb{H}_D$ with a dual vector $\tilde{\mathbf{q}} \in \mathbb{D}^4$. It will be denoted such an identification by the symbol " $\cong$ " i.e.,

$$\tilde{Q} = \tilde{a}_0 + \tilde{a}_1 i + \tilde{a}_2 j + \tilde{a}_3 k \cong \tilde{\mathbf{q}} = (\tilde{a}_0, \tilde{a}_1, \tilde{a}_2, \tilde{a}_3)',$$

where the prime superscript stands for the transpose. Then addition in $\mathbb{H}_D$ becomes the componentwise addition of vectors in $\mathbb{D}^4$, and multiplication can be represented by an ordinary matrix-by-vector product

$$\tilde{Q}\tilde{P} \cong \mathbf{L}_{\tilde{\mathbf{q}}}\tilde{\mathbf{p}} \ \text{ or } \ \tilde{P}\tilde{Q} \cong \mathbf{R}_{\tilde{\mathbf{q}}}\tilde{\mathbf{p}},$$

where the matrices $\mathbf{L}_{\tilde{\mathbf{q}}}$ and $\mathbf{R}_{\tilde{\mathbf{q}}}$ called Hamilton operators, are given by

$$\mathbf{L}_{\tilde{\mathbf{q}}} = \begin{pmatrix} \tilde{a}_0 & -\tilde{a}_1 & -\tilde{a}_2 & -\tilde{a}_3 \\ \tilde{a}_1 & \tilde{a}_0 & -\tilde{a}_3 & \tilde{a}_2 \\ \tilde{a}_2 & \tilde{a}_3 & \tilde{a}_0 & -\tilde{a}_1 \\ \tilde{a}_3 & -\tilde{a}_2 & \tilde{a}_1 & \tilde{a}_0 \end{pmatrix}, \quad \mathbf{R}_{\tilde{\mathbf{q}}} = \begin{pmatrix} \tilde{a}_0 & -\tilde{a}_1 & -\tilde{a}_2 & -\tilde{a}_3 \\ \tilde{a}_1 & \tilde{a}_0 & \tilde{a}_3 & -\tilde{a}_2 \\ \tilde{a}_2 & -\tilde{a}_3 & \tilde{a}_0 & \tilde{a}_1 \\ \tilde{a}_3 & \tilde{a}_2 & -\tilde{a}_1 & \tilde{a}_0 \end{pmatrix}. \tag{1}$$

Since these operators play a crucial role in the subsequent considerations, they will be called as the left and right fundamental matrices, respectively. It will be discussed their main features in this and next sections.

It can be written following identities as a direct consequence of the above fundamental matrices.

$$\mathbf{L}_1 = \mathbf{R}_1 = \mathbf{I}_4,$$

$$\mathbf{L}_i = \mathbf{E}_1,\ \mathbf{L}_j = \mathbf{E}_2,\ \mathbf{L}_k = \mathbf{E}_3,$$

$$\mathbf{R}_i = \mathbf{F}_1,\ \mathbf{R}_j = \mathbf{F}_2,\ \mathbf{R}_k = \mathbf{F}_3,$$

where $\mathbf{I}_4$ is a 4×4 identity matrix. Note that the properties of $\mathbf{E}_n$ and $\mathbf{F}_n$ $(n = 1, 2, 3)$ are identical to that of dual quaternionic units i, j, k. Since $\mathbf{L}_{\tilde{\mathbf{q}}}$ and $\mathbf{R}_{\tilde{\mathbf{q}}}$ are linear in their elements, it follows that

$$\mathbf{L}_{\tilde{\mathbf{q}}} = \tilde{a}_0\mathbf{L}_1 + \tilde{a}_1\mathbf{L}_i + \tilde{a}_2\mathbf{L}_j + \tilde{a}_3\mathbf{L}_k = \tilde{a}_0\mathbf{I}_4 + \tilde{a}_1\mathbf{E}_1 + \tilde{a}_2\mathbf{E}_2 + \tilde{a}_3\mathbf{E}_3 = \mathbf{L}_{\mathbf{q}} + \varepsilon\mathbf{L}_{\mathbf{q}^*}, \tag{2}$$

$$\mathbf{R}_{\tilde{\mathbf{q}}} = \tilde{a}_0\mathbf{R}_1 + \tilde{a}_1\mathbf{R}_i + \tilde{a}_2\mathbf{R}_j + \tilde{a}_3\mathbf{R}_k = \tilde{a}_0\mathbf{I}_4 + \tilde{a}_1\mathbf{F}_1 + \tilde{a}_2\mathbf{F}_2 + \tilde{a}_3\mathbf{F}_3 = \mathbf{R}_{\mathbf{q}} + \varepsilon\mathbf{R}_{\mathbf{q}^*}, \tag{3}$$

where $\mathbf{q} = (a_0, a_1, a_2, a_3)'$, $\mathbf{q}^* = (a_0^*, a_1^*, a_2^*, a_3^*)' \in \mathbb{R}^4$.

Using the definitions of the fundamental matrices, the multiplication of the two dual quaternions $\tilde{Q}$ and $\tilde{P}$ is given by

$$\tilde{\mathbf{r}} = \mathbf{L}_{\tilde{\mathbf{q}}}\tilde{\mathbf{p}} = \mathbf{R}_{\tilde{\mathbf{p}}}\tilde{\mathbf{q}}. \tag{4}$$

Real part, imaginary part, and conjugate of a dual quaternion $\tilde{Q}$ is shown as

$$\mathcal{R}\left(\tilde{Q}\right) \cong \tilde{a}_0\mathbf{e}_1, \qquad \mathbf{e}_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \Im\left(\tilde{Q}\right) \cong \tilde{\mathbf{q}}_* := \begin{pmatrix} 0 \\ \tilde{a}_1 \\ \tilde{a}_2 \\ \tilde{a}_3 \end{pmatrix},$$

and

$$\bar{\tilde{Q}} \cong \bar{\tilde{\mathbf{q}}} := \begin{pmatrix} \tilde{a}_0 \\ -\tilde{a}_1 \\ -\tilde{a}_2 \\ -\tilde{a}_3 \end{pmatrix} = \mathbf{C}\tilde{\mathbf{q}}, \qquad \mathbf{C} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

In the sequel, it will be represented a dual number $\tilde{a}$ by $\tilde{a}\mathbf{e}_1$ whenever appropriate.

In the following theorem, some properties associated with the dual fundamental matrices and some identities are presented.

Theorem 2.1. Let $\tilde{Q}$, $\tilde{P}$ be dual quaternions, $\tilde{\alpha}$, $\tilde{\beta}$ be dual numbers, and $\mathbf{L}$ and $\mathbf{R}$ be the dual fundamental matrices as defined in (1), then the following identities hold:

(i) $\tilde{Q} = \tilde{P} \Leftrightarrow \mathbf{L}_{\tilde{\mathbf{q}}} = \mathbf{L}_{\tilde{\mathbf{p}}} \Leftrightarrow \mathbf{R}_{\tilde{\mathbf{q}}} = \mathbf{R}_{\tilde{\mathbf{p}}}$.

(ii) $\mathbf{L}_{\tilde{\alpha}\tilde{\mathbf{q}}+\tilde{\beta}\tilde{\mathbf{p}}} = \tilde{\alpha}\mathbf{L}_{\tilde{\mathbf{q}}} + \tilde{\beta}\mathbf{L}_{\tilde{\mathbf{p}}}$, $\mathbf{R}_{\tilde{\alpha}\tilde{\mathbf{q}}+\tilde{\beta}\tilde{\mathbf{p}}} = \tilde{\alpha}\mathbf{R}_{\tilde{\mathbf{q}}} + \tilde{\beta}\mathbf{R}_{\tilde{\mathbf{p}}}$.

(iii) $\mathbf{L}_{\tilde{\mathbf{q}}}\mathbf{L'}_{\tilde{\mathbf{q}}} = \mathbf{L'}_{\tilde{\mathbf{q}}}\mathbf{L}_{\tilde{\mathbf{q}}}$, $\mathbf{R}_{\tilde{\mathbf{q}}}\mathbf{R'}_{\tilde{\mathbf{q}}} = \mathbf{R'}_{\tilde{\mathbf{q}}}\mathbf{R}_{\tilde{\mathbf{q}}}$, $\mathbf{L}_{\bar{\tilde{\mathbf{q}}}} = \mathbf{L'}_{\tilde{\mathbf{q}}}$, $\mathbf{R}_{\bar{\tilde{\mathbf{q}}}} = \mathbf{R'}_{\tilde{\mathbf{q}}}$.

(iv) $\det(\mathbf{L}_{\tilde{\mathbf{q}}}) = \det(\mathbf{R}_{\tilde{\mathbf{q}}}) = \|\tilde{\mathbf{q}}\|^4$, $\mathbf{L}_{\tilde{\mathbf{q}}}^{-1} = \frac{1}{\|\tilde{\mathbf{q}}\|^2}\mathbf{L'}_{\tilde{\mathbf{q}}}$, $\mathbf{R}_{\tilde{\mathbf{q}}}^{-1} = \frac{1}{\|\tilde{\mathbf{q}}\|^2}\mathbf{R'}_{\tilde{\mathbf{q}}}$, $\mathbf{0} \neq \tilde{\mathbf{q}} \in \mathbb{D}^4$ (where $\|\cdot\|$ denotes the Euclidean norm of a dual vector).

(v) $tr(\mathbf{L}_{\tilde{\mathbf{q}}}) = tr(\mathbf{R}_{\tilde{\mathbf{q}}}) = 4\tilde{a}_0$.

(vi) $\mathbf{R}_{\tilde{\mathbf{q}}} = \mathbf{C}\mathbf{L'}_{\tilde{\mathbf{q}}}\mathbf{C}$, $\mathbf{L}_{\tilde{\mathbf{q}}} = \mathbf{C}\mathbf{R'}_{\tilde{\mathbf{q}}}\mathbf{C}$, $\mathbf{C}^{-1} = \mathbf{C}' = \mathbf{C}$, $\mathbf{C}^2 = \mathbf{I}_4$.

(vii) $\tilde{Q}\bar{\tilde{Q}} = \left|\tilde{Q}\right|^2$, $\left|\tilde{Q}\tilde{P}\right|^2 = \left|\tilde{Q}\right|^2\left|\tilde{P}\right|^2$, $\overline{\tilde{Q}\tilde{P}} = \bar{\tilde{P}}\bar{\tilde{Q}}$.

(viii) $\mathbf{L}_{\tilde{\mathbf{q}}}\mathbf{L}_{\tilde{\mathbf{p}}} = \mathbf{L}_{\mathbf{L}_{\tilde{\mathbf{q}}}\tilde{\mathbf{p}}}$, $\mathbf{R}_{\tilde{\mathbf{q}}}\mathbf{R}_{\tilde{\mathbf{p}}} = \mathbf{R}_{\mathbf{R}_{\tilde{\mathbf{q}}}\tilde{\mathbf{p}}}$, $\mathbf{L}_{\tilde{\mathbf{q}}}\mathbf{R}_{\tilde{\mathbf{p}}} = \mathbf{R}_{\tilde{\mathbf{p}}}\mathbf{L}_{\tilde{\mathbf{q}}}$.

Proof. The parts (i)-(vi) can be proved by the using (1)-(4) and simple matrix computation.

Using the identification with dual vectors in $\mathbb{D}^4$, it is seen that

$$\tilde{Q}\bar{\tilde{Q}} \cong \mathbf{L}_{\tilde{\mathbf{q}}}\bar{\tilde{\mathbf{q}}} = \mathbf{R}_{\bar{\tilde{\mathbf{q}}}}\tilde{\mathbf{q}} = \|\tilde{\mathbf{q}}\|^2\mathbf{e}_1 \cong \left|\tilde{Q}\right|^2 = \tilde{a}_0^2 + \tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{a}_3^2,$$

$$\left|\tilde{Q}\tilde{P}\right|^2 = \|\mathbf{L}_{\tilde{\mathbf{q}}}\tilde{\mathbf{p}}\|^2 = \tilde{\mathbf{p}}'\mathbf{L'}_{\tilde{\mathbf{q}}}\mathbf{L}_{\tilde{\mathbf{q}}}\tilde{\mathbf{p}} = \tilde{\mathbf{p}}'\|\tilde{\mathbf{q}}\|^2\mathbf{L}_{\tilde{\mathbf{q}}}^{-1}\mathbf{L}_{\tilde{\mathbf{q}}}\tilde{\mathbf{p}} = \|\tilde{\mathbf{q}}\|^2\|\tilde{\mathbf{p}}\|^2 = \left|\tilde{Q}\right|^2\left|\tilde{P}\right|^2,$$

and

$$\overline{\tilde{Q}\tilde{P}} \cong \mathbf{C}\left(\mathbf{L}_{\tilde{\mathbf{q}}}\tilde{\mathbf{p}}\right) = \left(\mathbf{C}\mathbf{L}_{\tilde{\mathbf{q}}}\right)\tilde{\mathbf{p}} = \left(\mathbf{R'}_{\tilde{\mathbf{q}}}\mathbf{C}\right)\tilde{\mathbf{p}} = \mathbf{R'}_{\tilde{\mathbf{q}}}\left(\mathbf{C}\tilde{\mathbf{p}}\right) = \mathbf{R}_{\bar{\tilde{\mathbf{q}}}}\left(\mathbf{C}\tilde{\mathbf{p}}\right) = \mathbf{R}_{\bar{\tilde{\mathbf{q}}}}\bar{\tilde{\mathbf{p}}} \cong \bar{\tilde{P}}\bar{\tilde{Q}},$$

which completes the part (vii).

Moreover, using the associative property of dual quaternion's multiplication it is clear that the following identities hold:

$$\left(\tilde{Q}\tilde{P}\right)\tilde{R} = \tilde{Q}\left(\tilde{P}\tilde{R}\right) = \tilde{Q}\tilde{P}\tilde{R},$$

$$\tilde{R}\left(\tilde{P}\tilde{Q}\right) = \left(\tilde{R}\tilde{P}\right)\tilde{Q} = \tilde{R}\tilde{P}\tilde{Q},$$

$$\tilde{Q}\left(\tilde{R}\tilde{P}\right) = \left(\tilde{Q}\tilde{R}\right)\tilde{P} = \tilde{Q}\tilde{R}\tilde{P}.$$

In terms of the fundamental matrices, the above identities can be written as (5)(6)(7) respectively.

$$\left(\mathbf{L}_{\tilde{\mathbf{q}}}\tilde{\mathbf{p}}\right)\tilde{\mathbf{r}} = \mathbf{L}_{\mathbf{L}_{\tilde{\mathbf{q}}}\tilde{\mathbf{p}}}\tilde{\mathbf{r}} = \tilde{\mathbf{q}}\left(\mathbf{L}_{\tilde{\mathbf{p}}}\tilde{\mathbf{r}}\right) = \mathbf{L}_{\tilde{\mathbf{q}}}\left(\mathbf{L}_{\tilde{\mathbf{p}}}\tilde{\mathbf{r}}\right) = \mathbf{L}_{\tilde{\mathbf{q}}}\mathbf{L}_{\tilde{\mathbf{p}}}\tilde{\mathbf{r}}, \tag{5}$$

$$\left(\mathbf{R}_{\tilde{\mathbf{q}}}\tilde{\mathbf{p}}\right)\tilde{\mathbf{r}} = \mathbf{R}_{\mathbf{R}_{\tilde{\mathbf{q}}}\tilde{\mathbf{p}}}\tilde{\mathbf{r}} = \tilde{\mathbf{q}}\left(\mathbf{R}_{\tilde{\mathbf{p}}}\tilde{\mathbf{r}}\right) = \mathbf{R}_{\tilde{\mathbf{q}}}\left(\mathbf{R}_{\tilde{\mathbf{p}}}\tilde{\mathbf{r}}\right) = \mathbf{R}_{\tilde{\mathbf{q}}}\mathbf{R}_{\tilde{\mathbf{p}}}\tilde{\mathbf{r}}, \tag{6}$$

$$\tilde{\mathbf{q}}\left(\mathbf{R}_{\tilde{\mathbf{p}}}\tilde{\mathbf{r}}\right) = \mathbf{L}_{\tilde{\mathbf{q}}}\left(\mathbf{R}_{\tilde{\mathbf{p}}}\tilde{\mathbf{r}}\right) = \mathbf{L}_{\tilde{\mathbf{q}}}\mathbf{R}_{\tilde{\mathbf{p}}}\tilde{\mathbf{r}} = \left(\mathbf{L}_{\tilde{\mathbf{q}}}\tilde{\mathbf{r}}\right)\tilde{\mathbf{p}} = \mathbf{R}_{\tilde{\mathbf{p}}}\left(\mathbf{L}_{\tilde{\mathbf{q}}}\tilde{\mathbf{r}}\right) = \mathbf{R}_{\tilde{\mathbf{p}}}\mathbf{L}_{\tilde{\mathbf{q}}}\tilde{\mathbf{r}}. \tag{7}$$

Since the column $\tilde{\mathbf{r}}$ is arbitrary, (5), (6) and (7) relations employ the part (viii).

§3. Eigenvalues and eigenvectors of the fundamental matrices

Theorem 3.1. For $\tilde{\mathbf{q}} = (\tilde{a}_0, \tilde{a}_1, \tilde{a}_2, \tilde{a}_3)' \in \mathbb{D}^4$, the eigenvalues of the fundamental matrix $\mathbf{L}_{\tilde{\mathbf{q}}}$ are given by $\tilde{a}_0 \pm i\left\|\tilde{\mathbf{q}}_*\right\|$, where in case $\left\|\tilde{\mathbf{q}}_*\right\| \neq 0$ each eigenvalue occurs with algebraic multiplicity 2, and otherwise the eigenvalue $\tilde{a}_0$ has algebraic multiplicity 4.

Proof. For $\tilde{\mathbf{q}} = (\tilde{a}_0, \tilde{a}_1, \tilde{a}_2, \tilde{a}_3)' \in \mathbb{D}^4$ consider the eigenvalue-eigenvector equation

$$\mathbf{L}_{\tilde{\mathbf{q}}}\hat{\mathbf{z}} = \hat{\lambda}\hat{\mathbf{z}}, \qquad \hat{\mathbf{z}} \neq \mathbf{0},$$

where $\hat{\lambda} \in \mathbb{C}_D$ is an eigenvalue, and $\mathbf{0} \neq \hat{\mathbf{z}} \in \mathbb{C}_D^4$ is a corresponding eigenvector of $\mathbf{L}_{\tilde{\mathbf{q}}}$.

The matrix $\mathbf{L}_{\tilde{\mathbf{q}}}$ can be written as $\mathbf{L}_{\tilde{\mathbf{q}}} = \tilde{a}_0\mathbf{I}_4 + \mathbf{L}_{\tilde{\mathbf{q}}_*}$. Consequently, the eigenvalues of $\mathbf{L}_{\tilde{\mathbf{q}}}$ are obtained by adding $\tilde{a}_0$ to the eigenvalues of $\mathbf{L}_{\tilde{\mathbf{q}}_*}$. If $\hat{\mu}$ is an eigenvalue of $\mathbf{L}_{\tilde{\mathbf{q}}_*}$, then $\hat{\mu}^2$ is an eigenvalue of $\mathbf{L}^2_{\tilde{\mathbf{q}}_*}$. From

$$\mathbf{L}^{\mathbf{2}}_{\tilde{\mathbf{q}}_*} = -\left\|\tilde{\mathbf{q}}_*\right\|^2 \mathbf{I}_4,$$

we conclude that $\hat{\mu}^2 = -\left\|\tilde{\mathbf{q}}_*\right\|^2$. Hence, the eigenvalues of $\mathbf{L}_{\tilde{\mathbf{q}}_*}$ can only be $\hat{\mu} = i\left\|\tilde{\mathbf{q}}_*\right\|$ or $\hat{\mu} = -i\left\|\tilde{\mathbf{q}}_*\right\|$. But the dual complex eigenvalues of the dual matrix $\mathbf{L}_{\tilde{\mathbf{q}}_*}$ occur in conjugate pairs, so that $\mathbf{L}_{\tilde{\mathbf{q}}_*}$ has two eigenvalues $i\left\|\tilde{\mathbf{q}}_*\right\|$ and two eigenvalues $-i\left\|\tilde{\mathbf{q}}_*\right\|$. So, the proof is completed.

Corollary 3.1. For $\tilde{\mathbf{q}} = (\tilde{a}_0, \tilde{a}_1, \tilde{a}_2, \tilde{a}_3)' \in \mathbb{D}^4$, the eigenvalues of the fundamental matrix $\mathbf{R}_{\tilde{\mathbf{q}}}$ are given by $\tilde{a}_0 \pm i \|\tilde{\mathbf{q}}_*\|$, where in case $\|\tilde{\mathbf{q}}_*\| \neq 0$ each eigenvalue occurs with algebraic multiplicity 2, and otherwise the eigenvalue $\tilde{a}_0$ has algebraic multiplicity 4.

Proof. The eigenvalues of $\mathbf{R}_{\tilde{\mathbf{q}}} = \mathbf{C}\mathbf{L}'_{\tilde{\mathbf{q}}}\mathbf{C}$ coincide with the eigenvalues of $\mathbf{L}'_{\tilde{\mathbf{q}}}$, which in turn coincide with the eigenvalues of $\mathbf{L}_{\tilde{\mathbf{q}}}$. So, the proof is clear from Theorem 3.1.

Let us now turn our attention to the eigenvectors of $\mathbf{L}_{\tilde{\mathbf{q}}}$. From Theorem 2.1 (iii) $\mathbf{L}_{\tilde{\mathbf{q}}}$ is nondefective, which means that geometric and algebraic multiplicity of the eigenvalues of $\mathbf{L}_{\tilde{\mathbf{q}}}$ coincide. Therefore, in case $\|\tilde{\mathbf{q}}_*\| \neq 0$ the eigenspaces associated with the eigenvalues $\tilde{a}_0 + i \|\tilde{\mathbf{q}}_*\|$ and $\tilde{a}_0 - i \|\tilde{\mathbf{q}}_*\|$ both have dimension 2.

Theorem 3.2. Let $\|\tilde{\mathbf{q}}_*\| \neq 0$ for $\tilde{\mathbf{q}} = (\tilde{a}_0, \tilde{a}_1, \tilde{a}_2, \tilde{a}_3)' \in \mathbb{D}^4$. Then the eigenspaces of $\mathbf{L}_{\tilde{\mathbf{q}}}$ corresponding to $\tilde{a}_0 + i \|\tilde{\mathbf{q}}_*\|$ and $\tilde{a}_0 - i \|\tilde{\mathbf{q}}_*\|$ are:

$$\left\{\mathbf{L}_{\hat{\mathbf{g}}}\hat{\mathbf{y}} : \hat{\mathbf{y}} \in \mathbb{C}_D^4\right\} \text{ and } \left\{\mathbf{L}_{\hat{\mathbf{h}}}\hat{\mathbf{y}} : \hat{\mathbf{y}} \in \mathbb{C}_D^4\right\},$$

respectively, where $\hat{\mathbf{g}} = i \|\tilde{\mathbf{q}}_*\| \, \mathbf{e}_1 + \tilde{\mathbf{q}}_*$ and $\hat{\mathbf{h}} = -i \|\tilde{\mathbf{q}}_*\| \, \mathbf{e}_1 + \tilde{\mathbf{q}}_*$.

Proof. This can be verified by calculating $\mathbf{L}_{\tilde{\mathbf{q}}}\mathbf{L}_{\hat{\mathbf{g}}}\hat{\mathbf{y}} - (\tilde{a}_0 + i \|\tilde{\mathbf{q}}_*\|) \, \mathbf{L}_{\hat{\mathbf{g}}}\hat{\mathbf{y}}$ and $\mathbf{L}_{\tilde{\mathbf{q}}}\mathbf{L}_{\hat{\mathbf{h}}}\hat{\mathbf{y}} - (\tilde{a}_0 - i \|\tilde{\mathbf{q}}_*\|) \, \mathbf{L}_{\hat{\mathbf{h}}}\hat{\mathbf{y}}$, which both yield the zero vector for any $\hat{\mathbf{y}} \in \mathbb{C}_D^4$.

Observe that we admit dual complex entries in the matrices $\mathbf{L}_{\hat{\mathbf{g}}}$ and $\mathbf{L}_{\hat{\mathbf{h}}}$, as distinct from our former procedure where only dual entries were considered.

§4. Application to the equation $\tilde{R}\tilde{Q} = \tilde{P}\tilde{R} + \tilde{C}$

Two dual quaternions $\tilde{Q}$ and $\tilde{P}$ are called similar if there exists a nonzero dual quaternion $\tilde{U}$ such that

$$\tilde{U}^{-1}\tilde{P}\tilde{U} = \tilde{Q}.$$

Similarity will be denoted by $\tilde{Q} \sim \tilde{P}$ and it can be shown that " $\sim$ " is an equivalence relation on $\mathbb{H}$.

Now, let us consider the dual quaternion equation

$$\tilde{R}\tilde{Q} = \tilde{P}\tilde{R} + \tilde{C},$$

where $\tilde{Q}, \tilde{P}, \tilde{C} \in \mathbb{H}_D$ are given.

Using matrix representation, it is seen that the above equation is equivalent to $\mathbf{R}_{\tilde{\mathbf{q}}}\tilde{\mathbf{r}} = \mathbf{L}_{\tilde{\mathbf{p}}}\tilde{\mathbf{r}} + \tilde{\mathbf{c}}$, which can be written as

$$(\mathbf{R}_{\tilde{\mathbf{q}}} - \mathbf{L}_{\tilde{\mathbf{p}}}) \, \tilde{\mathbf{r}} = \tilde{\mathbf{c}}.$$

Lemma 4.1. The equation $\tilde{R}\tilde{Q} = \tilde{P}\tilde{R} + \tilde{C}$ is uniquely solvable with respect to $\tilde{R}$ if and only if $\tilde{Q} \nsim \tilde{P}$.

Proof. Transferring this notion to matrix notation with $\tilde{Q} \cong \tilde{\mathbf{q}}$ and $\tilde{P} \cong \tilde{\mathbf{p}}$, it is obtained that

$$\tilde{Q} \sim \tilde{P} \Leftrightarrow \exists \, \mathbf{0} \neq \tilde{\mathbf{u}} \in \mathbb{D}^4 : \mathbf{L}_{\tilde{\mathbf{p}}}\tilde{\mathbf{u}} = \mathbf{R}_{\tilde{\mathbf{q}}}\tilde{\mathbf{u}} \Leftrightarrow \mathbf{R}_{\tilde{\mathbf{q}}} - \mathbf{L}_{\tilde{\mathbf{p}}} \text{ is singular.} \tag{8}$$

Since the dual matrix equation $(\mathbf{R}_{\tilde{\mathbf{q}}} - \mathbf{L}_{\tilde{\mathbf{p}}}) \, \tilde{\mathbf{r}} = \tilde{\mathbf{c}}$ is uniquely solvable if and only if $\mathbf{R}_{\tilde{\mathbf{q}}} - \mathbf{L}_{\tilde{\mathbf{p}}}$ is nonsingular. So the proof is complete.

Lemma 4.2. Two dual quaternions $\tilde{Q}$ and $\tilde{P}$ are similar if and only if $\mathcal{R}\left(\tilde{Q}\right)=\mathcal{R}\left(\tilde{P}\right)$ and $\left|\Im\left(\tilde{Q}\right)\right|=\left|\Im\left(\tilde{P}\right)\right|$.

Proof. Since the two commuting normal matrices $\mathbf{R}_{\tilde{\mathbf{q}}}$ and $\mathbf{L}_{\tilde{\mathbf{p}}}$ can be simultaneously unitarily diagonalizable [14, Theorem 2.5.4, 2.5.5], each eigenvalue of $\mathbf{R}_{\tilde{\mathbf{q}}}-\mathbf{L}_{\tilde{\mathbf{p}}}$ is the difference of an eigenvalue of $\mathbf{R}_{\tilde{\mathbf{q}}}$ and an eigenvalue of $\mathbf{L}_{\tilde{\mathbf{p}}}$, i.e., the eigenvalues of the crucial matrix $\mathbf{R}_{\tilde{\mathbf{q}}}-\mathbf{L}_{\tilde{\mathbf{p}}}$ are given by

$$\left(\tilde{a}_0-\tilde{b}_0\right)\pm i\left(\|\tilde{\mathbf{q}}_*\|-\|\tilde{\mathbf{p}}_*\|\right) \text{ and } \left(\tilde{a}_0-\tilde{b}_0\right)\pm i\left(\|\tilde{\mathbf{q}}_*\|+\|\tilde{\mathbf{p}}_*\|\right).$$

Moreover, a matrix is singular if and only if at least one of its eigenvalues is 0. So, the proof is completed.

We will proceed by considering the solutions to the dual matrix equation $\left(\mathbf{R}_{\tilde{\mathbf{q}}}-\mathbf{L}_{\tilde{\mathbf{p}}}\right)\tilde{\mathbf{x}}=\tilde{\mathbf{c}}$ and then collect our findings in terms of dual quaternions. Since $\mathbf{R}_{\tilde{\mathbf{q}}}-\mathbf{L}_{\tilde{\mathbf{p}}}$ is normal, its rank equals the number of its nonzero eigenvalues. Hence,

$$rank\left(\mathbf{R}_{\tilde{\mathbf{q}}}-\mathbf{L}_{\tilde{\mathbf{p}}}\right)\in\{0,2,4\}. \tag{9}$$

Theorem 4.1. Let $\tilde{Q},\tilde{P},\tilde{C}\in\mathbb{H}_D$.

(i) The equation $\tilde{R}\tilde{Q}=\tilde{P}\tilde{R}+\tilde{C}$ is uniquely solvable with respect to $\tilde{R}$ if and only if $\tilde{Q}\nsim\tilde{P}$, in which case the solution is given by

$$\tilde{R}=\tilde{m}^{-1}\left(\tilde{C}\bar{\tilde{Q}}-\tilde{P}\tilde{C}\right),\qquad \tilde{m}=2\left[\mathcal{R}\left(\tilde{P}\right)-\mathcal{R}\left(\tilde{Q}\right)\right]\tilde{P}+\left|\tilde{Q}\right|^2-\left|\tilde{P}\right|^2.$$

(ii) If $\bar{\tilde{Q}}=\tilde{Q}\sim\tilde{P}$, a necessary and sufficient condition for solvability is $\tilde{C}=0$, in which case any $\tilde{R}\in\mathbb{H}_D$ is a solution.

(iii) If $\bar{\tilde{Q}}\neq\tilde{Q}\sim\tilde{P}$, a necessary and sufficient condition for solvability is $\tilde{C}\bar{\tilde{Q}}=\tilde{P}\tilde{C}$, in which case all solutions are given by

$$\tilde{R}=\frac{1}{4\left|\Im\left(\tilde{Q}\right)\right|^2}\left(\tilde{P}\tilde{C}-\tilde{C}\tilde{Q}\right)+\tilde{\mathrm{Z}}-\frac{1}{\left|\Im\left(\tilde{Q}\right)\right|^2}\Im\left(\tilde{P}\right)\tilde{\mathrm{Z}}\Im\left(\tilde{Q}\right),$$

where $\tilde{\mathrm{Z}}\in\mathbb{H}_D$ is arbitrary.

Proof. A matrix has the same eigenvalues with its transpose. From Theorem 2.1 (iii), the normal matrix $\mathbf{R}_{\bar{\tilde{\mathbf{q}}}}-\mathbf{L}_{\tilde{\mathbf{p}}}$ has the same eigenvalues as $\mathbf{R}_{\tilde{\mathbf{q}}}-\mathbf{L}_{\tilde{\mathbf{p}}}$ and therefore the same rank. From (8), $\tilde{Q}\nsim\tilde{P}$ if and only if $rank\left(\mathbf{R}_{\tilde{\mathbf{q}}}-\mathbf{L}_{\tilde{\mathbf{p}}}\right)=4$ and so the unique solution is given by $\tilde{\mathbf{r}}=\left(\mathbf{R}_{\tilde{\mathbf{q}}}-\mathbf{L}_{\tilde{\mathbf{p}}}\right)^{-1}\tilde{\mathbf{c}}$, where the matrix $\left(\mathbf{R}_{\tilde{\mathbf{q}}}-\mathbf{L}_{\tilde{\mathbf{p}}}\right)^{-1}$ may as well be expressed as

$$\begin{aligned}\left(\mathbf{R}_{\tilde{\mathbf{q}}}-\mathbf{L}_{\tilde{\mathbf{p}}}\right)^{-1}&=\left(\mathbf{R}_{\tilde{\mathbf{q}}}-\mathbf{L}_{\tilde{\mathbf{p}}}\right)^{-1}\left(\mathbf{R}_{\bar{\tilde{\mathbf{q}}}}-\mathbf{L}_{\tilde{\mathbf{p}}}\right)^{-1}\left(\mathbf{R}_{\bar{\tilde{\mathbf{q}}}}-\mathbf{L}_{\tilde{\mathbf{p}}}\right)\\&=\left[\left(\mathbf{R}_{\bar{\tilde{\mathbf{q}}}}-\mathbf{L}_{\tilde{\mathbf{p}}}\right)\left(\mathbf{R}_{\tilde{\mathbf{q}}}-\mathbf{L}_{\tilde{\mathbf{p}}}\right)\right]^{-1}\left(\mathbf{R}_{\bar{\tilde{\mathbf{q}}}}-\mathbf{L}_{\tilde{\mathbf{p}}}\right)\\&=\mathbf{L}_{\tilde{\mathbf{m}}}^{-1}\left(\mathbf{R}_{\bar{\tilde{\mathbf{q}}}}-\mathbf{L}_{\tilde{\mathbf{p}}}\right),\end{aligned}$$

$$\tilde{\mathbf{m}}=2\left(\tilde{b}_0-\tilde{a}_0\right)\tilde{\mathbf{p}}+\left(\|\tilde{\mathbf{q}}\|^2-\|\tilde{\mathbf{p}}\|^2\right)\mathbf{I}_4.$$

This completes the part (i). Hence, from (9) it remains to consider the cases $rank\left(\mathbf{R}_{\tilde{\mathbf{q}}}-\mathbf{L}_{\tilde{\mathbf{p}}}\right)=0$ or $rank\left(\mathbf{R}_{\tilde{\mathbf{q}}}-\mathbf{L}_{\tilde{\mathbf{p}}}\right)=2$. Let's consider $rank\left(\mathbf{R}_{\tilde{\mathbf{q}}}-\mathbf{L}_{\tilde{\mathbf{p}}}\right)=0$, then it is obvious that $\left(\mathbf{R}_{\tilde{\mathbf{q}}}-\mathbf{L}_{\tilde{\mathbf{p}}}\right)\tilde{\mathbf{x}}=$

$\tilde{\mathbf{c}}$ is solvable if and only if $\tilde{\mathbf{c}} = \mathbf{0}$, in which case any vector $\tilde{\mathbf{r}} \in \mathbb{D}^4$ is a solution. So, the part (ii) is completed.

Finally, let $rank\,(\mathbf{R}_{\tilde{\mathbf{q}}} - \mathbf{L}_{\tilde{\mathbf{p}}}) = 2$, then $\tilde{a}_0 = \tilde{b}_0$ and $\|\tilde{\mathbf{q}}_*\| = \|\tilde{\mathbf{p}}_*\|$ but $\|\tilde{\mathbf{q}}_*\| \neq 0$, since otherwise all eigenvalues of $\mathbf{R}_{\tilde{\mathbf{q}}} - \mathbf{L}_{\tilde{\mathbf{p}}}$ would be 0. Now, the matrix difference $\mathbf{R}_{\tilde{\mathbf{q}}} - \mathbf{L}_{\tilde{\mathbf{p}}}$ can be expressed as

$$\mathbf{R}_{\tilde{\mathbf{q}}} - \mathbf{L}_{\tilde{\mathbf{p}}} = \mathbf{R}_{\tilde{\mathbf{q}}_*} - \mathbf{L}_{\tilde{\mathbf{p}}_*},$$

where the commuting matrices $\mathbf{R}_{\tilde{\mathbf{q}}_*}$ and $\mathbf{L}_{\tilde{\mathbf{p}}_*}$ satisfy

$$\mathbf{R}_{\tilde{\mathbf{q}}_*} = -\mathbf{R}_{\bar{\tilde{\mathbf{q}}}_*} = -\|\tilde{\mathbf{q}}_*\|^2 \mathbf{R}_{\tilde{\mathbf{q}}_*}^{-1} \quad \text{and} \quad \mathbf{L}_{\tilde{\mathbf{p}}_*} = -\mathbf{L}_{\bar{\tilde{\mathbf{p}}}_*} = -\|\tilde{\mathbf{q}}_*\|^2 \mathbf{L}_{\tilde{\mathbf{p}}_*}^{-1}.$$

Using these properties and noting that also $\mathbf{R}_{\bar{\tilde{\mathbf{q}}}} - \mathbf{L}_{\tilde{\mathbf{p}}} = \mathbf{R}_{\bar{\tilde{\mathbf{q}}}_*} - \mathbf{L}_{\tilde{\mathbf{p}}_*}$, the following can be seen by exploiting simple matrix calculus.

If there exists a vector $\tilde{\mathbf{r}}$ such that $(\mathbf{R}_{\tilde{\mathbf{q}}} - \mathbf{L}_{\tilde{\mathbf{p}}})\,\tilde{\mathbf{r}} = \tilde{\mathbf{c}}$, then $\tilde{\mathbf{c}}$ satisfies $(\mathbf{R}_{\tilde{\mathbf{q}}} - \mathbf{L}_{\tilde{\mathbf{p}}})\,\tilde{\mathbf{c}} = \mathbf{0}$. Conversely, if $\tilde{\mathbf{c}}$ satisfies $\left(\mathbf{R}_{\bar{\tilde{\mathbf{q}}}} - \mathbf{L}_{\tilde{\mathbf{p}}}\right)\tilde{\mathbf{c}} = \mathbf{0}$, then it follows that

$$(\mathbf{R}_{\tilde{\mathbf{q}}} - \mathbf{L}_{\tilde{\mathbf{p}}})\,\tilde{\mathbf{r}} = \tilde{\mathbf{c}} \quad \text{for} \quad \tilde{\mathbf{r}} = -\frac{1}{4\,\|\tilde{\mathbf{q}}_*\|^2}\,(\mathbf{R}_{\tilde{\mathbf{q}}} - \mathbf{L}_{\tilde{\mathbf{p}}})\,\tilde{\mathbf{c}}.$$

Furthermore, every vector

$$\tilde{\mathbf{r}} = \left(\mathbf{I}_4 - \frac{1}{\|\tilde{\mathbf{q}}_*\|^2}\mathbf{L}_{\tilde{\mathbf{p}}_*}\mathbf{R}_{\tilde{\mathbf{q}}_*}\right)\tilde{\mathbf{w}}, \qquad \tilde{\mathbf{w}} \in \mathbb{D}^4,$$

satisfies $(\mathbf{R}_{\tilde{\mathbf{q}}} - \mathbf{L}_{\tilde{\mathbf{p}}})\,\tilde{\mathbf{r}} = \mathbf{0}$. On the other hand, if $(\mathbf{R}_{\tilde{\mathbf{q}}} - \mathbf{L}_{\tilde{\mathbf{p}}})\,\tilde{\mathbf{r}} = \mathbf{0}$, then

$$\tilde{\mathbf{r}} = \left(\mathbf{I}_4 - \frac{1}{\|\tilde{\mathbf{q}}_*\|^2}\mathbf{L}_{\tilde{\mathbf{p}}_*}\mathbf{R}_{\tilde{\mathbf{q}}_*}\right)\frac{1}{2}\tilde{\mathbf{r}}.$$

So, the proof is complete.

§5. Powers of a dual quaternion

Let's consider again the equation $(\mathbf{R}_{\tilde{\mathbf{q}}} - \mathbf{L}_{\tilde{\mathbf{p}}})\,\tilde{\mathbf{r}} = \mathbf{0}$ for the case $rank\,(\mathbf{R}_{\tilde{\mathbf{q}}} - \mathbf{L}_{\tilde{\mathbf{p}}}) = 2$ (which implies $\|\tilde{\mathbf{q}}_*\| \neq 0$). Then the dimension of the space of all its solutions with respect to $\tilde{\mathbf{r}}$, the null space of $\mathbf{R}_{\tilde{\mathbf{q}}} - \mathbf{L}_{\tilde{\mathbf{p}}}$ is 2. Hence, this space can be written as

$$\left\{\tilde{\mathbf{r}} \;:\; \tilde{\mathbf{r}} = \tilde{\lambda}\,\tilde{\mathbf{r}}_1 + \tilde{\mu}\,\tilde{\mathbf{r}}_2\,, \tilde{\lambda}\,, \tilde{\mu} \in \mathbb{D}\right\},$$

where $\tilde{\mathbf{r}}_1$ and $\tilde{\mathbf{r}}_2$ are two linearly independent nonzero vectors satisfying

$$(\mathbf{R}_{\tilde{\mathbf{q}}} - \mathbf{L}_{\tilde{\mathbf{p}}})\,\tilde{\mathbf{r}}_1 = (\mathbf{R}_{\tilde{\mathbf{q}}} - \mathbf{L}_{\tilde{\mathbf{p}}})\,\tilde{\mathbf{r}}_2 = \mathbf{0}.$$

In case $\tilde{\mathbf{p}} \neq \bar{\tilde{\mathbf{q}}}$ this is true for the orthogonal vectors

$$\tilde{\mathbf{r}}_1 = \|\tilde{\mathbf{q}}_*\|^2\,\mathbf{e}_1 - \mathbf{L}_{\tilde{\mathbf{p}}_*}\tilde{\mathbf{q}}_* \quad \text{and} \quad \tilde{\mathbf{r}}_2 = \tilde{\mathbf{q}}_* + \tilde{\mathbf{p}}_*.$$

In other words, for two given quaternions $\bar{\tilde{Q}} \neq \tilde{Q} \sim \tilde{P} \neq \bar{\tilde{Q}}$, all quaternions $\tilde{R}$ satisfying $\tilde{R}\tilde{Q} = \tilde{P}\tilde{R}$ are

$$\tilde{R} = \tilde{\lambda}\left[\left|\Im\left(\tilde{Q}\right)\right|^2 - \Im\left(\tilde{P}\right)\Im\left(\tilde{Q}\right)\right] + \tilde{\mu}\left[\Im\left(\tilde{Q}\right) + \Im\left(\tilde{P}\right)\right], \quad \tilde{\lambda}, \tilde{\mu} \in \mathbb{D}.$$

As a direct consequence, all quaternions $\tilde{R}$ which commute with a quaternion $\tilde{Q} \neq \bar{\tilde{Q}}$ are given by

$$\tilde{R} = \tilde{\gamma} + \tilde{\delta}\,\Im\left(\tilde{Q}\right),$$

where $\tilde{\gamma}$ and $\tilde{\delta}$ are arbitrary numbers in $\mathbb{D}$.

Theorem 5.1. For a dual quaternion $\tilde{Q} \in \mathbb{H}_D$, let $\hat{\alpha} = \mathcal{R}\left(\tilde{Q}\right) + i\left|\Im\left(\tilde{Q}\right)\right|$. Then the nth power, $n \in \mathbb{N}$, of $\tilde{Q}$ is given by

$$\tilde{Q}^n = \tilde{\lambda}_n + \tilde{\mu}_n\left|\Im\left(\tilde{Q}\right)\right|,$$

where $\tilde{\lambda}_n = \mathrm{Du}\left(\hat{\alpha}^n\right)$ and $\tilde{\mu}_n = \left(1\middle/\left|\Im\left(\tilde{Q}\right)\right|\right)\mathrm{Im}\left(\hat{\alpha}^n\right)$ in case $\tilde{Q} \neq \bar{\tilde{Q}}$, while $\tilde{\mu}_n$ can be chosen arbitrarily otherwise.

Proof. It is obvious that the nth power an of a dual quaternion $\tilde{Q}$ commutes with $\tilde{Q}$, where $n \in \mathbb{N}$ and $\tilde{Q}^0 := 1$. Hence, we can write

$$\tilde{Q}^n = \tilde{\lambda}_n + \tilde{\mu}_n\left|\Im\left(\tilde{Q}\right)\right|,$$

for some dual numbers $\tilde{\lambda}_n$ and $\tilde{\mu}_n$, where in the trivial case $\tilde{Q} = \bar{\tilde{Q}}$ we have $\tilde{\lambda}_n = \left(\mathrm{R}\left(\tilde{Q}\right)\right)^n$ and $\tilde{\mu}_n \in \mathbb{D}$ arbitrary.

For determining $\tilde{\lambda}_n$ and $\tilde{\mu}_n$ in the nontrivial case $\tilde{Q} \neq \bar{\tilde{Q}}$, it is seen from $\tilde{Q}^{n+1} = \tilde{Q}\tilde{Q}^n$ and the identification of $\tilde{Q}^n$ with its corresponding real vector $\tilde{\lambda}_n\mathbf{e}_1 + \tilde{\mu}_n\tilde{\mathbf{q}}_*$ for any $n \in \mathbb{N}$, that the pairs $\left(\tilde{\lambda}_n, \tilde{\mu}_n\right)$ obey the following system of linear homogeneous first-order difference equations

$$\tilde{\lambda}_{n+1} = \tilde{\lambda}_n\tilde{a}_0 - \tilde{\mu}_n\left\|\tilde{\mathbf{q}}_*\right\|^2, \qquad \tilde{\mu}_{n+1} = \tilde{a}_0\tilde{\mu}_n + \tilde{\lambda}_n,$$

with initial values $\tilde{\lambda}_0 = 1$ and $\tilde{\mu}_0 = 0$. Observe that $\|\tilde{\mathbf{q}}_*\| \neq 0$ due to $\tilde{Q} \neq \bar{\tilde{Q}}$. The two equations can be written as

$$\tilde{\mathbf{w}}_{n+1} = \mathbf{A}\tilde{\mathbf{w}}_n, \qquad \mathbf{A} = \begin{pmatrix} \tilde{a}_0 & -\left\|\tilde{\mathbf{q}}_*\right\|^2 \\ 1 & \tilde{a}_0 \end{pmatrix}, \qquad \tilde{\mathbf{w}}_n = \begin{pmatrix} \tilde{\lambda}_n \\ \tilde{\mu}_n \end{pmatrix}.$$

From Theorem 3.1, the eigenvalues of the nonsingular matrix $\mathbf{A}$ are $\hat{\sigma}_1 = \tilde{a}_0 + i\left\|\tilde{\mathbf{q}}_*\right\|$ and $\hat{\sigma}_2 = \tilde{a}_0 - i\left\|\tilde{\mathbf{q}}_*\right\|$ with corresponding eigenvectors $\hat{\mathbf{z}}_1 = \left(i\left\|\tilde{\mathbf{q}}_*\right\|, 1\right)'$ and $\hat{\mathbf{z}}_2 = \left(i\left\|\tilde{\mathbf{q}}_*\right\|, -1\right)'$. Using

$$\tilde{\mathbf{w}}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = k\left(\hat{\mathbf{z}}_1 + \hat{\mathbf{z}}_2\right), \qquad k = \frac{-i}{2\left\|\tilde{\mathbf{q}}_*\right\|},$$

it follows from Theorem 5.10.1 in [9] that $\tilde{\mathbf{w}}_n = k\left(\hat{\sigma}_1^n \hat{\mathbf{z}}_1 + \hat{\sigma}_2^n \hat{\mathbf{z}}_2\right), k = \frac{-i}{2\|\tilde{\mathbf{q}}_*\|}$. Thus, we arrive at

$$\begin{pmatrix} \tilde{\lambda}_n \\ \tilde{\mu}_n \end{pmatrix} = \begin{pmatrix} \mathrm{Du}\left[\left(\tilde{a}_0 + i\left\|\tilde{\mathbf{q}}_*\right\|^2\right)^n\right] \\ \left(1/\left\|\tilde{\mathbf{q}}_*\right\|\right)\mathrm{Im}\left[\left(\tilde{a}_0 + i\left\|\tilde{\mathbf{q}}_*\right\|^2\right)^n\right] \end{pmatrix},$$

where for a number $\hat{\alpha} = \tilde{\beta} + i\tilde{\gamma} \in \mathbb{C}_D$ we use $\mathrm{Du}\left(\hat{\alpha}\right) = \tilde{\beta}$ and $\mathrm{Im}\left(\hat{\alpha}\right) = \tilde{\gamma}$.

A further way of expressing the nth power of a quaternion is to directly exploit that the quaternion $\tilde{Q}$ is similar to $\hat{\alpha}$, namely

$$\tilde{Q} = \tilde{U}\hat{\alpha}\tilde{U}^{-1},$$

where in case $\tilde{Q} \neq \bar{\tilde{Q}}$ the dual quaternion $\tilde{U}$ may be chosen as nonzero

$$\tilde{U} = \tilde{\lambda}\left[\left|\Im\left(\tilde{Q}\right)\right| - \left|\Im\left(\tilde{Q}\right)\right| i\right] + \tilde{\mu}\left[\left|\Im\left(\tilde{Q}\right)\right| i + \left|\Im\left(\tilde{Q}\right)\right|\right]$$

with arbitrary $\tilde{\lambda},\ \tilde{\mu} \in \mathbb{D}$. Thus

$$\tilde{Q}^n = \tilde{U}\hat{\alpha}^n\tilde{U}^{-1}.$$

Writing $\hat{\alpha}^n = \mathrm{Du}\left(\hat{\alpha}^n\right) + i\mathrm{Im}\left(\hat{\alpha}^n\right)$ and utilizing

$$\tilde{U}i\tilde{U}^{-1} = \frac{1}{\left|\Im\left(\tilde{Q}\right)\right|}\left|\Im\left(\tilde{Q}\right)\right|, \quad \tilde{Q} \neq \bar{\tilde{Q}},$$

one easily obtains the assertion of Theorem 5.1.

References

[1] S. L. Adler, Quaternionic quantum mechanics and quantum fields, Oxford University Press inc., New York, 1995.

[2] O. P. Agrawal, Hamilton Operators and Dual-number-quaternions in Spatial Kinematics, Mech. Mach. Theory, **22**(1987), 569-575.

[3] E. Ata, Y. Yaylı, Dual unitary matrices and unit dual quaternions, Differential Geometry-Dynamical Systems, **10**(2008), 1-12.

[4] K. Bharathi, M. Nagaraj, Quaternion valued function of real variable Serret–Frenet formulae, Indian J. Pure Appl. Math., **16**(1985), 741-756.

[5] W. K. Clifford, Preliminary sketch of biquaternions, Proc. London Math. Soc., **4**(1873), 361-395.

[6] E. B. Dam, M. Koch, M. Lillholm, Quaternions, Interpolation and Animation, Technical Report, DIKUTR-98/5, University of Copenhagen, 1998.

[7] J. R. Dooley, J. M. McCarthy, Spatial Rigid Body Dynamics Using Dual Quaternion Components, Proc. of IEEE International Conf. on Robotics and Automation, Sacramento, CA, **1**(1991), 90-95.

[8] D. Finkelstein, J. M. Jaunch, S. Schiminovich, D. Speiser, Foundations of quaternion quantum mechanics, J. Math. Phys., 1962, 3207-3220.

[9] J. L. Goldberg, Matrix Theory with Applications, McGraw-Hill, New York, 1991.

[10] J. Groß, G. Trenkler, S. O. Troschke, Quaternions: further contributions to a matrix oriented approach, Linear Algebra Appl., **326**(2001), 205-213.

[11] H. H. Hacısalihoğlu, Acceleration Axes in Spatial Kinamatics Communications, **20A** (1971), 1-15.

[12] H. H. Hacısalihoğlu, Hareket Geometrisi ve Kuaternionlar Teorisi, Gazi Univ. Publishing, 1983.

[13] H. Halberstam, R. E. Ingram (Eds.), The mathematical papers of Sir William Rowan Hamilton, Algebra, Cambridge University Press, Cambridge, MA, **III**(1967).

[14] R. A. Horn, C. R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, UK, 1985.

[15] A. P. Kotelnikov, Screw calculus and some of its applications in geometry and mechanics. Annals of the Imperial University, Kazan (in Russian), 1895.

[16] G. R. Pennock, A. T. Yang, Dynamic Analysis of Multi-Rigid-Body Open-Chain System, Trans. ASME, J. of Mechanisms, Transmissions, and Automation in Design, **105**(1983), 28-34.

[17] G. R. Pennock, A. T. Yang, Application of Dual Number Matrices to the Inverse Kinematics Problem of Robot Manipulators, Trans. ASME, J. of Mechanisms, Transmissions and Automation in Design, **107**(1985), 201-208.

[18] B. Ravani, Q. J. Ge, Kinematic Localization for World Model Calibration in Off-Line Robot Programming Using Clifford Algebra, Proc of IEEE International Conf. on Robotics and Automation., Sacramento CA., **1**(1991), 584-589.

[19] J. Schmidt, H. Niemann, Using Quaternions for Parameterizing 3-D Rotations in Unconstrained Nonlinear Optimization, Vision Modeling and Visualization, Stuttgart, Germany, **11**(2001), 399-406.

[20] E. Study, Geometrie der Dynamen, Leipzig, 1903.

[21] G. R. Veldkamp, On the use of dual numbers, vectors and matrices in instantaneous, Spatial Kinematics, Mech. Mach. Theory, **11**(1976), 141-156.

[22] J. P. Ward, Quaternions and Cayley Numbers, Kluwer Academic Publisher, The Netherlands, 1997.

[23] A. T. Yang, F. Freudenstein, Application of dual-number quaternion algebra to the analysis of spatial mechanisms, Transactions of the ASME, 1964, 300-308.

[24] F. Zhang, Quaternions and matrices of quaternions, Linear Algebra Appl., **251**(1997), 21-57.

Scientia Magna
Vol. 7 (2011), No. 1, 12-17

Generalized Weyl's theorem for Class A operators

Junhong Tian†, Wansheng He‡ and Cuiqin Guo♯

† ‡ College of Mathematics and Statistics, Tianshui Normal University,
Tianshui 741000, Gansu, China
♯ Yuan Long Middle School of TianShui, 741033, Gansu, China
E-mail: tianjh_2006@126.com

Abstract Two variants of the Weyl spectrum are discussed, we prove that Class A operators satisfies the generalized Weyl's theorem, hence Weyl's theorem holds for Class A operators.

Keywords Weyl spectrum, generalized Weyl's theorem, operators.

§1 Introduction

H. Weyl [20] examined the spectra of all compact perturbations of a hermitian operator on Hilbert space and found in 1909 that their intersection consisted precisely of those points of the spectrum which were not isolated eigenvalues of finite multiplicity. This Weyl's theorem has since been extended to hyponormal and to Toeplitz operators (Coburn [8]), to seminormal and other operators (Berberian [2], [3]) and to Banach spaces operators (Istrătescu [12], Oberai [16]). Variants have been discussed by Harte and Lee [11] and Rakočevič [17], M. Berkani and J. J. Koliha [6]. In this note we show how generalized Weyl's theorem follows from the equality of the Drazin spectrum and a variant of the Weyl's spectrum.

Recall that the Weyl's spectrum of a bounded linear operator T on a Banach space X is the intersection of the spectra of its compact perturbations:

$$\sigma_w(T) = \bigcap \{\sigma(T+K) : K \in K(X)\} \ . \tag{1}$$

Equivalently $\lambda \in \sigma_w(T)$ iff $T - \lambda I$ fails to be Fredholm of index zero. The Browder spectrum is the intersection of the spectra of its commuting compact perturbations:

$$\sigma_b(T) = \bigcap \{\sigma(T+K) : K \in K(X) \cap \mathrm{comm}(T)\} \ . \tag{2}$$

Equivalently $\lambda \in \sigma_b(T)$ iff $T - \lambda I$ fails to be Fredholm of finite ascent and descent. The Weyl's theorem holds for T iff

$$\sigma(T) \backslash \sigma_w(T) = \pi_{00}(T) \ , \tag{3}$$

where we write

$$\pi_{00}(T) = \{\lambda \in \text{iso } \sigma(T) : \ 0 < \dim N(T - \lambda I) < \infty\} \tag{4}$$

for the isolated points of the spectrum which are eigenvalues of finite multiplicity. Harte and Lee [11] have discussed a variant of Weyl's theorem: the Browder's theorem holds for T iff

$$\sigma(T) = \sigma_w(T) \cup \pi_{00}(T) \ . \tag{5}$$

What is missing is the disjointness between the Weyl spectrum and the isolated eigenvalues of finite multiplicity: equivalently

$$\sigma_w(T) = \sigma_b(T) \ . \tag{6}$$

For a bounded linear operator T and a nonnegative integer n define $T_{[n]}$ to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_{[0]} = T$). If for some integer n the range space $R(T^n)$ is closed and $T_{[n]}$ is upper (resp. a lower) semi-Fredholm operator, then T is called an upper (resp. lower) semi-B-Fredholm operator. Moreover if $T_{[n]}$ is a Fredholm (Weyl or Browder) operator, then T is called a B-Fredholm (B-Weyl or B-Browder) operator. Similarly, we can define the upper semi-B-Fredholm, B-Fredholm, B-Weyl, and B-Browder spectrums $\sigma_{SF_+}(T)$, $\sigma_{BF}(T)$, $\sigma_{BW}(T)$, $\sigma_{BB}(T)$. A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator.

(See [13]) Let $T \in B(X)$ and let

$$\triangle(T) = \{n \in \mathbb{N} : \forall m \in \mathbb{N}, m \geq n \Rightarrow [R(T^n) \cap N(T)] \subseteq [R(T^m) \cap N(T)]\}.$$

Then the degree of stable iteration dis(T) of T is defined as $dis(T) = inf \triangle (T)$.

Let T be a semi-B-Fredholm operator and let d be the degree of the stable iteration of T. It follows from [4, Proposition 2.1] that if $T_{[d]}$ is a semi-Fredholm operator, and $ind(T_{[m]}) = ind(T_{[d]})$ for each $m \geq d$. This enables us to define the index of a semi-B-Fredholm operator T as the index of the semi-Fredholm operator $T_{[d]}$.

In the case of a normal operator T acting on a Hilbert space, Berkani [5, Theorem 4.5] showed that

$$\sigma_{BW}(T) = \sigma(T) \backslash E(T),$$

$E(T)$ is the set of all eigenvalues of T which are isolated in the spectrum of T. This result gives a generalization of the classical Weyl's theorem. We say T obeys generalized Weyl's theorem if $\sigma_{BW}(T) = \sigma(T) \backslash E(T)$ ([6, Definition 2.13]).

In this paper, first we describe Browder's theorem and generalized Weyl's theorem using two new spectrum sets which we define in section 1; In section 2, we prove that Class A operators satisfies the generalized Weyl's theorem, hence Weyl's theorem holds for Class A operators.

§2 Some known results

Using Corollary 4.9 in [10], we can say that $\sigma_{BB}(T) = \sigma_D(T)$, where $\sigma_D(T) = \{\lambda \in \sigma(T) : \lambda$ is not a pole of T $\}$. We call $\sigma_D(T)$ the Drazin spectrum of T. We can prove that the Drazin spectrum satisfies the spectral mapping theorem, and the Drazin spectrum of a direct sum is the union of the Drazin spectrum of the components.

In [19], We proved that the follow result is true:

Lemma 2.1. Browder's theorem holds for T if and only if $\sigma_{BW}(T) = \sigma_D(T)$.

Lemma 2.2. If Browder's theorem holds for $T \in B(X)$ and $S \in B(X)$, and p is a polynomial, then Browder's theorem holds for

$$p\ (T) \Longleftrightarrow p\ (\sigma_{BW}(T)) = \sigma_{BW}(p\ (T));$$

$$T \oplus S \Longleftrightarrow \sigma_{BW}(T \oplus S) = \sigma_{BW}(T) \cup \sigma_{BW}(S).$$

Lemma 2.3. If $T \in B(X)$, then $\mathrm{ind(T-\lambda I)ind(T-\mu I)} \geq 0$ for each pair $\lambda,\ \mu \in \mathbb{C}\backslash\sigma_{\mathrm{e}}(\mathrm{T})$ if and only if $\mathrm{p(\sigma_{BW}(T)) = \sigma_{BW}(p(T))}$ for each polynomial p.

Lemma 2.4. $T \in B(X)$ is isoloid and generalized Weyl's theorem holds for T if and only if $\sigma_1(T) = \sigma_D(T)$.

Lemma 2.5. Let suppose $T,\ S \in B(X)$ are all isoloid. If generalized Weyl's theorem holds for T and S and if p is a polynomial, then generalized Weyl's theorem holds for

$$p\ (T) \Longleftrightarrow \sigma_1(p\ (T)) = p(\sigma_1(T))$$

and

$$T \oplus S \Longleftrightarrow \sigma_1(T \oplus S) = \sigma_1(T) \cup \sigma_1(S).$$

Lemma 2.6. $T \in B(X)$, then $\mathrm{ind(T-\lambda I)ind(T-\mu I)} \geq 0$ for each pair $\lambda,\ \mu \in \mathbb{C}\backslash\sigma_{\mathrm{e}}(\mathrm{T})$ if and only if $\mathrm{f(\sigma_1(T)) \subseteq \sigma_1(f(T))}$ for any $\mathrm{f \in H(T)}$.

For the converse, if there exist $\lambda,\ \mu \in \mathbb{C}\backslash\sigma_{\mathrm{e}}(\mathrm{T})$ for which $\mathrm{ind(T-\lambda I) = -m < 0 < k = ind(T-\mu I)}$, let $f(T) = (T-\lambda I)^k(T-\mu I)^m$. Then $0 \in f(\sigma_1(T))$ but 0 is not in $\sigma_1(f(T))$. It is a contradiction. The proof is completed.

Lemma 2.7. If $T \in B(X)$ is isoloid and generalized Weyl's theorem holds for T, then the following statements are equivalent:

(1) $ind(T-\lambda I)ind(T-\mu I) \geq 0$ for each pair $\lambda,\ \mu \in \mathbb{C}\backslash\sigma_e(T)$;

(2) $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$ for every $f \in H(\sigma(T))$;

(3) generalized Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$;

(4) $\sigma_1(f(T)) = f(\sigma_1(T))$ for every $f \in H(\sigma(T))$.

§3 Generalized Weyl's theorem for Class A operators.

In the following, let X denote a complex Hilbert space. If for all $x \in X$, $\|Tx\|^2 \leq \|T^2x\|$, then we say that T is paranormal. It is well known that if T is paranormal, then $\|T\| = \{|\lambda| : \lambda \in \sigma(T)\}$. We say that an operator $T \in B(X)$ belongs to the class A if $|T^2| \geq |T|^2$. Class A operator was first introduced by Furuta-Ito-Yamazaki [9] as a subclass of paranormal operators which includes the classess of p-hyponormal and log-hyponormal operators. The following Lemma is due to [9] and [19]:

Lemma 3.1. (1) If T is a class A operator and M is an invariant subspace of T, then $T|_M$ is also a class A operator;

(2) If T belongs to the class A and $\sigma(T) = \{0\}$, then $T = 0$;

(3) If T belongs to the class A, then T is isoloid;

(4) If T belongs to the class A and λ is non-zero complex number, then $(T-\lambda I)x = 0$ implies that $(T-\lambda I)^*x = 0$.

In [19], A. Uchiyama showed the following results:

Theorem 1. If T belongs to the class A and $\text{Ker}T|_{[TH]} = \{0\}$, then Weyl's theorem holds for T.

Theorem 2. If T belongs to the class A and $\text{Ker}T|_{[TX]} = \{0\}$ and f is an analytic function on an open neighborhood of $\sigma(T)$, then Weyl's theorem holds for $f(T)$.

In fact in these Theorems, the results are true without the condition $\text{Ker}T|_{[TH]} = \{0\}$. In the following we will show that for class A operator T, generalized Weyl's theorem holds for $f(T)$ for any $f \in H(\sigma(T))$, hence Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$. The main theorem in this section is:

Theorem 3.2. If T belongs to the class A, then generalized Weyl's theorem holds for T. Hence Weyl's theorem holds for T.

Proof. We need to prove $\sigma(T)\backslash\sigma_{BW}(T) = E(T)$.

Let $\lambda_0 \in \sigma(T)\backslash\sigma_{BW}(T)$, that is $T - \lambda_0 I$ is B-Weyl. Then there exists $\epsilon > 0$ such that $T - \lambda I$ is Weyl and $N(T - \lambda I) \subseteq \bigcap\limits_{n=1}^{\infty} R[(T - \lambda I)^n]$ if $0 < |\lambda - \lambda_0| < \epsilon$ ([7, Remark iii]). We can take $\lambda \neq 0$ if $0 < |\lambda - \lambda_0| < \epsilon$. Suppose that there exists λ such that $\lambda \in \sigma(T)$ and $0 < |\lambda - \lambda_0| < \epsilon$. By Lemma 3.1 (4), we have $N(T - \lambda I) = N[(T - \lambda I)^*]$ and it is a reducing subspace of T. Let E be the orthogonal projection onto $N(T - \lambda I)$. Then $T = \lambda E \oplus T(I - E)$ on $E(H) \oplus E(X)^{\perp}$ and $\sigma(T) = \{\lambda\} \cup \sigma(T(I - E)|_{E(X)^{\perp}})$. Since E is a finite rank projection, $ind(T(I - E) - \lambda(I - E)) = ind(T - \lambda I) = 0$ and since $[T(I - E) - \lambda(I - E)]|_{E(H)^{\perp}}$ is one-one, $[T(I - E) - \lambda(I - E)]|_{E(H)^{\perp}}$ is invertible. This implies that λ is not in $\sigma(T(I - E)|_{E(H)^{\perp}})$ and $\lambda \in iso\ \sigma(T)$. Then $T - \lambda I$ is Browder and hence $N(T - \lambda I) = N(T - \lambda I) \cap \bigcap\limits_{n=1}^{\infty} R[(T - \lambda I)^n] = \{0\}$, which means that $T - \lambda I$ is invertible. It is in contradiction to the fact that $\lambda \in \sigma(T)$. Now we have proved that $\lambda_0 \in iso\ \sigma(T)$. Then $\lambda_0 \in E(T)$. Thus $\sigma(T)\backslash\sigma_{BW}(T) \subseteq E(T)$.

Conversely, suppose $\lambda_0 \in E(T)$, that is $\lambda_0 \in iso\ \sigma(T)$ which is an eigenvalue of T.

Case 1 suppose $\lambda_0 = 0$. Using the spectral projection $P = \frac{1}{2\pi i}\int_{\partial B_0}(T - \lambda I)^{-1}d\lambda$, where B_0 is an open disk of center 0 which contains no other points of $\sigma(T)$, we can represent T as the direct sum

$$T = T_1 \oplus T_2, \text{ where } \sigma(T_1) = \{0\} \text{ and } \sigma(T_2) = \sigma(T)\backslash\{0\}.$$

Then T_2 is invertible. By Lemma 3.1 (1) and (2), $T_1 = 0$, then $T = 0 \oplus T_2$. Thus T is B-Weyl.

Case 2 suppose $\lambda_0 \neq 0$. By Lemma 3.1 (4), we have $T = \lambda_0 \oplus T_2$ on $H = N(T - \lambda_0 I) \oplus [N(T - \lambda_0 I)]^{\perp}$ and the isolatedness of $\lambda_0 \in \sigma(T)$ implies either $\lambda_0 \in iso\ \sigma(T_2)$ or $T_2 - \lambda_0 I$ is invertible. Since T_2 is a class A operator (hence T_2 is isoloid) with $N(T_2 - \lambda_0 I) = \{0\}$, then λ_0 is not in $iso\ \sigma(T_2)$, that is $T_2 - \lambda_0 I$ is invertible. By $T - \lambda_0 I = 0 \oplus (T_2 - \lambda_0 I)$, then $T - \lambda_0 I$ is B-Weyl.

From Case 1 and Case 2, we get that $E(T) \subseteq \sigma(T)\backslash\sigma_{BW}(T)$.

Now we have proved that generalized Weyl's theorem holds for T, hence Weyl's theorem holds for T.

Corollary 3.1. If T belongs to the class A, then for any $f \in H(\sigma(T))$, generalized Weyl's theorem holds for $f(T)$. Hence for every $f \in H(\sigma(T))$, Weyl's theorem holds for $f(T)$.

Proof. From Lemma 2.7 and Theorem 3.2, we only need to $T \in A_1(X)$, where $A_1(X) = \{S \in B(H) : ind(S - \lambda I)ind(S - \mu I) \geq 0 \text{ for all } \lambda, \mu \in \mathbb{C}\backslash\sigma_e(S)\}$. If $\lambda_0 \in \mathbb{C}\backslash\sigma_e(T)$, then

$T-\lambda I$ is Fredholm operator and $ind(T-\lambda I) = ind(T-\lambda_0 I)$ if $|\lambda-\lambda_0|$ is sufficiently small. We can suppose that $\lambda \neq 0$. By Lemma 3.1 (4), we know that $ind(T-\lambda I) = dimN(T-\lambda I) - dimN[(T-\lambda I)^*] \leq 0$, then $ind(T-\lambda_0 I) \leq 0$, that is $T \in A_1(X)$.

In [21], Xia proved that if T is semi-hyponormal, then $\sigma(T) = \{\lambda : \overline{\lambda} \in \sigma_a(T^*)\}$. In [1, Corollary 3.5], A. Aluthge and Derming Wang proved that if T is w-hyponormal, then $\sigma(T) - \{0\} = \{\lambda : \overline{\lambda} \in \sigma_a(T^*)\} - \{0\}$. We know that if T is w-hyponormal then T belongs to class A. We extend [21, Corollary 3.5] to the following result:

Corollary 3.2. If T belongs to the class A, then $\sigma(T) = \{\lambda : \overline{\lambda} \in \sigma_a(T^*)\}$.

Proof. We only need to prove that $\sigma(T) \subseteq \{\lambda : \overline{\lambda} \in \sigma_a(T^*)\}$. Let $\lambda_0 \in \sigma(T)$ but $\overline{\lambda_0}$ is not in $\sigma_a(T^*)$, that is $T^* - \overline{\lambda_0} I$ is bounded form below. Then $R(T-\lambda_0 I) = X$. By perturbation theorem of lower semi-Fredholm, then $R(T-\lambda I) = X$ if $|\lambda-\lambda_0|$ is sufficiently small. Thus $N[(T-\lambda I)^*] = [R(T-\lambda I)]^{\perp} = \{0\}$. Lemma 3.1 (4) asserts that $N(T-\lambda I) = \{0\}$, then $T-\lambda I$ is invertible if $|\lambda-\lambda_0|$ is sufficiently small. Thus $\lambda_0 \in iso\ \sigma(T)$. [10, page 332, Theorem 10.5] tells us that $\alpha(T-\lambda_0 I) = \beta(T-\lambda_0 I) = 0$, that is $T-\lambda_0 I$ is invertible. It is a contradiction.

References

[1] A. Aluthge and Derming Wang, w-hyponormal operators, Integr. equ. oper. theory, **36**(2000), 1-10.

[2] S. K. Berberian, An extension of Weyl's theorem to a class of not necessarily normal operators, Michigan Math. J., **16**(1969), 273-279.

[3] S. K. Berberian, The Weyl spectrum of an operator, Indiana Univ. Math. J., **20**(1970), 529-544.

[4] M. Berkani and M. Sarih, On semi B-Fredholm operator, Glasgow Math. J., **43**(2001), 457-465.

[5] M. Berkani, Index of B-Fredholm operators and generalization of a Weyl's theorem, Proc. Amer. Math. Soc., **130**(2001), 1717-1723.

[6] M. Berkani and J. J. Koliha, Weyl type theorems for bounded linear operators, Acta. Sci. Math. (Szeged), **69**(2003), 379-391.

[7] M. B erkani, On a class of quasi-Fredholm operator, Integral Equations and operator Theory, **34**(1999), 244-249.

[8] L. A. Coburn, Weyl's theorem for nonnormal operators, Michigan Math. J., 1966, 285-288.

[9] T. Furuta, M. Ito and T. Yamazaki, A subclass of paranormal operators including class of log-hyponormal and several related classes, Scientiae Mathematicas, **1**(1998), 389-403.

[10] S. Grabiner, Uniform ascent and descent of bounded operators, J. Math. Soc. Japan, **34**(1982), 317-337.

[11] R. Harte and Woo Young Lee, Another note on Weyl's theorem, Trans. Amer. Math. Soc., **349**(1997), 2115-2124.

[12] V. I. Istrătescu, On Weyl's spectrum of an operator I, Rev. Ronmaine Math. Pure Appl., **17**(1972), 1049-1059.

[13] J. Ph. Labrousse, Les opérateurs quasi-Fredholm: une gén éralisation des opérateurs semi-Fredholm, Rend. Circ. Math. Palermo, **29**(1980), No. 2, 161-258.

[14] D. C. Lay, Spectral analysis using ascent, descent, nullity and defect, Math. Ann., 1970, 197-214.

[15] M. Mbekhta and V. Mŭller, On the axiomatic theory of spectrum II, Studia Math., **119**(1996), No. 2, 129-147.

[16] K. K. Oberai, On the Weyl spectrum, Illinois J. Math., **18**(1974), 208-212.

[17] V. Rakočevič, Operators obeying a-Weyl's theorem, Rev. Roumaine Math. Pures Appl., **34**(1989), 915-919.

[18] A. E. Taylor, Theorems on ascent, descent, nullity and defect of linear operators, Math. Ann., **163**(1966), 18-49.

[19] Junhong Tian and Wansheng He, Browder's theorem and generalized Weyl's theorem, Scientia Magna, **5**(2009), 1, 58-63.

[20] A. Uchiyama, Weyl's theorem for class A operators, Math. Ineq. Appl., **4**(2001), No. 1, 143-150.

[21] H. Weyl, Über beschränkte quadratische Formen, deren Differenz vollstetig ist, Rend. Circ. Mat. Palermo, **27**(1909), 373-392.

[22] D. Xia, Spectral theory of hyponormal operators, Birkhauser Verlag, Boston, 1983.

Scientia Magna
Vol. 7 (2011), No. 1, 18-37

Weakly convex domination in graphs

R. Poovazhaki† and V. Swaminathan‡

† Department of Mathematics, EMG Yadava Women's College,
Madurai 625014, India
‡ Department of Mathematics, Saraswathi Narayan College,
Madurai 625022, India
E-mail: rpoovazhaki@yahoo.co.in sulanesri@yahoo.com

Abstract A set $D \subseteq V(G)$ is said to be a dominating set if every vertex $v \in V(G)$ is either in D or has an adjacency in D. The minimum cardinality among the dominating sets is called the domination number of G and is denoted by $\gamma(G)$. In this paper, a new parameter, called **weakly convex domination number** is being introduced and its basic properties are analysed.

Keywords Domination number, connected domination number, wcd set, weakly convex do -mination number.

§1. Introduction and preliminaries

A dominating set D is said to be a **connected dominating** set if for every $u, v \in D$, there exists an $u - v$ path in $\langle D \rangle$.

The cardinality of a minimum connected dominating set is called the **connected domination number** of G and is denoted by $\gamma_c(G)$.

A dominating set D is said to be a **weakly convex dominating** set (wcd set) if for every $u, v \in D$, either u and v are not connected or there exists a $u - v$ shortest path (of G), in $\langle D \rangle$.

The cardinality of a minimum wcd set is called the **weakly convex domination number** of G and is denoted by $\gamma_{wc}(G)$.

$\gamma(G) \leq \gamma_c(G) \leq \gamma_{wc}(G)$. In this paper, graphs with $\gamma(G) = \gamma_{wc}(G)$ and $\gamma_c(G) = \gamma_{wc}(G)$ are characterized.

§2. Weakly convex domination in graphs

A dominating set D is said to be a weakly convex dominating set (wcd set) if for every $u, v \in D$, either u and v are not connected or there exists a $u - v$ shortest path (of G), in $\langle D \rangle$.

The cardinality of a minimum wcd set is called the weakly convex domination number of G and is denoted by $\gamma_{wc}(G)$.

Observation. For any graph G,

(i) $\gamma_t(G) \leq \gamma_{wc}(G)$, where $\gamma_t(G)$ is the total domination number of G.

(ii) $\gamma(G) \leq \gamma_c(G) \leq \gamma_{wc}(G)$, where $\gamma(G)$ is the domination number and $\gamma_c(G)$ is the connected domination number of G.

Example.

(i) $\gamma_{wc}(P_2) = 1$.

(ii) $\gamma_{wc}(P_n) = n - 2$ for any positive integer integer $n \geq 3$.

(iii) $\gamma_{wc}(K_n) = 1$ for any positive integer n.

(v) $\gamma_{wc}(K_{m,n}) = 2$ for any positive integer m and n.

(vi) $\gamma_{wc}(W_n) = 1$ for any positive integer n.

Notion. The length of a smallest cycle in G is called the girth of G and is denoted by $g(G)$.

Lemma. If G is a graph with $\delta(G) \geq 2$ and $g(G) \geq 7$ then $\gamma_{wc}(G) = n$.

Proof. Let if possible there exist a proper wcd set D. Then $V - D \neq \phi$. Then for every $u \in V - D$ there exists $u_1 \in D$ such that $uu_1 \in E(G)$. $\delta(G) \geq 2$ implies there exists $u_2 (\neq u_1) \in N(u)$. Therefore, $d(u_1, u_2) \leq 2$.

If $u_2 \in D$, then $u_1, u_2 \in D$ implies there exists $u_1 \ldots u_2$ shortest path in D. This shortest path (or geodesic) must have length at most two whence it follows that G has a cycle C_n for $n \in \{3, 4\}$, a contradiction to our hypothesis that $g(G) \geq 7$. Therefore, $u_2 \notin D$ and hence $u_2 \in V - D$.

Then $N(u_2) \cap D \neq \phi$. Therefore u_2 may be adjacent to u_1 or there exists $u_3 (\neq u_1) \in D$ such that $u_3 u_2 \in E(G)$.

If $u_2 u_1 \in E(G)$ then there will exist a 3 cycle.

If there exists $u_3 (\neq u_1) \in D$ such that $u_3 u_2 \in E(G)$, then $d(u_1, u_3) \leq 3$. $u_1, u_3 \in D$ implies there exists an $u_1 \ldots u_3$ shortest path in $\langle D \rangle$.

If $d(u_1, u_3) = 1$ then there will exist a 4 cycle.

If $d(u_1, u_3) = 2$ then there will exist a 5 cycle.

If $d(u_1, u_3) = 3$ then there will exist a 6 cycle.

$\ldots$

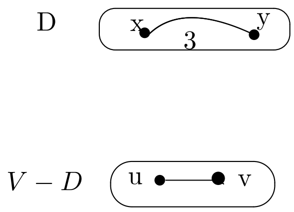
A contradiction. (since $g(G) \geq 7$) and therefore G cannot have a proper wcd set. (i.e.) $\gamma_{wc}(G) = n$.

Corollary. If $\gamma_{wc}(G) < n$ then either $\delta \leq 1$ (or) $g(G) \leq 6$.

Remark. From Lemma 1 we can conclude that there exist infinitely many number of graphs G with $\gamma_{wc}(G) = n$.

Observation. $\gamma_{wc}(C_n) = n - 2$ for any integer $3 \leq n \leq 6$ and is equal to n for $n \geq 7$.

Proof. (i) If $3 \leq n \leq 6$, then choose any $u, v \in V(C_n)$ with $uv \in E(G)$ and consider $D = V(C_n) - \{u, v\}$. Then D is a dominating set of C_n. For any $x, y \in D$, if there exists no $x \ldots y$ shortest path in $\langle D \rangle$, then $d_{\langle D \rangle}(x, y) > d(x, u) + d(u, v) + d(v, y) \geq 3$. Therefore $d_{\langle D \rangle}(x, y) > 3$. (ie) $d_{\langle D \rangle}(x, y) \geq 4$.



Then $x..yvu..x$ is a cycle of length atleast 7, a contradiction. Hence, for any two $x, y \in D$ there exists an $x \ldots y$ shortest path in D. (i.e.) D is a wcd set. Therefore $\gamma_{wc}(C_n)) \leq n-2$. $\gamma_c(C_n) = n - \epsilon_T(G)$ where $\epsilon_T(G)$ is the maximum number of pendant vertices in any spanning tree T_G of G. (i.e.) $n-2 = \gamma_c(C_n) \leq \gamma_{wc}(C_n) \leq n-2$. Hence $\gamma_{wc}(C_n) = n-2$ for n, $3 \leq n \leq 6$.

(ii) If $n \geq 7$, by Lemma 1 we can conclude that $\gamma_{wc}(C_n) = n$.

Remark. From the above observation we can conclude that the following disconnected graph has no proper wcd set.

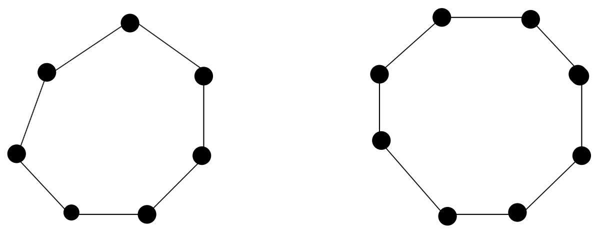


fig. 3(a)

Remark. If G is a disconnected graph and D_1 is a proper wcd of a component C_1 of G, then $(V(G) - V(C_1)) \cup D_1$ is a proper wcd set of G. And if no component of G has a proper wcd set, then G cannot have a proper wcd set. (i.e.) A disconnected graph G has a proper wcd set if and only if there exists a component C_1 of G with proper wcd set. Hence wcd sets in a disconnected graph can be analysed by studying the wcd sets of it's components. For this purpose first we make a complete study of wcd sets in connected graphs. In the following discussion by a graph we always mean a connected graph.

Remark. From the Lemma 1 we can conclude that $1 \leq \gamma_{wc}(G) \leq n$.

Remark. $\gamma_{wc}(G) = 1$ if and only if G has a vertex of full degree.

Remark. Every graph with $diam(G) \leq 2$ has a proper wcd set.

Proof. If $diam(G) = 1$ with $n \geq 2$ then G is K_n. Therefore $D = \{u\}$ is a wcd set for any $u \in V(G)$. If $diam(G) = 2$ then for any $u \in V(G)$ with $deg u = \Delta$, $N[u]$ is a wcd set. For if $x, y \in N[u]$ either $xy \in E(G)$ (or) $\langle x, u, y\rangle$ is connected. Therefore $N[u]$ is weakly convex. Let $v \in V - N[u]$. Then v must be adjacent to atleast one $v_1 \in N(u)$. For otherwise $d(u, v) \geq 3$. Hence for every $v \in V - N[u]$ there exists $v_1 \in N(u)$ such that $vv_1 \in E(G)$. Therefore $N[u]$ is a dominating set.

Remark. All graphs (with $diam(G) > 2$) need not have a proper wcd set.

Example. C_n with $n \geq 7$ has no proper wcd set.

Observation. For any tree T of order n, $\gamma_{wc}(T) = n - \epsilon$ where ϵ denotes the number of pendant vertices of the given tree T.

Proof. $D = V(T) - A$, where A is the set of all pendant vertices of T, is a wcd set of T. Hence $\gamma_{wc}(T) \leq n - \epsilon$. Therefore $n - \epsilon = \gamma_c(T) \leq \gamma_{wc}(T) \leq n - \epsilon$. (i.e.) $\gamma_{wc}(T) = n - \epsilon$.

Remark. $\gamma_{wc}(T) = n - 2$ if and only if T is a path.

Proof. Let $\gamma_{wc}(T) = n - 2$ for a tree.

Claim. T is a path. (i.e.) To prove that T has exactly two pendant vertices.

Let A be the set of all pendant vertices of T. Then $D = V(T) - A$ is a wcd set and hence $\gamma_{wc}(T) \leq n - |A| \leq n - 2$ (since $|A| \geq 2$ for any non trivial tree). If $|A| > 2$ then $n - 2 = \gamma_{wc}(T) \leq n - |A| < n - 2$ ... a contradiction. Hence T is a tree with exactly two pendant vertices. (i.e.) T is a path.

Conversely, if T is a path then $\gamma_{wc}(T) = n - 2$.

Notation. $\epsilon_T(G)$ denote the number of pendant vertices of a spanning tree T_G (of a connected graph) G with maximum number of pendant edges.

Observation. If G is a unicyclic graph then $\gamma_{wc}(G) = n - \epsilon_T(G)$ (or) $n - \epsilon_T(G) + 1$ (or) $n - \epsilon_T(G) + 2$.

Proof. If A is the set of all pendant vertices of a spanning tree T_G of G with maximum number of pendant edges, then $|A| = \epsilon_T(G)$. Let C_r be the unique cycle of G where r denote the length of the cycle C_r. Let $B = \{u \in V(C_r)/N(u) \cap (V(G) - V(C_r)) = \phi\}$.

case(i): $B = \phi$.

In this case for any $r \geq 3$, $N(u) \cap (V(G) - V(Cr)) \neq \phi$ for each $u \in V(C_r)$. Then $A \cap B = \phi$ and $D = V(G) - A$ is a wcd set. Therefore $\gamma_{wc}(G) \leq n - \epsilon_T(G)$. Hence $n - \epsilon_T(G) = \gamma_c(G) \leq \gamma_{wc}(G) \leq n - \epsilon_T(G)$. (i.e.) $\gamma_{wc}(G) = n - \epsilon_T(G)$.

Case(ii): $B \neq \phi$ and B is independent.

(a) $r = 3$

B is an independent subset of a 3 cycle implies $|B| = 1$. Let $B = \{x\}$ where $x \in V(C_3) = \{x, y, z\}$. In this case G can be obtained by taking a $C_3 : xyzx$ and attaching one or more nontrivial trees at two vertices y and z.

Therefore $N(y) \cap (V(G) - V(C_3)) \neq \phi$, $N(z) \cap (V(G) - V(C_3) \neq \phi$ and

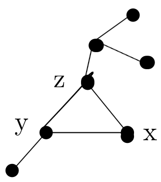


fig. 4

$N(x) \cap (V(G) - V(C_3)) = \phi$. (i.e.) x is a pendant vertex in any spanning tree of G. Therefore $x \in A$, and $D = V(G) - A$ is a wcd set. Therefore $\gamma_{wc}(G) = n - \epsilon_T(G)$ as in case (i).

(b) $r = 4$

B is an independent subset of a 4 cycle implies $|B| \leq 2$.

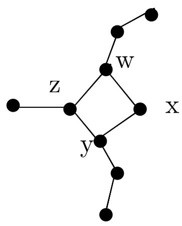


fig. 5

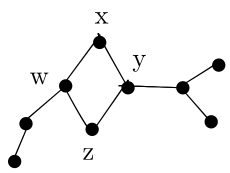


fig. 6

If $|B| = 1$, then let $B = \{x\}$ where $x \in V(C_4) = \{x, y, z, w\}$. In this case G can be obtained by taking a 4 cycle $C_4 : xyzwx$ and attaching one or more nontrivial trees at three consecutive vertices y, z, w. Then as in case (i) $\gamma_{wc}(G) = n - \epsilon_T(G)$.

If $|B| = 2$, then let $B = \{x, z\}$. In this case G can be obtained by taking a 4 cycle $C_4 : xyzwx$ and attaching one or more nontrivial trees at two non consecutive vertices y and w. Then either x (or) z is in A. Both x and z cannot be in A. Since the possible spanning trees are:

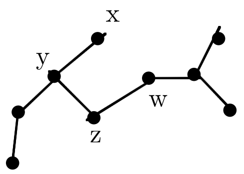


fig. 7

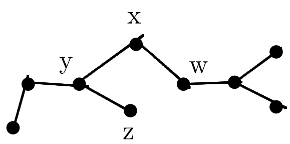
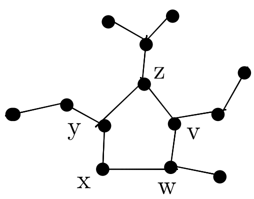
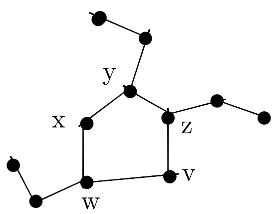


fig. 8

Again as in case (i) $\gamma_{wc}(G) = n - \epsilon_T(G)$.

(c) $r = 5$





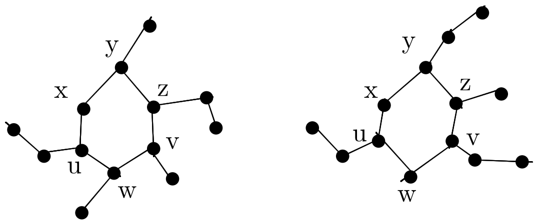
B is an independent subset of a 5 cycle implies $|B| \leq 2$.

If $|B| = 1$ let $B = \{x\}$ where $x \in V(C_5) = \{x, y, z, v, w\}$. In this case G can be obtained by taking a 5 cycle $C_5 : xyzvwx$ and attaching one or more nontrivial trees at four consecutive vertices y, z, v, and w. $N(x) \cap (V(G) - V(C_5)) = \phi$. (i.e.) x is a pendant vertex in any spanning tree of G and therefore $x \in A$. But $D = V(G) - A$ is not a wcd set. (since $d_{\langle D \rangle}(y, w) = 3 > d_G(y, w) = 2$). But $D' = (V(G) - A) \cup \{x\}$ is a wcd set. Hence $\gamma_{wc}(G) = n - \epsilon_T(G) + 1$.

If $|B| = 2$, let $B = \{x, v\}$. In this case G can be obtained by taking $C_5 : xyzvwx$ and attaching one or more nontrivial trees at y, z, w. Then either x or v is in A. Both x and v cannot be in A. If $x \in A$, then $d_{\langle D \rangle}(w, y) = 3 > d_G(w, y) = 2$ and hence $D = V(G) - A$ is not a wcd set. But $D' = (V(G) - A) \cup \{x\}$ is a wcd set. Similarly $D' = (V(G) - A) \cup \{y\}$ is a wcd set. Therefore $\gamma_{wc}(G) = n - \epsilon_T(G) + 1$.

(d) $r = 6$

B is an independent subset of a 6 cycle implies $|B| \leq 3$.



Then the possible independent sets are: $\{x\}$, $\{x, w\}$ and $\{x, z, v\}$. Arguing as in the previous case we get

$$\gamma_{wc}(G) = n - \epsilon_T(G) + 1$$

.

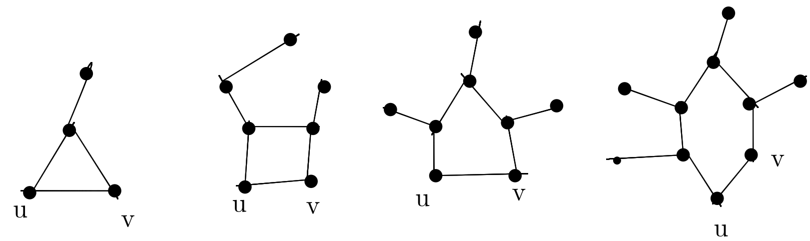
(e) $r \geq 7$

B is independent subset of C_r with $r \geq 7$ implies $|B| = [r/2]$ and $|A \cap B| = 1$. If $x \in A \cap B$ and $x_1, x_2 \in N(x) \cap V(C_r)$. $r \geq 7$ implies there exists no $x_1 \dots x_2$ shortest path in $V(G) - \{x\}$. Therefore $D = V(G) - A$ is not a wcd set. But $D' = (V(G) - A) \cup \{x\}$ is a wcd set. Hence $\gamma_{wc}(G) = n - \epsilon_T(G) + 1$.

Case(iii): B is not independent.

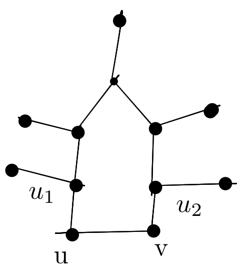
Then there exists $u, v \in B$ with $uv \in E(C_r)$. Therefore $N(u) \cap (V(G) - V(C_r)) = \phi$ and $N(v) \cap (V(G) - V(C_r)) = \phi$. Therefore u, v are pendant vertices in any spanning tree of G. (i.e.) $u, v \in A$.

(iii) - (a): $3 \leq r \leq 6$.



Then $D = V(G) - A$ is a wcd set of G and hence $\gamma_{wc}(G) = n - \epsilon_T(G)$.

(iii) - (b): $r \geq 7$.



Then $D = V(G) - A$ is not a wcd set. (since $d_{\langle D \rangle}(u_1, v_1) = 4 > d_G(u_1, v_1) = 3$ where $u_1 \in N(u)$ and $v_1 \in N(v)$). But $D' = (V(G) - A) \cup \{u, v\}$ is a wcd set. Therefore $\gamma_{wc}(G) = n - \epsilon_T(G) + 2$.

If G is a graph with $\delta(G) = 1$, then $D = V(G) - \{u\}$ where $\deg u = \delta$ is a wcd set of G (i.e). G has a proper wcd set.

Lemma. If B is a block of a separable graph G with wcd set B' containing all cut vertices belonging to B then $(V - B) \cup B'$ is a wcd set of G.

Proof. Let $D = (V - B) \cup B'$. Then for each $u \in V - D = V - [(V - B) \cup B'] = B - B'$ there exists $v \in B'$ such that $uv \in E(G)$ (since B' is a wcd set of B). Therefore D is a dominating set of G.

Let $x, y \in (V - B) \cup B'$.

Case I: Every block of G is incident at the same cut vertex. (i.e.) G has exactly one cut vertex, say w. As B' contains all cut vertices belonging to B, $w \in B'$.

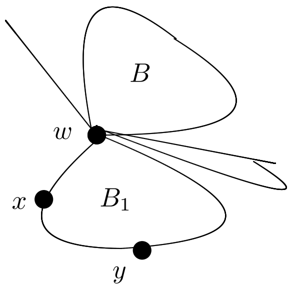


fig. 9

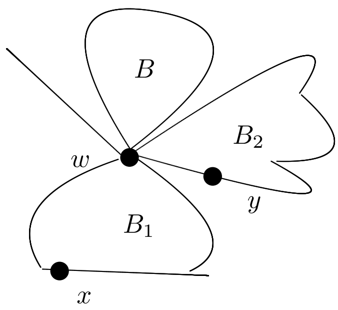


fig. 10

I - (a): $x, y \in V - B$.

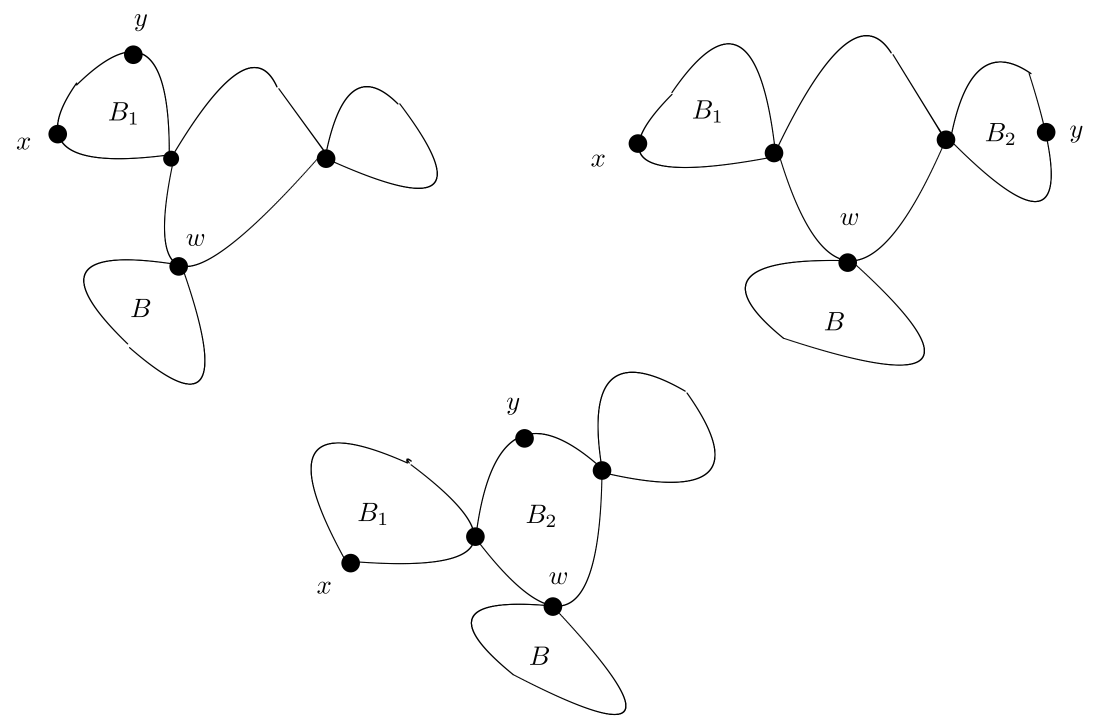
First, we observe that every $x - y$ shortest path (of G) in $\langle (V - B) \cup \{w\} \rangle$ has no vertex from $B - \{w\}$ and it may or may not contain w. (i.e.) there exists an $x - y$ shortest path (of G) in $\langle (V - B) \cup \{w\} \rangle$ not containing w or containing w. If this $x - y$ shortest path does not contain w then this path completely lies in $\langle V - B \rangle$. If it conains w then this path is contained in $\langle (V - B) \cup B' \rangle$.

I - (b): $x \in V - B$ and $y \in B'$.

Then the $x - w$ shortest path of G in $\langle (V - B) \cup \{w\} \rangle$ together with the $w - y$ shortest path in B' gives an $x - y$ shortest path of G in $\langle (V - B) \cup B' \rangle$.

Case II: G has at least two cut vertices.

II - (a): If B is an end block then B has exactly one cut vertex, say w. Then arguing as in the previous case we get the result.



II - (b): If B is not an end block then B may have more than one cut vertex. Let $\{w_1, w_2, \ldots, w_r\}$ be the set of cut vertices belonging to B. Then $\{w_1, w_2, \ldots, w_r\} \subseteq B'$. As B' is a wcd set of B there exists a shortest path connecting any two cut vertices w_i and w_j in $\langle B' \rangle$. Again, for any two $x, y \in (V - B)$, every $x - y$ shortest path (of G) in $\langle (V - B) \cup \{w_1, w_2, \ldots, w_r\} \rangle$ has no vertex from $B - \{w_1, w_2, \ldots, w_r\}$ and it may or may not contain some or all of $w_1, w_2, \ldots, w_r$. (i.e.) there exists an $x - y$ shortest path (of G) in $V - B$ not containing any of the cut vertices $w_1, w_2, \ldots, w_r$ or containing some or all of $w_1, w_2, \ldots, w_r$.

II - (b) - (i): $x, y \in (V - B)$.

If the $x - y$ shortest path has no w_i then the $x - y$ shortest path completely lies in $\langle V - B \rangle$. If it contain some or all of w_i then the $x - y$ shortest path lies in $\langle (V - B) \cup B' \rangle$.

II - (b) - (ii): $x \in (V - B)$ and $y \in B'$.

Then there exists an $x - w_i$ shortest path in $\langle (V - B) \cup \{w_i\} \rangle$ for every i. $w_i, y \in B'$ implies there exists an $w_i - y$ shortest path in $\langle B' \rangle$. Hence $x - w_i - y$ is an $x - y$ shortest path in $\langle (V - B) \cup B' \rangle$.

In both cases I and II, if $x, y \in B'$ then as B' is a wcd set of B there exists an $x - y$ shortest path (of B) which is also an $x - y$ shortest path of G is in B'.

Hence in all cases $(V - B) \cup B'$ is a wcd set of G.

Notation. The length of a longest cycle in G is called the circumference of G and is denoted by $c(G)$.

Definition. In a separable graph G, a block with at most one cut vertex is called an end block.

lemma. If G is a block with $3 \leq c(G) \leq 6$, then $\gamma_{wc}(G) < n$.

Proof. Case (i): $c(G) = 3$.

Let C_r be a cycle with $r = 3$. If $G = C_r$, then obviously $\gamma_{wc}(G) < n$. If $G \neq C_r$ then there exists an edge $uv \in E(G)$ such that $u \in V(C_r)$ and $v \in V(G) - V(C_r)$. Take any other $u' \in V(C_r)$. Then uv is an edge and u' is another vertex and G is a block implies there exists a cycle containing uv and u'. But if there exists a cycle containing uvu' then $c(G) > 3$ which is not true. Therefore there cannot exist $v \in V(G) - V(C_r)$. (i.e.) $G = C_3$ and $D = \{u\}$ is a wcd set. Hence $\gamma_{wc}(G) = 1 < n$.

Case (ii): $c(G) = 4$.

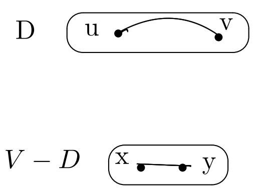
G is a block with $c(G) = 4$ implies $\mathrm{diam}(G) \leq 2$. If $\mathrm{diam}(G) = 1$, then $D = \{u\}$ is a wcd set of G for any $u \in V(G)$. If $\mathrm{diam}(G) = 2$, then for any $u \in V(G)$, $D = N[u]$ is a wcd set. For if, $v \in V(G) - N[u]$ then v cannot be adjacent to u. Therefore v must be adjacent to some $v_1 \in N(u)$. (otherwise $d(v, u) > 2$). (i.e.) D is a dominating set. For any $x, y \in N[u]$ there exists $x \ldots y$ shortest path in $N[u]$. (i.e.) $D = N[u]$ is a wcd set. Therefore $\gamma_{wc}(G) \leq degu + 1 < n$.

Case (iii): $c(G) = 5$.

G is a block with $c(G) = 5$ implies $\mathrm{diam}(G) \leq 2$. Then G has a proper wcd set as in the previous case. Therefore $\gamma_{wc}(G) < n$.

Case (iv): $c(G) = 6$.

G is a block with $c(G) = 6$ implies $\mathrm{diam}(G) \leq 3$. If $\mathrm{diam}(G) \leq 2$, then G has a proper wcd set as in the previous cases. If $\mathrm{diam}(G) = 3$, choose any $x, y \in V(G)$ with $xy \in E(G)$ and consider $D = V(G) - \{x, y\}$. Then $\langle D \rangle$ is connected and D is a dominating set (since G is a block). Therefore for any $u, v \in D$ there exists an $u \ldots v$ path in $\langle D \rangle$ Suppose there exists no $u \ldots v$ shortest path in $\langle D \rangle$. Then every $u \ldots v$ shortest path must pass through x, y.



Therefore $d_{\langle D \rangle}(u, v) > d_G(u, x) + d_G(x, y) + d_G(y, v) > 3$. (i.e.) $d_{\langle D \rangle}(u, v) \geq 4$. Hence there will exist a cycle of length at least $7 \ldots$a contradiction (since $c(G) = 6$). Therefore there exists an $u \ldots v$ shortest path in D. D is a wcd set. Hence $\gamma_{wc}(G) < n$

Lemma. If G is a separable graph with $\delta(G) \geq 2$ and $3 \leq c(G) \leq 6$, then $\gamma_{wc}(G) < n$.

Proof. $c(G) \leq 6$ implies $c(B) \leq 6$ for any block B of G. Choose an end block B. Then B has at most one cut vertex.

Case (i): $c(B) = 3$.

In this case $B' = \{u\}$ is a proper wcd set of B for any $u \in V(B)$ (By the previous lemma). Choose u to be a cut vertex belonging to B. As B can contain at most one cut vertex, B' is a proper wcd set containing all cut vertices belonging to B. Therefore $D = (V - B) \cup B'$ is a wcd set of G. (i.e.) $\gamma_{wc}(G) \leq n - |B - B'| < n$.

Case (ii): $c(B) = 4$.

Then $\mathrm{diam}(B) \leq 2$. If $\mathrm{diam}(B) = 1$, then $B' = \{u\}$ or $B' = N_B[u]$ is a wcd set for B for any cut vertex $u \in B$. (i.e) B' is a wcd set of B containing all cut vertices belonging to B. Therefore $D = (V - B) \cup B'$ is a wcd set of G. Hence $\gamma_{wc}(G) \leq n - |B - B'| < n$.

Case (iii): $c(B) = 5$.

Then $\mathrm{diam}(B) \leq 2$. Then also $B' = \{u\}$ or $B' = N_B[u]$ is a proper wcd set for any cut vertex u belonging to B. Hence $\gamma_{wc}(G) \leq |D| \leq n - |B - B'| < n$.

Case (iv): $c(B) = 6$.

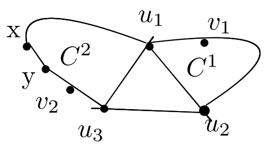
Choose any two vertices $x, y \in V(B)$ with $xy \in E(B)$ and neither x nor y is a cut vertex. Such an edge exists since B is an end block with $\delta(G) \geq 2$. Let $B' = V(B) - \{x, y\}$. Then B' is a wcd set of B containing the cut vertex belonging to B. Therefore $D = (V - B) \cup B'$ is a wcd set of G. Therefore $\gamma_{wc}(G) \leq |D| \leq n - |B - B'| < n$.

Observation. If G is a block with $g(G) = 3$ and $c(G) \leq 12$ then $\gamma_{wc}(G) < n$.

Proof. Let C be a cycle of length 3 with $V(C) = \{u_1, u_2, u_3\}$.

Case (i): $N(u_i) \cap (V(G) - V(C)) = \phi$ for some $u_i \in V(C)$, $i = 1, 2, 3$, then $D = V(G) - \{u_i\}$ is a wcd set of G and hence $\gamma_{wc}(G) < n$.

Case (ii): $N(u_i) \cap (V(G) - V(C)) \neq \phi$ for each $i \in \{1, 2, 3\}$. Let $v_1 \in N(u_1) \cap (V(G) - V(C))$.



G is a block implies there exists a cycle C^1 containing $u_1v_1 (= e)$ and u_2. Also $N(u_3) \cap (V(G) - V(C)) \neq \phi$. Let $v_2 \in N(u_3) \cap (V(G) - V(C))$. Then there exists a cycle C^2 containing u_1v_2 and u_3. $c(G) \leq 12$ implies $l(C^1)$ (or) $l(C^2)$ is less than or equal to 6. If $l(C_2) \leq 6$ then choose $x, y \in V(C^2)$ with $xy \in E(C^2)$ and $x, y \neq \{u_1, u_3\}$. Then $D = V(G) - \{x, y\}$ is a wcd set. Hence $\gamma_{wc}(G) < n$.

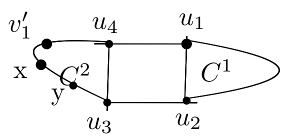
Observation. If G is a block with $g(G) = 4$ and $c(G) \leq 13$ then $\gamma_{wc}(G) < n$.

Proof. Let C be a cycle of length 4 with $V(C) = \{u_1, u_2, u_3, u_4\}$.

Case (i): $N(u_i) \cap (V(G) - V(C)) = \phi$ for some $u_i \in V(C)$.

Then $D = V(G) - \{u_i\}$ is a wcd set of G and hence $\gamma_{wc}(G) < n$.

Case (ii): $N(u_i) \cap (V(G) - V(C)) \neq \phi$ for each $i \in \{1, 2, 3, 4\}$. Let $v_1 \in N(u_1) \cap (V(G) - V(C))$. G is a block implies there exists a cycle C^1 containing $e = u_1v_1$ and u_2.



$N(u_4) \cap (V(G) - V(C)) \neq \phi$ implies there exists $v_1' \in N(u_4) \cap (V(G) - V(C))$. Therefore there exists a cycle C^2 containing u_4v_1' and u_3. $c(G) \leq 13$ implies $l(C^1)$ or $l(C^2) \leq 6$. Let

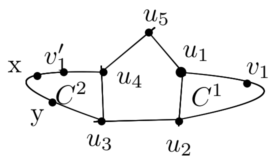
$l(C^2) \leq 6$. Then $D = V(G) - \{x, y\}$ for any two $x, y \in V(C^2)$ with $xy \in E(C^2)$ and $x, y \notin \{u_4, u_3\}$ is a wcd set. Therefore $\gamma_{wc}(G) < n$.

Observation. If G is block with $g(G) = 5$ and $c(G) \leq 14$, then $\gamma_{wc}(G) < n$.

Proof. Let C be a cycle of length 5 with $V(C) = \{u_1, u_2, u_3, u_4, u_5\}$.

Case (i): $N(u) \cap (V(G) - V(C)) = \phi$ and $N(v) \cap (V(G) - V(C)) = \phi$ for some $u, v \in V(C)$ with $uv \in E(C)$ then, $D = V(G) - \{u, v\}$ is a wcd set. Therefore $\gamma_{wc}(G) < n$.

Case (ii): $N(u) \cap (V(G) - V(C)) = \phi$ and $N(v) \cap (V(G) - V(C)) = \phi$ for no two adjacent vertices $u, v \in V(C)$. Let $N(u_5) \cap (V(G) - V(C)) = \phi$. Then $N(u_1) \cap (V(G) - V(C)) \neq \phi$ and $N(u_4) \cap (V(G) - V(C)) \neq \phi$. Let $v_1 \in N(u_1) \cap (V(G) - V(C))$. G is a block implies there exists a cycle C^1 containing $e = u_1v_1$ and u_2. $N(u_4) \cap (V(G) - V(C)) \neq \phi$ implies there exists $v_1' \in N(u_4) \cap (V(G) - V(C))$.



Therefore there exists a cycle C^2 containing u_4v_1' and u_3. $c(G) \leq 14$ implies $l(C^1)$ or $l(C^2) \leq 6$. Let $l(C^2) \leq 6$. Then $D = V(G) - \{x, y\}$ for any two $x, y \in V(C^2)$ with $xy \in E(C^2)$ and $x, y \in \{u_4, u_3\}$ is a wcd set. Therefore $\gamma_{wc}(G) < n$.

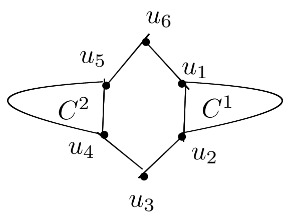
Observation. If G is a block with $g(G) = 6$ and $c(G) \leq 15$ then $\gamma_{wc}(G) < n$.

Proof. Let C be a cycle of length 6 with $V(C) = \{u_1, u_2, u_3, u_4, u_5, u_6\}$.

Case (i): $N(u) \cap (V(G) - V(C)) = \phi$ and $N(v) \cap (V(G) - V(C)) = \phi$ for some $u, v \in V(C)$ with $uv \in E(C)$, then, $D = V(G) - \{u, v\}$ is a wcd set. Therefore $\gamma_{wc}(G) < n$.

Case (ii): $N(u) \cap (V(G) - V(C)) = \phi$ and $N(v) \cap (V(G) - V(C)) = \phi$ for no two adjacent vertices $u, v \in V(C)$. Let $N(u_6) \cap (V(G) - V(C)) = \phi$. Then $N(u_1) \cap (V(G) - V(C)) \neq \phi$ and $N(u_5) \cap (V(G) - V(C)) \neq \phi$.

Let $v_1 \in N(u_1) \cap (V(G) - V(C))$. G is a block implies there exists a cycle C^1 containing $e = u_1v_1$ and u_2.



$N(u_5) \cap (V(G) - V(C)) \neq \phi$ implies there exists $v_1' \in N(u_5) \cap (V(G) - V(C))$. Therefore there exists a cycle C^2 containing u_5v_1' and u_4. $c(G) \leq 14$ implies $l(C_1)$ or $l(C^2) \leq 6$. Let $l(C^2) \leq 6$. Then $D = V(G) - \{x, y\}$ for any two $x, y \in V(C^2)$ with $xy \in E(C^2)$ and $x, y \in \{u_4, u_5\}$ is a wcd set. Therefore $\gamma_{wc}(G) < n$.

Definition.[7] A graph G is distance-hereditary if for all connected induced subgraphs F of G, $d_F(u, v) = d_G(u, v)$ for all $u, v \in V(F)$.

Observation. For every distance hereditary graph G, $\gamma_{wc}(G) < n$.

Proof. Consider any spanning tree T_G of T_G. Then $\langle V(T_G) - A\rangle$ where A is the set of all pendant vertices of T_G is distance preserving. (i.e.) $d_{\langle V(T_G)-A\rangle}(x, y) = d_G(x, y)$ for all $x, y \in V(T_G) - A$. Therefore for every $x, y \in V(T_G) - A$ there exists an $x \dots y$ shortest path in $\langle V(T_G) - A\rangle$. Also $V(T_G) - A$ is a dominating set. Hence $\gamma_{wc}(G) < n$

CHARACTERIZATIONS

We use the following notations throught the remaining discussion.

Notation. T_G denote any spanning tree of G.

A - denote the set of all pendant vertices in T_G.

T'_G denote the subtree obtained by deleting all pendant vertices from T_G. (i.e.) $T'_G = T_G - A$.

T''_G denote the subtree obtained by deleting a proper subset $A' \subset A$ from T_G. (i.e.) $T''_G = T_G - A'$.

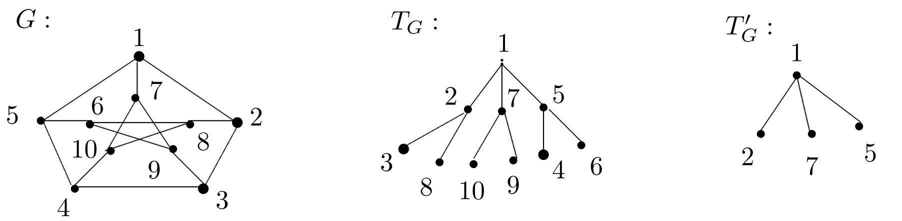
$\langle V(T'_G)\rangle$ denote the subgraph of G induced by $V(T'_G)$.

$\langle V(T''_G)\rangle$ denote the subgraph induced by $V(T''_G)$.

Definition. A subset $V(H)$ of $V(G)$ is said to be distance preserving if $d_{\langle V(H)\rangle}(x, y) = d_G(x, y)$ for all $x, y \in V(H)$.

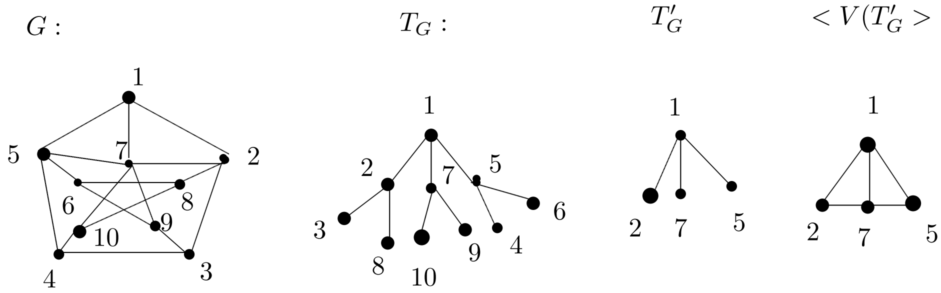
Remark. If G has a wcd set D, then for every $x, y \in D$ there exists an $x \dots y$ shortest path in $\langle D\rangle$. $d_{\langle D\rangle}(x, y) = d_G(x, y)$ for all $x, y \in D$ and hence D is distance preserving.

Example.

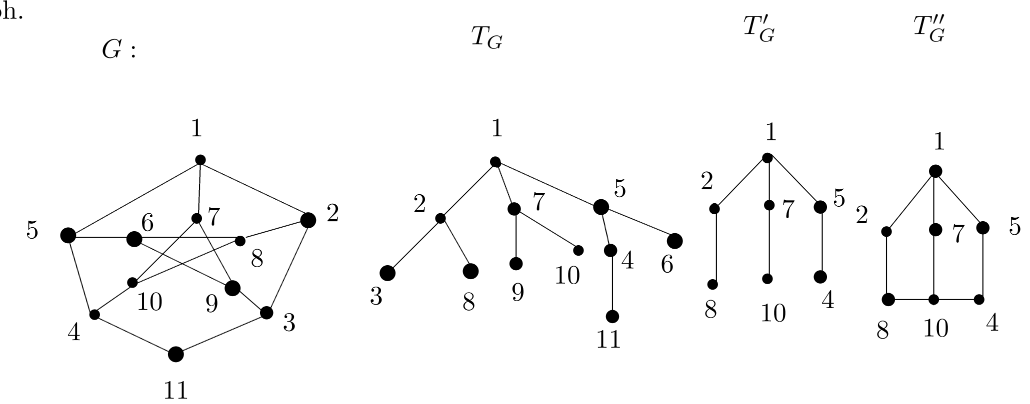


Here $A = \{3, 8, 10, 9, 4, 6\}$ $d_{T'_G}(x, y) = d_G(x, y)$ for all $x, y \in V(T'_G)$.

Example.



$d_{T'_G}(2,7) = 2 \neq d_G(2,7) = 1$. Therefore $d_{T'_G}(x,y) \neq d_G(x,y)$ for all $x, y \in V(T'_G)$. But $d_{\langle V(T'_G)\rangle}(x,y) = d_G(x,y)$ for all $x, y \in \langle V(T'_G)\rangle$ and hence $\langle V(T'_G)\rangle$ is a distance preserving subgraph.



Example.

Here $A = \{3, 8, 9, 10, 11, 6\}$. $d_{T'_G}(x,y) \neq d_G(x,y)$ for all $x, y \in V(T'_G)$ (since $d_{T'_G}(4,7) = 3$ and $d_G(4,7) = 2$). $A' = \{3, 9, 11, 6\}$ then $A' \subset A$. T''_G is not distance preserving. But $\langle V(T''_G)\rangle$ is distance preserving.

Observation. If $\delta(G) = 1$ then $\gamma_{wc}(G) < n$ as $D = V(G) - \{u \in V(G)/degu = 1\}$ is a wcd set.

Observation. If G is a graph with $\delta(G) \geq 2$, then $\gamma_{wc}(G) < n$ if and only if G is a tree or G has a spanning tree T_G satisfying one of the following three conditions:

(i) T_G has a subtree T'_G such that $d_{T'_G}(x,y) = d_G(x,y)$ for all $x, y \in V(T'_G)$.

(ii) T_G has a subtree T'_G such that $\langle V(T'_G)\rangle$ is distance preserving.

(iii) T_G has a subtree T''_G such that $\langle V(T''_G)\rangle$ is distance preserving.

Proof. $\gamma_{wc}(G) < n$ implies that there exists a proper wcd set D. Then D is a dominating set with $\langle D\rangle$ is connected and distance preserving. $\langle D\rangle$ is connected implies $\langle D\rangle$ has a spanning tree $T_{\langle D\rangle}$. Hence $D = V(T_{\langle D\rangle})$ and each vertex in $V - D$ is adjacent to some vertex in $V(T_{\langle D\rangle})$. Deleting the edges in $\langle V - D\rangle$ we get a spanning tree T_G of G. (i.e.) there exists a spanning tree T_G of G such that $D = V(T_G) - A'$ where $A' \subseteq A$ and $\langle D\rangle = \langle V(T_G) - A'\rangle$ is distance preserving.

Case (I): $\langle D\rangle = \langle V(T_G - A')\rangle = T'_G$ with T'_G is distance preserving and $A = A'$.

I - (i): $\langle A\rangle$ is independent.

If $u \in A$ then there exists $u_1 \in D = V(T'_G)$ such that $uu_1 \in E(G)$. $\delta(G) \geq 2$ implies there exists $u_2 (\neq u_1) \in V(G)$ such that $u_2 \in N(u)$. A is independent implies $u_1, u_2 \in D = V(T'_G)$. Therefore there exists an $u_1 \ldots u_2$ shortest path in $\langle D\rangle = T'_G$. (i.e.) there exists an $u_1 \ldots u_2$ shortest path in the subtree T'_G.

Therefore if $u, v \in A$ then $|N(u) \cap V(T'_G)| > 1$ and $|N(v) \cap V(T'_G)| > 1$. Hence if $u_1 \in |N(u) \cap V(T'_G)| > 1$ and $v_1 \in |N(v) \cap V(T'_G)| > 1$ then there exists an $u_1 \ldots v_1$ shortest path in T'_G. (i.e) every shortest path connecting u snd v should pass through the points of T'_G and T'_G is a distance preserving subtree. Therefore $d_{T_G}(u,v) = d_G(u,v)$ for all $u, v \in A$. T'_G is a distance preserving subtree implies $d_{T'_G}(x,y) = d_G(x,y)$ for all $x, y \in V(T'_G)$. Hence $d_{T_G}(x,y) = d_G(x,y)$ for all $x, y \in V(G)$. (i.e.) T_G is a distance preserving spanning tree of G. Hence no two pendant vertices of T_G are adjacent in G (since if u, v are pendant vertices in T_G with $uv \in E(G)$ then $1 = d_G(u, v) < d_{T_G}(u, v)$). T'_G is distance preserving implies no two pendant vertices of T'_G are adjacent in G. Similarly no two pendant vertices in the tree obtained by removing the pendant vertices of T'_G are adjacent. Proceeding like this we get G as an acyclic connected graph. Hence G is a tree.

I - (ii): $\langle A\rangle$ is not independent.

$\langle A\rangle$ is not independent implies there exist at least two adjacent vertices u, v in $\langle A\rangle$. Then $d_{T_G}(u, v) > d_G(u, v) = 1$. (i.e.) T_G is not distance preserving. But T'_G is distance preserving. Hence G has a spanning tree T_G with distance preserving subtree T'_G. Hence condition (ii) is satisfied.

Case (II): $D = V(T'_G)$ and T'_G is not distance preserving. A' may or may not be independent and $A = A'$.

$\langle D\rangle = \langle V(T_G) - A\rangle = \langle V(T_{G'})\rangle$ is distance preserving. (i.e.) G has a spanning tree T_G with a subtree T'_G such that the subgraph induced by $V(T'_G)$ is distance preserving. Hence condition (iii) is satisfied.

Case (III): T'_G is not distance preserving and $A' \subset A$ and A' may or may not be independent.

Then $\langle D\rangle = \langle V(T_G) - A'\rangle$ is distance preserving. (i.e) $\langle V(T''_G)\rangle$ is distance preserving. (i.e.) G has a spanning tree T_G with subtree T''_G such that $\langle V(T''_G)\rangle$ is distance preserving. Hence condition (iv) is satisfied.

Conversely, suppose G is a tree or G has a spanning tree T_G satisfying one of the conditions (i) to (iii).

If G is a tree then $D = V(G) - A$ where A is the set of all pendant vertices of G is a wcd set of G.

If G has a spanning tree T_G satisfying condition (i) or (ii) then $D = V(T_G) - A$ is a wcd set of G.

If G has a spanning tree T_G satisfying condition (iii) then $D = V(T_G) - A'$ is a wcd set of G.

Hence $\gamma_{wc}(G) < n$ in all cases.

CONSTRUCTION OF A GRAPH G WITH $\gamma_{wc}(G) < n$.

Observation. $\gamma_{wc}(G) < n$ if and only if G can be constructed as follows:

Take any connected graph H.

(i) Attach one or more pendant vertices at each (or at some) vertices of H.

(ii) Identify an edge of a 3 cycle at each (or some) edges of H.

(iii) Identify two consecutive edges of a 4 cycle at each (or some) paths of length 2 in H.

(iv) Do any one or both of the two operations (i) and (ii) at each (or some) vertices of H and do operation.

(iii) At each (or at some) paths of length 2 in H.

(v) Attach one or more triangles at each (or at some) vertices of $V(H)$.

(vi) Identify an edge of a 4 cycle at each (or some) edge of H.

(vii) Identify two consecutive edges of a 5 cycle at each (or some) paths of length 2 in H.

(viii) Identify three consecutive edges of a 6 cycle at each (or some) paths of length 3 in H.

(ix) Do any one of the operations (v) to (viii) at each (or some) vertex, edge, paths of length 2 or 3.

(x) Do all (or some) of the operations (i) to (ix) at each (or some) vertex, edge, paths of length 2, paths of length 3.

(xi) Take any two connected graphs H and H' with $V(H) = \{u_1, u_2, \ldots, u_r\}$ and $V(H') = \{v_1, v_2, \ldots, v_s\}$. First join v_1 to u_1. Join each $v_i \in N_{H'}(v_1)$ to u_1 or to all or some vertices which are at distance atmost 3 from u_1 in H. (i.e.) each $v_i \in N_{H'}(v_1)$ can be joined to u_1 or to all or some u_i with $d_H(u_1, u_i) \leq 3$. Do the same operation for each $v_j \in N(v_i)$ and repeat this process until each v_i is joined to some u_i.

(xii) Take any connected graph H and any number of connected graphs $\{H_i'\}$ and perform the operation (xi) to each H_i'.

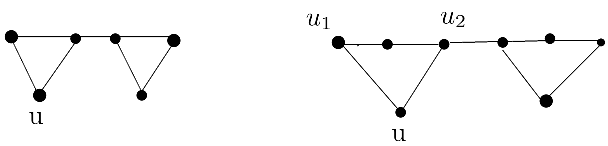
Proof. Let $\gamma_{wc}(G) < n$ and let D be a γ_{wc} set of G. Then $\langle D \rangle$ is connected and distance preserving.

Case (i): $\delta(G) = 1$ and $\langle V - D \rangle$ is independent.

Hence in this case G can be obtained by taking $H = \langle D \rangle$ and attaching one or more pendant vertices at each or some vertices of H.

Case (ii): $\delta(G) \geq 2$ and $\langle V - D \rangle$ is independent.

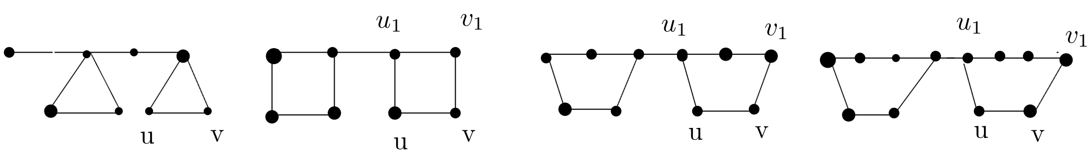
$\delta(G) \geq 2$ and $\langle V - D \rangle$ is independent implies for every $u \in V - D$, $|N(u) \cap D| \geq 2$. Let $u_1, u_2 \in N(u) \cap D$. Then $d(u_1, u_2) \leq 2$. $\langle D \rangle$ is a wcd set implies there exists an $u_1 \ldots u_2$ shortest path of length ≤ 2 in $\langle D \rangle$.



Therefore, G can be obtained by taking a connected graph $\langle D \rangle$ and performing one of the operations (ii) to (iv).

Case (ii): The components of $\langle V - D \rangle$ are K_2 alone.

For any two $u, v \in V - D$ with $uv \in E(G)$ both of them may be adjacent to a same vertex or adjacent to two different vertices in D. If each pair of adjacent vertices in $V - D$ are adjacent to the same vertex in D then in this case G can be obtained by taking $H = \langle D \rangle$ and performing the operation (v). If there exists $u, v \in V - D$ with $uv \in E(G)$ and adjacent to two different vertices u_1 and v_1 respectively in D, then $d(u_1, v_1) \leq 3$. D is a wcd set implies there exists an $u_1 \ldots v_1$ shortest path of length ≤ 3 in $\langle D \rangle$.



Therefore, G can be obtained by taking a connected graph $H = \langle D \rangle$ and performing one of the operations (v) to (ix).

Case (iii): The components of $\langle V - D \rangle$ are isolates and K_2s.

In this case G can be obtained by taking $H = \langle D \rangle$ and performing the operation (ix).

Case (iv): $\langle V - D \rangle$ is connected.

In this case, G can be obtained by taking $H = \langle D \rangle$ and performing one of the operations (x).

Case (v): $\langle V - D \rangle$ is disconnected and has more than one connected components.

In this case, G can be obtained by taking $H = \langle D \rangle$ and performing operation (xi) to each connected graph H_i'.

Lemma. If G is any simple graph with $n \geq 3$ and $\delta(G) \geq n/2$ then $\gamma_{wc}(G) < n$.

Proof. For any vertex u of degree δ consider $D = N[u]$. Then any $v \in V - D$ must be adjacent to some $u_i \in N(u)$, since v is not adjacent to u. Otherwise $\delta(G) \leq \deg v \leq n - 1 - |N(u)| = n - 1 - \delta \leq n - 1 - n/2 < n/2 \ldots$ a contradiction to the fact that $\delta \geq n/2$. Therefore, each $v \in V - D$ is adjacent to some $u_i \in N(u)$ and hence D is a dominating set. And $D = N[u]$ implies $d_{\langle D \rangle}(x, y) = d_{\langle G \rangle}(x, y)$ for all $x, y \in D$. (i.e.) D is a wcd set and hence $\gamma_{wc}(G) < n$.

Corollary. If $\gamma_{wc}(G) = n$ then $\delta(G) < n/2$.

Proof. For if $\delta(G) \geq n/2$ then by the previous lemma $\gamma_{wc}(G) < n \ldots$ a contradiction (since $\gamma_{wc}(G) = n$).

Lemma. If $\delta(G) \geq n/2$, then $\gamma_{wc}(G) \leq \delta + 1$ or $\gamma_{wc}(G) = 2$.

Proof. $\delta(G) \geq n/2$ implies G is Hamiltonian [6]. And $m = 1/2 \sum \deg u = n^2/4$. Therefore, G is pancyclic (or) G is $K_{n/2,n/2}$ [6]. If G is pancyclic, then $D = N[u]$ is a wcd set with deg $u = \delta$. Therefore, $\gamma_{wc}(G) \leq \delta + 1$. If G is $K_{n/2,n/2}$ then $\gamma_{wc}(G) = 2$.

Observation. $\gamma_{wc}(G) = n$ if and only if G is not a tree and for every spanning tree T_G and for every subset $A' \subseteq A$ there exists at least one pair of points $x, y \in V(T_G) - A'$ such that $d_{\langle V(T_G) - A' \rangle}(x, y) > d_G(x, y)$.

Proof. Let $\gamma_{wc}(G) = n$. Then G cannot be a tree. If there exists a spanning tree T_G and a set of pendant vertices $A' \subseteq A$ such that for every $x, y \in V(T_G) - A'$, $d_{\langle V(T_G) - A' \rangle}(x, y) \leq d_G(x, y)$ then $D = V(T_G) - A'$ is a proper wcd set of $G \ldots$ a contradiction.

Conversely, suppose G is not a tree and for every spanning tree T_G and for every subset $A' \subseteq A$ there exists at least one pair of points $x, y \in V(T_G) - A'$ such that $d_{\langle V(T_G) - A' \rangle}(x, y) > d_G(x, y)$. Let if possible $\gamma_{wc}(G) < n$. Then G is a tree (or) has a spanning tree T_G with a set of pendant vertices $A' \subseteq A$ such that for every pair of points $x, y \in V(T_G) - A'$, $d_{\langle V(T_G) - A' \rangle}(x, y) = d_G(x, y)$ (by observation (12) $\ldots$ a contradiction).

Observation. $\gamma(G) = \gamma_{wc}(G)$ if and only if G is a caterpillar (or) G has a spanning tree T_G with maximum number of pendant edges $\epsilon_T(G)$ satisfying one of the two conditions (a) or (b) and both the coditions (c) and (d).

(a) T_G has a distance preserving subtree T_G'.

(b) T_G has a subtree T_G' with $\langle V(T_G') \rangle$ is distance preserving.

(c) Each $u \in V(G)$ is either a support or a pendant vertex in T_G.

(d) The domination number of G is the number of supports in T_G.

Proof. Let $\gamma(G) = \gamma_{wc}(G)$ and let D be a γ_{wc} set. Then D is a connected dominating set. Therefore, $\gamma_c(G) \leq \gamma_{wc}(G) = \gamma(G)$. But $\gamma(G) \leq \gamma_c(G) \leq \gamma_{wc}(G)$. Therefore, $\gamma_c(G) = \gamma_{wc}(G) = \gamma(G)$. $\gamma_c(G) = n - \epsilon_T(G)$ where $\epsilon_T(G)$ [11] is the number of pendant vertices in a spanning tree of G with maximum number of pendant edges. Therefore, $n - \epsilon_T(G) = \gamma_c(G) = \gamma_{wc}(G) = \gamma(G) \ldots.$

(i) G has a proper wcd set. Therefore, G is a tree or has a spanning tree T_G satisfying one of the three conditions of observation 12. (i.e.) G is a tree or there exists a spanning tree T_G such that $D = V(T_G) - A'$ for some $A' \subseteq A$ with $\langle V(T_G) - A' \rangle$ is distance preserving.

(ii) $|D| = \gamma_{wc}(G) = \gamma(G)$ implies $N(u) \cap (V - D) \neq \phi$ for each $u \in D$ (since D is a minimum dominating set and G is connected).

(iii) By (i) and (ii) $n - \epsilon_T(G) = |D| = |V(T_G) - A'|$. Therefore $|A'| = \epsilon_T(G)$. (i.e) $\epsilon_T(G) = |A'| \leq |A| \leq \epsilon_T$. Hence $\epsilon_T(G) = |A'| = |A|$. (i.e.) G is a tree or has a spanning tree T_G with maximum number of pendant edges equal to $|A'|$ and $\langle D \rangle = \langle V(T_G) - A' \rangle = \langle V(T_G) - A \rangle$ is distance preserving. Hence by (iii) each $u \in D = V(T_G) - A$ is a support and each $u \in V - D = A$ is a pendant vertex in T_G. (i.e.) G is a tree or G has a spanning tree T_G with distance preserving subtree T'_G (or) has a subtree T'_G with $\langle V(T'_G) \rangle$ is distance preserving. (ie) G is a tree (or) satisfies one of the two conditions (a) and (b).

If $u \in V(G)$ then either $u \in D$ (or) $u \in V - D$. (i.e.) either u is a support (or) a pendant vertex in G (or) T_G. Hence, if G is a tree then G is a caterplillar. If G is not a tree then T_G satisfy condition (c). $\gamma(G) = \gamma_{wc}(G) = |D| =$ number of supports in T_G. Hence condition (d) is satisfied.

Conversely, suppose G is a tree or has a spanning tree satisfying one of the two conditions (a) or (b) and both the coditions (c) and (d).

If G is a tree and satisfying conditions (c) and (d), then $D = V(G) - A$ where A is the set of all pendant vertices is a wcd set and hence $\gamma_{wc}(G) \leq |D|$. G satisfies conditions (c) and (d) implies each $u \in D = V(G) - A$ is a support and $\gamma(G) = |D|$. Hence $\gamma_{wc}(G) \leq |D| = \gamma(G)$. (i.e.) $\gamma(G) = \gamma_{wc}(G)$.

If G satisfies conditions (a), (c) and (d), then $\langle D \rangle = \langle V(T_G) - A \rangle = T'_G$ is a distance preserving subtree of T_G with A as the set of all pendant vertices of T_G. (i.e) D is a wcd set and hence $\gamma_{wc}(G) \leq |D|$. That T_G satisfies condition (c) and (d) implies each $u \in D = V(T_G) - A$ is a support and $\gamma(G) = |D|$. Hence $\gamma_{wc}(G) \leq |D| = \gamma(G)$. (i.e.) $\gamma(G) = \gamma_{wc}(G)$.

If G satisfy conditions (b), (c) and (d), then $\langle D \rangle = \langle V(T_G) - A \rangle = \langle V(T'_G) \rangle$ is distance preserving. (i.e) D is a wcd set and hence $\gamma_{wc}(G) \leq |D|$. T_G satisfy conditions (c) and (d) implies each $u \in D = V(T_G) - A$ is a support and $\gamma(G) = |D|$. Hence $\gamma_{wc}(G) \leq |D| = \gamma(G)$. (i.e.) $\gamma(G) = \gamma_{wc}(G)$.

CONSTRUCTION OF GRAPHS WITH $\gamma(G) = \gamma_{wc}(G)$.

Observation. $\gamma(G) = \gamma_{wc}(G)$ if and only if G is a caterpillar (or) G can be obtained as follows:

Take any connected graph which is not a path H and consider a spanning tree T_H of H with maximum number of pendant edges. Let $\{u_1, u_2, \ldots, u_r\}$ be the set of pendant vertices of T_H. Attach one or more pendant vertices at each $u \in V(G) - \{u_1, u_2, \ldots, u_r\}$ and let $\{v_1, v_2, \ldots, v_s\}$ be the set of pendant vertices thus attached and T_G be the resulting tree.

If $u \in N(v_i) \cap V(H)$ then v_i can be joined to each or some $v \in V(H)$ with $d_H(u,v) \leq 2$. Any two v_is can be joined if both of them are adjacent to a same vertex in H. Join v_i and v_j if $d_H(u,v) \leq 3$ where $u \in N(v_i) \cap V(H)$ and $v \in N(v_j) \cap V(H)$.

Proof. Let $\gamma(G) = \gamma_{wc}(G)$. Then, by the previous observation, G is a caterpillar (or) G has a spanning tree T_G satisfying conditions (a) (or) (b) and both the conditions (c) and (d). Let $H = \langle V(T'_G)\rangle$ - the set of pendant vertices of T'_G. Then H satisfy the required conditions.

Conversely, if G is constructed by the above method then let $D = V(H) - \{u_1, u_2, \ldots, u_r\}$. Then T_G is a spanning tree of G with maximum number of pendant edges and D is a wcd set. Hence $\gamma_{wc}(G) \leq |D| = n - \epsilon_T$. Therefore $\gamma_{wc}(G) = n - \epsilon_T \leq \gamma(G)$. (i.e.) $\gamma(G) = \gamma_{wc}(G)$.

Observation. $\gamma_c(G) = \gamma_{wc}(G)$ if and only if G is a tree or has a spanning tree T_G with maximum number ϵ_T of pendant edges satisfying one of the following two conditions:

(i) T_G has a distance preserving subtree T'_G.

(ii) T_G has a subtree T'_G such that $\langle V(T'_G)\rangle$ is distance preserving.

Proof. Let $\gamma_c(G) = \gamma_{wc}(G)$. If D is a γ_{wc} set then G satisfy one of the four conditions of observation 12. (i.e.) G is a tree (or) $\langle D\rangle = \langle V(T_G) - A'\rangle$ where $A' \subseteq A$ and $\langle D\rangle$ is distance preserving. $n - \epsilon_T = \gamma_c(G) = \gamma_{wc}(G) = |D| = |V(T_G) - A'| = n - |A'|$ which implies $\epsilon_T = |A'| \leq |A| \leq \epsilon_T$. Therefore $\epsilon_T = |A'| = |A|$. (i.e.) T_G is a spanning tree of G with maximum number of pendant edges. Hence G has a spanning tree $T_G$with maximum number of pendant vertices satisfying one of the two conditions.

Conversely, if G is a tree then $\gamma_c(G) = \gamma_{wc}(G)$.

If G has a spanning tree with maximum number of pendant edges satisfying one of the two conditions then $D = V(T_G) - A$ is a wcd set with $|A| = \epsilon_T$ and hence $\gamma_{wc}(G) \leq |D| = |V(T_G) - A| = n - |A| = n - \epsilon_T = \gamma_c(G)$. Hence $\gamma_c(G) = \gamma_{wc}(G)$.

CONSTRUCTION OF A GRAPH WITH $\gamma_c(G) = \gamma_{wc}(G)$.

$\gamma_c(G) = \gamma_{wc}(G)$ if and only if G is either G_1 (or) G_2 which are obtained as follows:

Take any connected graph H and consider a spanning tree T_H with maximum number of pendant edges and let $B = \{u_1, u_2, \ldots, u_r\}$ be the set of pendant vertices of T_H. Attach one or more pendant vertices at each u_i, $1 \leq i \leq r$. Let $A = \{v_1, v_2, \ldots, v_s\}$ with $s \geq r$ be that set of pendant vertices thus attached. Let G_1 be the resulting graph.

In G_1 connect $v_i \in N(u_i)$ and $v_j \in N(u_j)$ by a path such that $d(u_i, u_j) \leq d(v_i, v_j)$ and let G_2 be the resulting graph.

Proof. Let $\gamma_c(G) = \gamma_{wc}(G)$. Then by the previous observation G is a tree or G has a spanning tree satisfying the two conditions.

If G is a tree then G is of the form G_1.

If G has a spanning tree T_G with maximum number of pendant edges and a distance preserving subtree T'_G, then $\langle D\rangle = \langle V(T_G) - A\rangle = T'_G$. Let $A = \{v_1, v_2, \ldots, v_s\}$. Now let $B = \{u_1, u_2, \ldots, u_r\}$ be the set of pendant vertices of T'_G. In this case $\langle A\rangle$ is not independent. Therefore G can be obtained by taking a tree and attaching one or more triangles at each u_i, $1 \leq i \leq r$. (i.e.) G is of the form G_2.

If G has a spanning tree T_G with maximum number of pendant edges and a subtree T'_G, with $\langle V(T'_G)\rangle$ is distance preserving. Then $\langle D\rangle = \langle V(T_G) - A\rangle = T'_G$. Let $A = \{v_1, v_2, \ldots, v_s\}$

and let $B = \{u_1, u_2, \ldots, u_r\}$ be the set of pendant vertices of T'_G. Then T'_G is a spanning tree of $\langle D\rangle$ with maximum number of pendant edges. For if there exists any other spanning tree $T'_{\langle D\rangle}$ of $\langle D\rangle$ with t number of pendant vertices and $t > r$. Each v_i is adjacent to some vertex in $T'_{\langle D\rangle}$. $t > r$ implies there exists a spanning tree T_{1G} of G with more number of pendant vertices than the number of pendant vertices in T_G which is a contradiction to the fact that T_G is a spanning tree with maximum number of pendant vertices.

Now let $\langle D\rangle = H$. If $v_i \in N(u_i)$ then $d(u_i, u_j) \leq d(v_i, v_j)$ (since D is a wcd set).

If $\langle V - D\rangle$ is independent then G is of the form G_1.

If $\langle V - D\rangle$ is not independent then G is of the form G_2.

Conversely, suppose G is of the form G_1 or G_2.

Then let $D = V(H)$. By construction $\gamma_c(G) = n - \epsilon_T(G) = |V(H)| = |D|$ and D is a wcd set. Therefore $\gamma_{wc}(G) \leq |D| = \gamma_c(G)$. Hence $\gamma_c(G) = \gamma_{wc}(G)$.

Definition. For any connected graph G we define,

$\epsilon_{max} = Max\left\{|A'|\,/A' \subseteq A, d_{\langle V(T_G)-A'\rangle}(x, y) = d_G(x, y), \forall x, y \in V(T_G) - A'\right\}$, where A is the set of all pendant vertices in a spanning tree T_G of G.

Observation. $\gamma_{wc}(G) \leq n - \epsilon_{max}$.

Observation. If G is a block with proper wcd set then $\gamma_{wc}(G) = n - \epsilon_{max}$.

Proof. If D is a γ_{wc} set of G, then $D = V(T_G) - A'$ with $A' \subseteq A$. Therefore $\gamma_{wc}(G) = |D| = |V(T_G) - A'| \geq n - \epsilon_{max}$ (since $|A'| \leq \epsilon_{max}$). Therefore by the previous observation $\gamma_{wc}(G) = n - \epsilon_{max}$.

References

[1] Acharya. B. D., Purnima Gupta, On Point - set domination in graphs, Proc. Nat. Seminar on Recent Development in Mathematics, N. M. Bujurke (Ed.), Karnataka University Press, Dharwad, 1996, 106-108.

[2] Acharya. B. D., Purnima Gupta, On Point - set dominiation in graphs: II minimum psd - sets, J. Combin. Theory Ser. B, 1996.

[3] Acharya. B. D., Purnima Gupta, On Point set domination in graphs: III Further details submitted.

[4] Acharya. B. D., Purnima Gupta, On Point set domination in graphs: IV Seperable graphs with unique minimum psd-sets, Discrete Mathematics, **195**(1999), 1-13.

[5] S. Arumugam and R. Kala, Domination Parameters of a star Graph, ARS Combinatoria, **44**(1996), 93-96.

[6] Balakrishnan. R., Ranganathan. K., A text book of Graph Theory, Springer Verlag, New York, 2000.

[7] Fred Buckley, Fred Harary, Distance in Graphs, Addission - Wesley Publishing Company, 1990.

[8] Gary Chartrand, Curtiss E. Wall and Ping Zhamg, The convexity Number of a Graph, Graphs and Combinatorics, **18**(2002), 209-217.

[9] Gary Chartrand, Frank Harary, Henda C. Swart, Geodomination in Graphs, Bulletin of the ICA, **31**(2001), 51-59.

[10] F. Harary, Graph Theory, Addison - Wesley, 1969.

[11] T. Haynes, S. T. Hedetniemi and P. J. Slater, Advances in the Theory of Domination in Graphs, Marcel Dekker Incl, New York, 1997.

[12] S. M. Hedetniemi, S. T Hedetniem, D. F. Rall, Acyclic domination, Discrete Mathematics, **222**(2000), 151-165.

[13] V. R Kulli and B. Janakiram, The Non Split domination number of a graph, Indian J. Pure appl. Math, 2000, 441-447.

[14] Parthasarathy. K. R., Basic Graph Theory, Tat McGraw-Hill Publishing Company Limited, 1994.

[15] Sampathkumar. E., Point set domination Number of a graph, Indian J. Pure Appl. Math., **4**(1993), 225-229.

[16] E. Sampathkumar (Ed.), Graph Theory Research Report, Karnatak University, Dharwad, No. 3, 1972.

[17] H. B. Walikar, B. D. Acharya and E. Sampathkumar, Recent Advances in the theory of domination in graphs and its applications, MRI Lecture Notes in Mathematics, Mehta Research Institute of Mathematics and Mathematical Physics, Allahabad, No. 1, 1979.

[18] Wayne Goddard and Michael A. Henning, Domination in planar Graphs with small Diameter, Journal of Graph Theory, 2002, 1-25.

Scientia Magna
Vol. 7 (2011), No. 1, 38-53

New hybrid filtering techniques for removal of speckle noise from ultrasound medical images

Gnanambal Ilango† and R. Marudhachalam‡

† Post Graduate and Research Department of Mathematics,
Government Arts College (Autonomous),
Coimbatore 641018, Tamilnadu, India
‡ Department of Mathematics, Kumaraguru College of Technology,
Coimbatore 641006, Tamilnadu, India
E-mail: gnanamilango@yahoo.co.in marudhu14@gmail.com

Abstract The most significant feature of diagnostic medical images is to reduce speckle noise which is commonly found in ultrasound medical images and make better image quality. In recent years, technological development has improved significantly in analyzing medical imaging. This paper proposes different hybrid filtering techniques for the removal of speckle noise, from ultrasound medical images, by topological approach. The filters are treated in terms of a finite set of certain estimation and neighborhood building operations. A set of such operations is suggested on the base of the analysis of a wide variety of nonlinear filters described in the literature. The quality of the enhanced images is measured by the statistical quantity measures: Root Mean Square Error (RMSE) and Peak Signal-to-Noise Ratio (PSNR).
Keywords Digital topological neighborhood, ultrasound image, speckle noise, RMSE, PSNR.

§1. Introduction

Ultrasound imaging is widely used in the field of medicine. It is used for imaging soft tissues in organs like liver, kidney, spleen, uterus, heart, brain etc. Ultrasound or ultrasonography is a medical imaging technique that uses high frequency sound waves and their echoes. It allows one to visualize and therefore examine a part of the human anatomy in medicine. It is a widely used medical imaging procedure because it is economical, comparatively safe, transferable, and adaptable. A major disadvantage with ultrasound imaging is the presence of noise, which perturbs feature location and creates artifacts. The acquired image is corrupted by a random granular pattern, called speckle, which delays the interpretation of the image content. The existence of speckle is unattractive because of its disgrace image quality and it affects the tasks of individual interpretation and diagnosis. Accordingly, speckle filtering is a central preprocessing step for feature extraction, analysis, and recognition of medical imagery measurements. Previously, a number of schemes have been proposed for speckle mitigation. Median filter has been introduced by Tukey [14] in 1970. It is a special case of non-linear filters used for smoothing signals. Median filter now is broadly used in reducing noise and smoothing the images. Hakan et al.[4] have used Topological Median filter to improve conventional Median filter. The better performance of the Topological Median filters over conventional Median filters is in maintaining edge sharpness. Yanchun et al.[16] proposed an algorithm for image denoising based on Average filter with maximization and minimization for the smoothness of the region, unidirectional Median filter for edge region and Median filter for the indefinite region. It was discovered that when the image is corrupted by both Gaussian and impulse noises, neither Average filter nor Median filter algorithm will obtain a result good enough to filter the noises because of their algorithm. Sudha et al.[12] recommend a novel thresholding algorithm for denoising speckle in ultrasound with wavelets. An improved adaptive median filtering method for denoising impulse noise reduction was carried out by Mamta Juneja et al.[6] An adaptive median filter (AMF) is the best filter to remove salt and pepper noise of image sensing was shown by Salem Saleh Al-amri et al.[10]. Thangavel et al.[13] showed that the M3-filter performed better than mean filter, median filter, max filter, min filter and various other filters. The objective of this study was to develop new hybrid speckle reduction techniques and investigate their performance on Ultrasound images.

This work is organized as follows: Section 2 discusses types of noises involved in medical imaging. In Section 3 basic definitions are introduced. Section 4 discusses the various existing filtering techniques for de-noising the speckle noise in the ultrasound medical image. Section 5 deals with proposed hybrid filtering techniques for de-noising the speckle noise in the ultrasound medical images. In Section 6, both quantitative (RMSE & PSNR) and qualitative comparisons have been provided. Section 7 puts forward the conclusion drawn by this paper.

§2. Types of noises

§2.1. Salt and pepper noise

Salt and pepper noise is a form of noise typically seen on images. It represents itself as randomly occurring white and black pixels. A spike or impulse noise drives the intensity values of random pixels to either their maximum or minimum values. The resulting black and white flecks in the image resemble salt and pepper. This type of noise is also caused by errors in data transmission.

§2.2. Speckle noise

Speckle noise affects all inherent characteristics of coherent imaging, including medical ultra sound imaging. It is caused by coherent processing of backscattered signals from multiple distributed targets. Speckle noise is caused by signals from elementary scatterers. In medical literature, speckle noise is referred to as texture and may possibly contain useful diagnostic information. For visual interpretation, smoothing the texture may be less desirable. Physicians generally have a preference for the original noisy images, more willingly, than the smoothed versions because the filter, even if they are more sophisticated, can destroy some relevant image details. Thus it is essential to develop noise filters which can preserve the features that are of interest to the physician. Several different methods are used to eliminate speckle noise, based upon different mathematical models of the phenomenon. In our work, we recommend hybrid filtering techniques for removing speckle noise in ultrasound images. The speckle noise model has the following form ('$\cdot$' denotes multiplication). For each image pixel with intensity value f_{ij} ($1 \le i \le m, 1 \le j \le n$ for an $m \times n$ image), the corresponding pixel of the noisy image g_{ij} is given by,

$$g_{i,j} = f_{i,j} + f_{i,j} \cdot n_{i,j}. \tag{1}$$

Where each noise value n is drawn from uniform distribution with mean 0 and variance σ^2.

§2.3. Gaussian noise

Gaussian noise is a statistical noise that has a probability density function (abbreviated pdf) of the normal distribution (also known as Gaussian distribution). In other words, the values that the noise can take on are Gaussian-distributed. Gaussian noise is properly defined as the noise with a Gaussian amplitude distribution. Noise is modeled as additive white Gaussian noise (AWGN), where all the image pixels deviate from their original values following the Gaussian curve. That is, for each image pixel with intensity value f_{ij} ($1 \le i \le m, 1 \le j \le n$ for an $m \times n$ image), the corresponding pixel of the noisy image g_{ij} is given by,

$$g_{i,j} = f_{i,j} + n_{i,j}, \tag{2}$$

where each noise value n is drawn from a zero-mean Gaussian distribution.

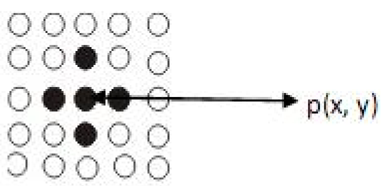
§3. Basic definitions

This section presents some general definitions and digital topological results, which will be used along the development of this paper.

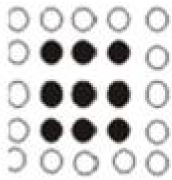
Definition 3.1.[9] A digital image is a function $f : Z \times Z \to [0, 1, \ldots, N-1]$ in which $N-1$ is a positive whole number belonging to the natural interval $[1, 256]$. The functional value of 'f' at any point $p(x, y)$ is called the intensity or gray level of the image at that point and it is denoted by $f(p)$.

Definition 3.2.[9] A neighborhood of a point $p \in X$ is a subset of X which contains an open set containing p. It is denoted by $N(p)$.

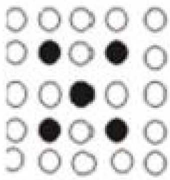
Definition 3.3.[9] The 4-neighbours of a point $p(x, y)$ are its four horizontal and vertical neighbours $(x \mp 1, y)$ and $(x, y \pm 1)$. It is denoted by $N_4(p)$.



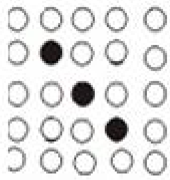
Definition 3.4.[9] The 8-neighbours of a point $p(x, y)$ consist of its 4-neighbours together with its four diagonal neighbours $(x + 1, y \mp 1)$ and $(x - 1, y \mp 1)$. It is denoted by $N_8(p)$.



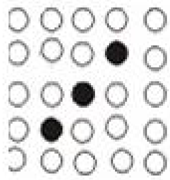
Definition 3.5. The cross neighbours of a point $p(x, y)$ consists of the neighbours $(x + 1, y \mp 1)$ and $(x - 1, y \mp 1)$. It is denoted by $C_4(p)$.



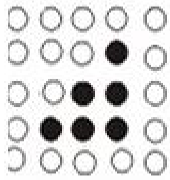
Definition 3.6. The *LT* neighbours of a point $p(x, y)$ consists of the neighbours $(x - 1, y - 1)$ and $(x + 1, y + 1)$. It is denoted by $L_3(p)$.



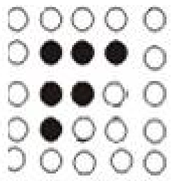
Definition 3.7. The *RT* neighbours of a point $p(x, y)$ consists of the neighbours $(x - 1, y + 1)$ and $(x + 1, y - 1)$. It is denoted by $R_3(p)$.



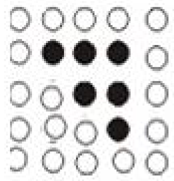
Definition 3.8. The R_L neighbours of a point $p(x, y)$ consists of its *RT* neighbours together with the neighbours $(x + 1, y)$, $(x + 1, y + 1)$ and $(x, y + 1)$. It is denoted by $R_{L_6}(p)$.



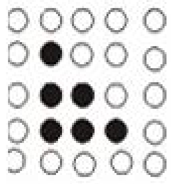
Definition 3.9. The R_U neighbours of a point $p(x, y)$ consists of its *RT* neighbours together with the neighbours $(x, y - 1)$, $(x - 1, y - 1)$ and $(x - 1, y)$. It is denoted by $R_{U_6}(p)$.



Definition 3.10. The L_U neighbours of a point $p(x, y)$ consists of its LT neighbours together with the neighbours $(x-1, y)$, $(x-1, y+1)$ and $(x, y+1)$. It is denoted by $L_{U_6}(p)$.



Definition 3.11. The L_L neighbours of a point $p(x, y)$ consists of its LT neighbours together with the neighbours $(x, y-1)$, $(x+1, y-1)$ and $(x+1, y)$. It is denoted by $L_{L_6}(p)$.



§4. Some existing filtering techniques

In this section, we provide the definitions of some existing filters. The image processing function in a spatial domain can be expressed as

$$g(p) = \gamma(f(p)). \tag{3}$$

where γ is the transformation function, $f(p)$ is the pixel value (intensity value or gray level value) of the point $p(x, y)$ of input image and $g(p)$ is the pixel value of the corresponding point of the processed image.

§4.1. Mean filter (MNF)

Mean Filter [2] is a simple linear filter, intuitive and easy to implement method of smoothing images, i.e., reducing the amount of intensity variation between one pixel and the next. It is often used to reduce noise in images. In mean filter the pixel value of a point p is replaced by the mean of pixel value of 8-neighborhood of a point 'p'. The operation of this filter can be expressed as:

$$g(p) = mean\{f(p), \text{ where } p \in N_8(p)\}. \tag{4}$$

§4.2. Median filter (MF)

The best-known order-statistic filter in digital image processing is the median filter. It is a useful tool for reducing salt-and- pepper noise in an image. The median filter [14] plays a key role in image processing and vision. In median filter, the pixel value of a point p is replaced by the median of pixel value of 8-neighborhood of a point 'p'. The operation of this filter can be expressed as:

$$g(p) = median\{f(p), \text{ where } p \in N_8(p)\}. \tag{5}$$

The median filter is popular because of its demonstrated ability to reduce random impulsive noise without blurring edges as much as a comparable linear lowpass filter. However, it often fails to perform as well as linear filters in providing sufficient smoothing of nonimpulsive noise components such as additive Gaussian noise. One of the main disadvantages of the basic median filter is that it is location-invariant in nature, and thus also tends to alter the pixels not disturbed by noise.

§4.3. M3 filter

The M3 filter [13] is hybridization of mean and median filter. This replaces the central pixel by the maximum value of mean and median for 8-neighborhood of central pixel. It is expressed as M3-filter, the intensity values are reduced in the adjacent pixel and it preserves the high frequency components in image. This filter is defined as

$$g(p) = max\left\{\begin{array}{c} mean\{f(p), p \in N_8(p)\}, \\ median\{f(p), p \in N_8(p)\}. \end{array}\right\} \tag{6}$$

§4.4. Hybrid median filter (HMF)

Hybrid Median filter [5] is of nonlinear class that easily removes impulse noise while preserving edges. The hybrid median filter plays a key role in image processing and vision. In comparison with basic version of the median filter hybrid one has better corner preserving characteristics. This filter is defined as

$$g(p) = median\left\{\begin{array}{c} median\{f(p), p \in N_4(p)\}, \\ median\{f(p), p \in C_4(p)\}, \\ f(p). \end{array}\right\} \tag{7}$$

A hybrid median filter preserves edges much better than a median filter. In hybrid median filter the pixel value of a point p is replaced by the median of median pixel value of 4-neighborhood of a point 'p', median pixel value of cross neighbours of a point 'p' and pixel value of 'p'.

§4.5. Hybrid mean filter (HMNF)

Hybrid Mean Filter [5] is a simple non linear filter. It is intuitive and easy to implement method of smoothing images. In hybrid mean filter, the pixel value of a point p is replaced by the mean of the mean pixel value of 4-neighborhood of a point 'p', mean pixel value of cross neighbours of a point 'p' and pixel value of 'p'. The operation of this filter can be expressed as:

$$g(p) = mean \left\{ \begin{array}{c} mean\{f(p), p \in N_4(p)\}, \\ mean\{f(p), p \in C_4(p)\}, \\ f(p). \end{array} \right\} \tag{8}$$

§5. Proposed hybrid filtering techniques

In this section, we will provide the definition of proposed hybrid filters. These filters are not yet applied by researchers to remove the speckle noise in the ultrasound medical images.

§5.1. Hybrid cross median filter (H_1F)

The hybrid cross median filter is a nonlinear filtering technique for image enhancement. It is proposed for speckle noise removal from the ultrasound medical image. It is expressed as:

$$g(p) = median \left\{ \begin{array}{c} median\{f(p), p \in L_3(p)\}, \\ median\{f(p), p \in R_3(p)\}, \\ f(p). \end{array} \right\} \tag{9}$$

In hybrid cross median filter, the pixel value of a point p is replaced by the median of median pixel value of LT neighbours of a point 'p', median pixel value of RT neighbours of a point 'p' and pixel value of 'p'.

§5.2. Hybrid min filter (H_2F)

Hybrid min filter plays a significant role in image processing and vision. Hybrid min filter is not a usual min filter. Min filter [2] recognizes the darkest pixels gray value and retains it by performing min operation. In min filter each output pixel value can be calculated by selecting minimum gray level value of the $N_8(p)$. H_2F filter is also used for removing the salt noise from the image. Salt noise has very high values in images. It is proposed for speckle noise removal from the ultrasound medical image. It is expressed as:

$$g(p) = min \left\{ \begin{array}{c} median\{f(p), p \in L_3(p)\}, \\ median\{f(p), p \in R_3(p)\}, \\ f(p). \end{array} \right\} \tag{10}$$

In hybrid min filter, the pixel value of a point p is replaced by the minimum of median pixel value of LT neighbours of a point 'p', median pixel value of RT neighbours of a point 'p' and pixel value of 'p'.

§5.3. Hybrid max filter (H_3F)

Hybrid max filter is not a usual max filter. Hybrid max filter plays a key role in image processing and vision. The brightest pixel gray level values are identified by max filter. In max filter [2] each output pixel value can be calculated by selecting maximum gray level value of the $N_8(p)$. H_3F filter is also used for removing the pepper noise from the image. It is proposed for speckle noise removal from the ultrasound medical image. It is expressed as:

$$g(p) = max\left\{\begin{array}{c} median\{f(p), p \in L_3(p)\}, \\ median\{f(p), p \in R_3(p)\}, \\ f(p). \end{array}\right\} \tag{11}$$

In hybrid max filter, the pixel value of a point p is replaced by the maximum of median pixel value of LT neighbours of a point 'p', median pixel value of RT neighbours of a point 'p' and pixel value of 'p'.

§5.4. Hybrid TMN filter (H_4F)

The hybrid tmn filter is a nonlinear filtering technique for image enhancement. It is proposed for speckle noise removal from the ultrasound medical image. It is expressed as:

$$g(p) = median\left\{\begin{array}{c} median\{f(p), p \in R_{U_6}(p)\}, \\ median\{f(p), p \in R_{L_6}(p)\}, \\ median\{f(p), p \in L_{L_6}(p)\}, \\ median\{f(p), p \in L_{U_6}(p)\}, \\ f(p). \end{array}\right\} \tag{12}$$

In hybrid tmn filter, the pixel value of a point p is replaced by the median of median pixel value of R_U neighbours of a point 'p', median pixel value of R_L neighbours of a point 'p', median pixel value of L_L neighbours of a point 'p', median pixel value of L_U neighbours of a point 'p' and pixel value of 'p'.

§5.5. Hybrid TM filter (H_5F)

The hybrid tm filter is a nonlinear filtering technique for image enhancement. It is proposed for speckle noise removal from the ultrasound medical image. It is expressed as:

$$g(p) = mean\left\{\begin{array}{c} mean\{f(p), p \in R_{U_6}(p)\}, \\ mean\{f(p), p \in R_{L_6}(p)\}, \\ mean\{f(p), p \in L_{L_6}(p)\}, \\ mean\{f(p), p \in L_{U_6}(p)\}, \\ f(p). \end{array}\right\} \tag{13}$$

In hybrid tm filter, the pixel value of a point p is replaced by the mean of mean pixel value of R_U neighbours of a point 'p', mean pixel value of R_L neighbours of a point 'p', mean pixel value of L_L neighbours of a point 'p', mean pixel value of L_U neighbours of a point 'p' and pixel value of 'p'.

§5.6. Maximum of hybrid median and hybrid mean filter (H_6F)

The maximum of hybrid median and hybrid mean filter is a nonlinear filtering technique for image enhancement. It plays a significant role in image processing and vision. It is proposed for speckle noise removal from the ultrasound medical image. It is expressed as:

$$g(p) = max \left\{ \begin{array}{c} median \left\{ \begin{array}{c} median\{f(p), p \in N_4(p)\}, \\ median\{f(p), p \in C_4(p)\}, \\ f(p) \end{array} \right\}, \\ mean \left\{ \begin{array}{c} mean\{f(p), p \in N_4(p)\}, \\ mean\{f(p), p \in C_4(p)\}, \\ f(p). \end{array} \right\}. \end{array} \right\} \tag{14}$$

In H_6F filter, each output pixel value is calculated by the maximum of output pixel value of hybrid median filter and hybrid mean filter.

§5.7. Average of hybrid median and hybrid mean filter (H_7F)

The average of hybrid median and hybrid mean filter is an another nonlinear filtering technique for image enhancement. It is proposed for speckle noise removal from the ultrasound medical image. It is expressed as:

$$g(p) = \frac{1}{2} \left\{ \begin{array}{c} median \left\{ \begin{array}{c} median\{f(p), p \in N_4(p)\}, \\ median\{f(p), p \in C_4(p)\}, \\ f(p) \end{array} \right\} + \\ mean \left\{ \begin{array}{c} mean\{f(p), p \in N_4(p)\}, \\ mean\{f(p), p \in C_4(p)\}, \\ f(p). \end{array} \right\}. \end{array} \right\} \tag{15}$$

In H_7F filter, each output pixel value is calculated by the average of output pixel value of hybrid median filter and hybrid mean filter.

§6. Experimental result analysis and discussion

The proposed hybrid filtering techniques have been implemented using MATLAB 7.0. The performance of various hybrid filtering techniques is analyzed and discussed. The measurement of ultrasound image enhancement is difficult and there is no unique algorithm available to measure enhancement of ultrasound image. We use statistical tool to measure the enhancement of ultrasound images. The Root Mean Square Error (RMSE) and Peak Signal-to-Noise Ratio (PSNR) are used to evaluate the enhancement of ultrasound images.

$$RMSE = \sqrt{\frac{\Sigma(f(i,j) - g(i,j))^2}{mn}}, \tag{16}$$

$$PSNR = 20 \log_{10} \frac{255}{RMSE}. \tag{17}$$

Here $f(i,j)$ is the original ultrasound image, $g(i,j)$ is enhanced ultrasound image and m and n are the total number of pixels in the horizontal and the vertical dimensions of the image. If the value of RMSE is low and value of PSNR is high then the enhancement approach is better. The original ultrasound image and filtered ultrasound image of liver tumor obtained by various hybrid filtering techniques are shown figure 1. Table 1 shows the proposed hybrid filtering techniques, that are compared with some existing filtering techniques namely, HMF, $HMNF$, M_3F, MF, MNF, with regard to ultrasound medical images for liver tumor.

Filters	H_1F	H_2F	H_3F	H_4F	H_5F	H_6F	H_7F	HMF	$HMNF$	$M3F$	MF	MNF
RMSE	2.885	4.9457	1.362	3.8874	4.1455	2.8849	4.2599	3.2134	3.5241	4.2674	4.5939	4.7789
PSNR	38.9284	34.2466	45.4475	36.3381	35.7797	38.9288	35.5432	37.9919	37.1902	35.5279	34.8876	34.5447

Table 1: RMSE and PSNR values for original image.

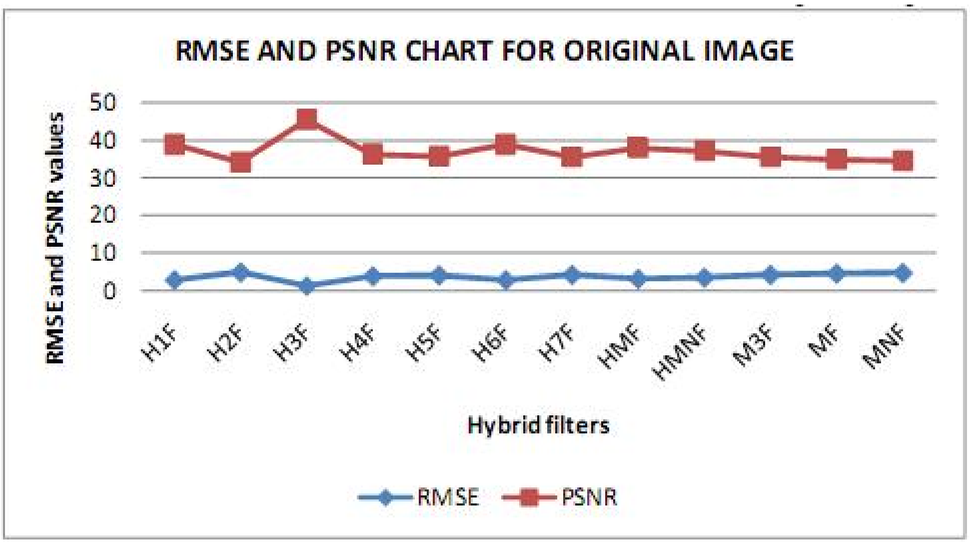


Chart 1: Analysis of RMSE & PSNR values of Ultrasound liver tumor image

The figure 2 shows denoising of ultrasound images corrupted by speckle noise of variance 0.07. The table 2 shows the comparison of RMSE & PSNR values of different denoising filters for ultrasound liver tumor images corrupted by speckle noise of variance 0.07.

Filters	H_1F	H_2F	H_3F	H_4F	H_5F	H_6F	H_7F	HMF	$HMNF$	$M3F$	MF	MNF
RMSE	4.1662	7.4766	1.3455	6.1125	6.1835	4.3792	5.6785	5.1226	5.6369	6.4303	6.828	6.824
PSNR	35.7365	30.6572	45.554	32.4069	32.3066	34.6172	33.0466	33.9415	33.1104	31.9666	31.4453	31.4505

Table 2: RMSE and PSNR values for noisy image of variance 0.07.

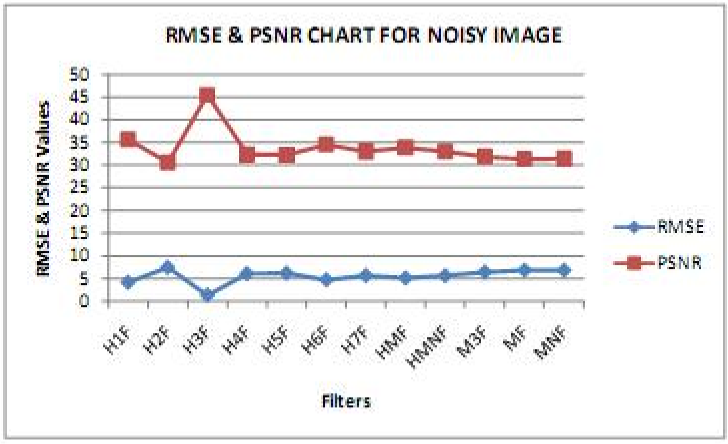


Chart 2: Analysis of RMSE & PSNR values of Ultrasound liver tumor images corrupted by speckle noise of variance 0.07.

Filters	H_1F	H_2F	H_3F	H_4F	H_5F	H_6F	H_7F	HMF	$HMNF$	$M3F$	MF	MNF
RMSE	1.776	2.4993	0.0846	1.2985	0.7449	0.9924	1.0525	1.6103	1.0099	0.6103	0.8438	0.6862
PSNR	43.1424	40.1749	69.5814	45.8626	50.6896	48.198	47.6871	43.9931	48.046	52.4199	49.6065	51.4019

Table 3: RMSE and PSNR values for noisy image of variance 0.07, after 9^{th} iteration.

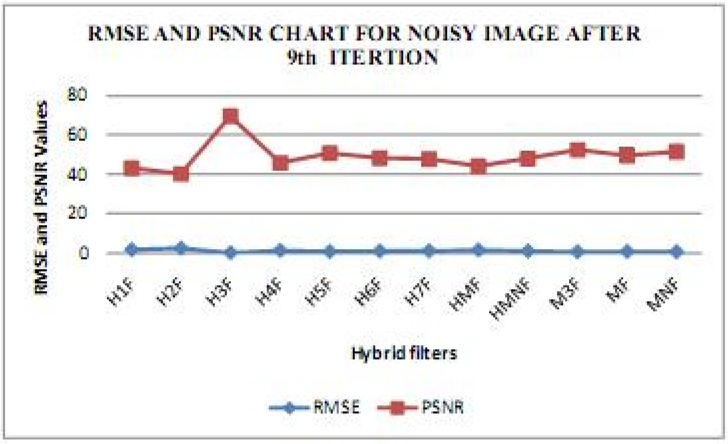


Chart 3: Analysis of RMSE & PSNR values of Ultrasound liver tumor images corrupted by speckle noise of variance 0.07, after 9^{th} iteration.

The table 4 shows the comparison of RMSE & PSNR values of different denoising filters for ultrasound liver tumor images corrupted by speckle noise.

Filters	Variance, σ^2											
	0.02		0.03		0.04		0.05		0.06		0.07	
	RMSE	PSNR	RMSE	PSNR	RMSE	PSNR	RMSE	PSNR	RMSE	PSNR	RMSE	PSNR
H_1F	3.5586	37.1057	3.7266	36.705	3.8571	36.406	3.9817	36.1299	4.0631	35.9542	4.1662	35.737
H_2F	6.3061	32.136	6.666	31.6539	6.9287	31.3182	7.1688	31.0223	7.3408	30.8164	7.4766	30.657
H_3F	**1.3682**	**45.408**	**1.3659**	**45.423**	**1.3587**	**45.469**	**1.3502**	**45.5237**	**1.3442**	**45.5619**	**1.3455**	**45.554**
H_4F	5.0432	34.0772	5.3421	33.577	5.5976	33.1711	5.8221	32.8296	5.9919	32.5799	6.1125	32.407
H_5F	5.1083	33.9657	5.4141	33.4607	5.6598	33.0752	5.8498	32.7884	6.0264	32.5301	6.1835	32.307
H_6F	3.8546	36.4117	4.1421	35.7869	4.3235	35.4145	4.497	35.0727	4.6296	34.8204	4.7392	34.617
H_7F	4.9308	34.2729	5.1596	33.8789	5.3122	33.6257	5.439	33.4208	5.5866	33.1883	5.6785	33.047
HMF	4.2884	35.4853	4.5411	34.9881	4.7383	34.6188	4.8887	34.3474	5.0248	34.1089	5.1226	33.942
$HMNF$	4.5041	35.0592	4.8193	34.4715	5.0842	34.0067	5.2996	33.6464	5.4865	33.3453	5.6369	33.11
$M3F$	5.4277	33.439	5.7095	32.9992	5.9594	32.6271	6.1453	32.3604	6.3147	32.1242	6.4303	31.967
MF	5.7884	32.88	6.1149	32.4034	6.3529	32.0717	6.5308	31.8319	6.6941	31.6174	6.828	31.445
MNF	5.7857	32.884	6.0835	32.4482	6.3112	32.1289	6.501	31.8716	6.6745	31.6429	6.824	31.451

Table 4: Comparison of RMSE & PSNR values of different denoising filters for Ultrasound liver tumor images corrupted by speckle noise.

Conclusion

In this work, we have introduced various hybrid filtering techniques for removal of speckle noise from ultrasound medical images. To demonstrate the performance of the proposed techniques, the experiments have been conducted on ultrasound image with liver tumor to compare our methods with many other well known techniques. The performance of speckle noise removing hybrid filtering techniques is measured using quantitative performance measures such as RMSE and PSNR. The experimental results indicate that one of the proposed hybrid filters, Hybrid Max Filter performs significantly better than many other existing techniques and it gives best the results after successive iterations. The proposed method is simple and easy to implement.

References

[1] L. Busse, T. R. Crimmins and J. R. Fienup, A model based approach to improve the performance of the Geometric filtering speckle reduction algorithm, IEEE Ultrasonic Symposium, 1995, 1353-1356.

[2] R. Gonzalez and R. Woods, Digital Image Processing, Adison-Wesley, New York, 1992.

[3] H. Hu and G. de Haan, Classification-based hybrid filters for image processing, Proc. SPIE, Visual Communications and Image Processing, **6077**(2006), 607711.1-607711.10.

[4] Hakan Güray Senel, Richard Alan Peters and Benoit Dawant, Topological Median Filter, IEEE Trans On Image Processing, **11**(2002), No. 2, 89-104.

[5] http://www.librow.com/articles/article-8.

[6] Mamta Juneja and Rajni Mohana, An Improved Adaptive Median Filtering Method for Impulse Noise Detection, International Journal of Recent Trends in Engineering, **1**(2009), No. 1, 274-278.

[7] S. Peng and L. Lucke, A hybrid filter for image enhancement, Proc. of International Conference on Image Processing (ICIP), **1**(1995), 163-166.

[8] T. Ratha, Jeyalakshmi and K. Ramar, A Modified Method for Speckle Noise Removal in Ultrasound Medical Images, International Journal of Computer and Electrical Engineering, **2**(2010), No. 1, 54-58.

[9] A. Rosenfeld, Digital topology, Amer. Math. Monthly, **86**(1979), 621-630.

[10] Salem Saleh Al-amri, N. V. Kalyankar and S. D. Khamitkar, A Comparative Study of Removal Noise from Remote Sensing Image, International Journal of Comp. Science, No. 1, **7**(2010), 32-36.

[11] Su Cheol Kang and Seung Hong, A Speckle Reduction Filter using Wavelet-Based Methods for Medical Imaging Application, Journal of Digital Signal Processing, No. 2, **18**(2002), 1169-1172.

[12] S. Sudha, G. R. Suresh and R. Sukanesh, Speckle Noise Reduction in Ultrasound Images by Wavelet Thresholding based on Weighted Variance, International Journal of Comp. Theory and Engg, No. 1, **1**(2009), 7-12.

[13] K. Thangavel, R. Manavalan and I. Laurence Aroquiaraj, Removal of Speckle Noise from Ultrasound Medical Image based on Special Filters: Comparative Study, International Conference on Graphics, Vision and Image Processing, **9**(2009), 25-32.

[14] J. W. Tukey, Nonlinear (nonsuperposable) methods for smoothing data, Proc. Congr. Rec. EASCOM '74, 1974, 673-681.

[15] Xin Wang, Optimal Edge-Preserving Hybrid Filters, IEEE Transactions on Image Processing, No. 6, **3**(1994), 862-865.

[16] Yanchun Wang, Dequn Liang, Heng Ma and Yan Wang, An Algorithm for Image Denoising Based on Mixed Filter, Proceedings of the 6^{th} World Congress on Intelligent Control and Automation, June 21-23, 2006, 9690-9693.

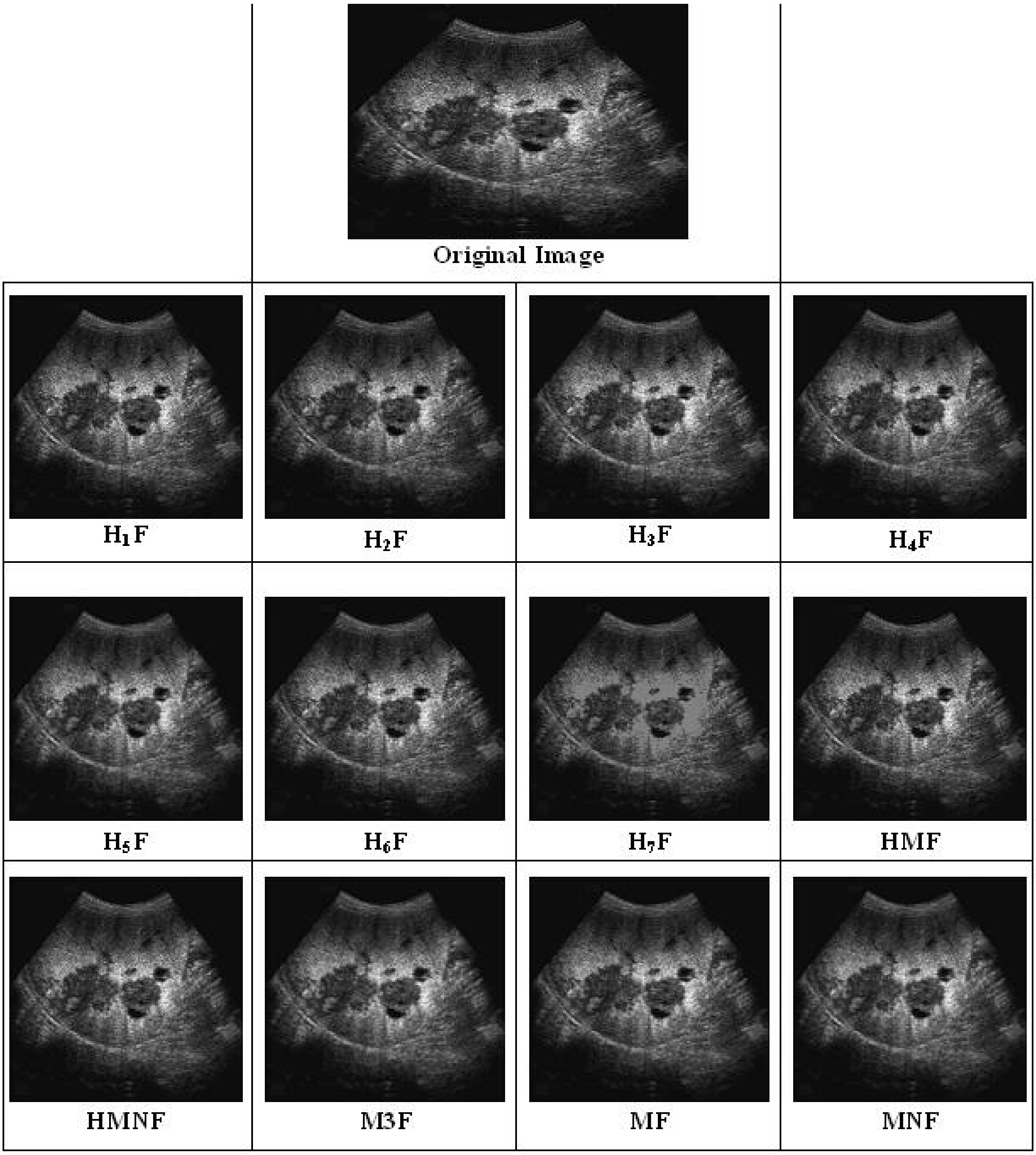


Figure 1: Denoising of Ultrasound liver tumor image.

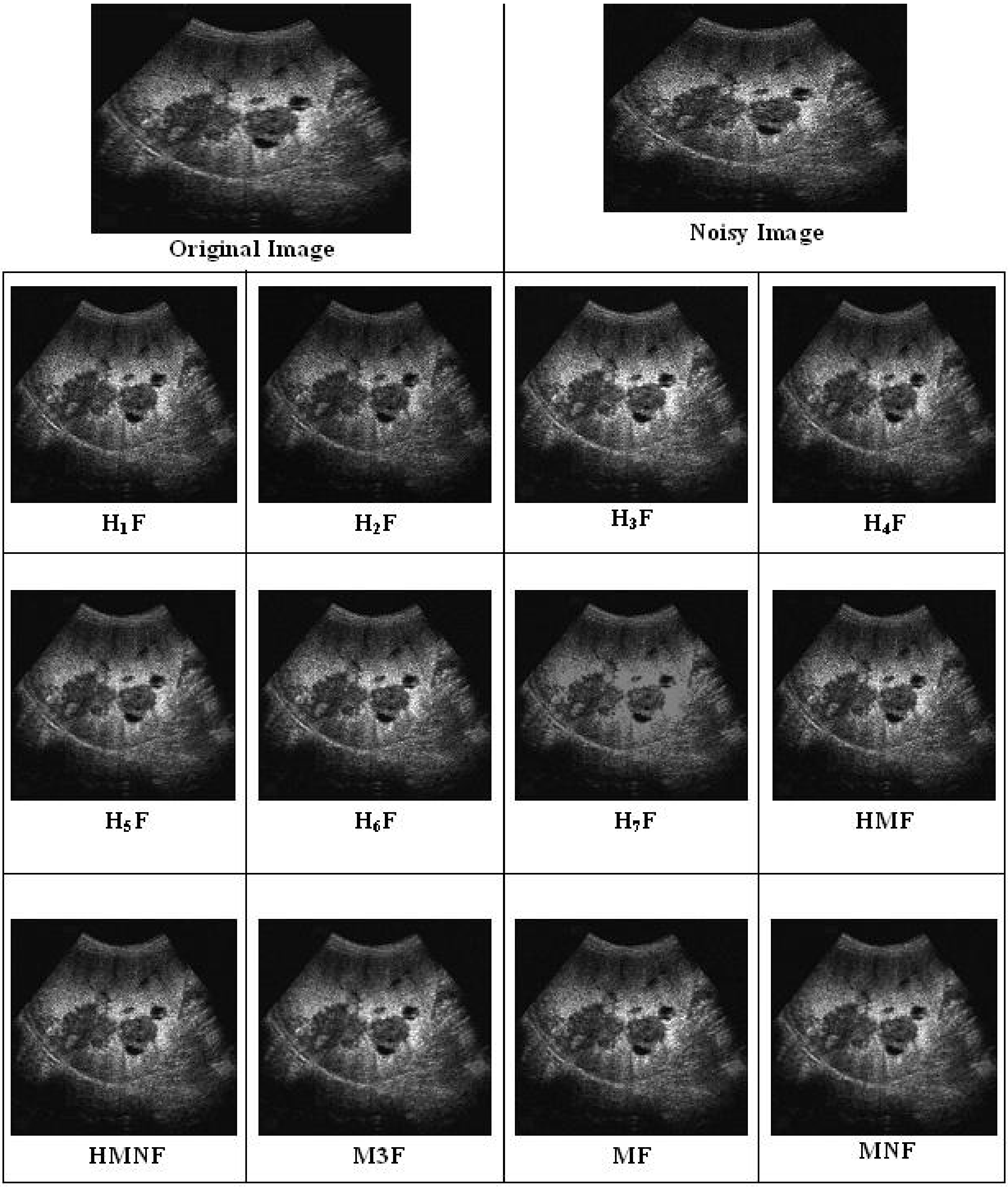


Figure 2: Denoising of Ultrasound liver tumor image corrupted by speckle noise of variance of 0.07.

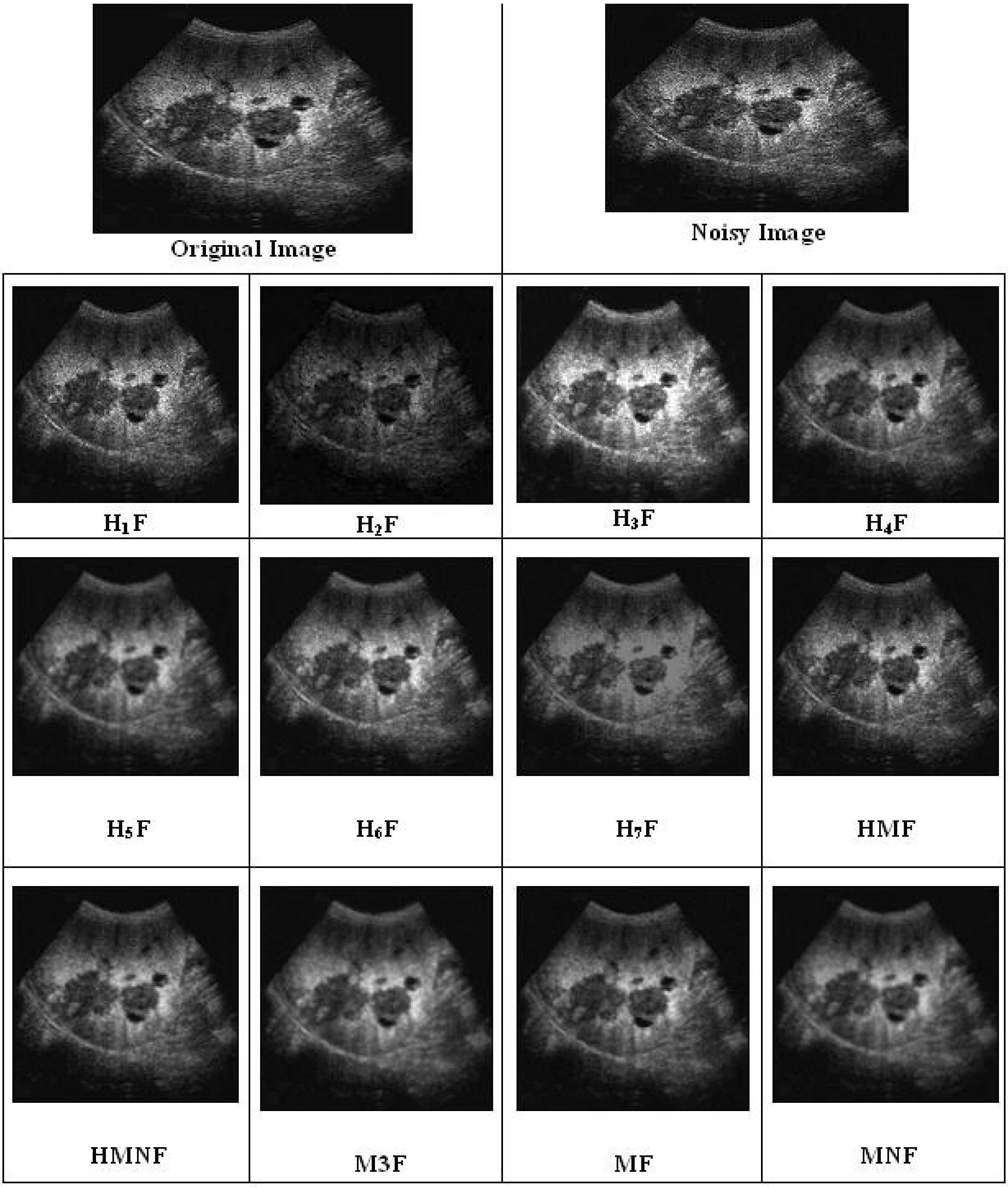


Figure 3: Denoising of Ultrasound liver tumor image corrupted by speckle noise of variance of 0.07, after 9^{th} iteration.

Scientia Magna
Vol. 7 (2011), No. 1, 54-60

Introduction of eigen values on relative character graphs

R. Stella Maragatham† and A. Vincent Jeyakumar‡

† Department of Mathematics, Queen Mary's College,
Chennai-4, Tamil Nadu, India
‡ Department of Computer Science, Periyar Maniammai University,
Tanjore Tamil Nadu, India
E-mail: r_stellae@yahoo.co.in Avjeyakumar2004@yahoo.com

Abstract RC-graphs are special type of graphs. This paper deals with the Eigen values of adjacency matrices and other related matrices of RC-graphs. Many interesting results on eigen values apart from eigen values of general graphs are obtained for RC-graphs. Several directions are initiated towards possible future study.

Keywords Relative character graph, adjacency matrix, Frobenious group, Borel subgroup, Dihedral group.

§1. Introduction

In this paper, we consider the adjacency matrix for RC-graphs and the eigen values are taken into account and a handful of results are obtained. An RC-graph is a finite simple graph $\Gamma(G, H)$ whose vertices are the complex irreducible characters of G and any two vertices ϕ and ψ are adjacent if and only if their restriction ϕ_A and ψ_H contain at least one irreducible character θ of H in common. The definition of adjacency matrix for RC-graphs is defined as follows.

§2. Definition of RC-graphs and main results

Let G denote any finite group and $Irr(G)$ denote the set of distinct complex irreducible characters of G. Let H be any subgroup of G. Then the Relative Character graph $\Gamma(G, H)$ (RC-graph) is defined as follows:

The vertex set V is $IrrG$. Two vertices and are adjacent if and only if their restrictions and to H contain at least one irreducible (complex) character of H is common (note that and need not be irreducible, but break into a direct sum of H-irreducibles). We refer to [5] for character theory of groups an [4] for graph theory.

§3. Basic properties of RC-graphs

(3.1) $\Gamma(G, H)$ is the null - graph if and only if $H = G$.

(3.2) $\Gamma(G, (1))$ is complete. However, the converse is not true.

(3.3) $\Gamma(G, H)$ is regular if and only if it is complete.

(3.4) $\Gamma(G, H)$ is connected if and only if $Core_G H = (1)$, where $Core_G H$ is the largest normal subgroup of G contained in H.

(3.5) The connected components of $\Gamma(G, H)$ are completely studied. Two vertices ϕ, ψ lie in the same component if and only if $\phi \subset \psi\psi^s$ for some $s \geq 1$, where $\psi =$ the induced character $1 \overset{G}{H}$.

(3.6) A group G is Frobenius if there is a nontrivial proper subgroup H such that $H \cap H^x = (1)$ for all $x \notin H$ $(H^x = xHx^{-1})$.

Then there exists a normal subgraph N such that G is the semidirect product NH. N is called the (unique) Frobenius kennel and H is called the (unique, upto contingency) Frobenius complement.

(3.7) $\Gamma(G, H)$ is a tree if and only if $G = NH$ is Frobenius and N is elementary abelian of order p^m and $O(H) = p^m - 1$.

(3.8) $\Gamma(G, H)$ is always triangulated.

(3.9) $\Gamma(G, H)$ is a naturally (edge) signed graph.

§4. The eigen value problem for RC-graphs

The adjacency matrix

For any finite, simple, undirected graph $\Gamma = \Gamma(V, E)$, the adjacency matrix $A = (a_{ij})$ is defined as follows:

$$a_{ij} = \begin{cases} 0, & \text{if } v_i \text{ and } v_j \text{ are not adjacent,} \\ 1, & \text{if } v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

It is well known that since A is a real symmetric matrix all its eigen values must be real. Order the eigen values as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, where $|V| = n$.

Before seeing where we digress for these special RC-graphs, we shall start from where the original results coincide for RC-graphs.

Recall from (3.2) that when (1) denote the trivial subgroup of G, then $\Gamma(G, (1))$ is the complete graph K_n.

Theorem 4.1. The eigen values of $\Gamma(G, (1))$ are - 1 (repeated $n-1$ times), $n-1$. (Notice that the sum must be 0 since this sum equals the sum of diagonal entries of A which is trivially 0.)

Theorem 4.2. Let G be an abelian group of order G and let H be a subgroup of order h. Then the distinct eigen values of $\Gamma(G, H)$ are -1 and $(g/h) - 1$ where -1 is repeated $g - h$ times and $(g/h) - 1$ is repeated h times.

Proof. First note that by Lagrange's theorem g/h is an integer. It is known that (i) $\Gamma(G, H)$ is a graph with g vertices and h connected components and (ii) each component is the complete graph $K_{g/h}$ (see eg. [3]).

We can rearrange the vertices of V such that the matrix A breaks into h blocks, where the i^{th} block is the $g/h \times g/h$ adjacency matrix of the i^{th} component K_i and the other block are 0's, $1 \leq i \leq h$.

Clearly the eigen values of $\Gamma(G, H)$ are those of $K_{g/h}$, repeated h times. Since the eigen values of $K_{g/h}$ are $-1, -1, \ldots, -1$, $(g/h)-1$ (-1 repeated $(g/h)-1$ times), overall, -1 is repeated $h(g/h-1) = g-h$ times and $(g/h)-1$ is repeated h times.

Hence the theorem.

§5. The case when G is non-abelian

When H is the trivial subgroup, then $\Gamma(G, H)$ is complete and this case is already taken care of. Hence we can assume that H is non-trivial.

There are two approaches. First, to settle with reasonable bounds for the eigen values for arbitrary pair (G, H). Second is to take special groups and subgroups and actually compute the eigen values. We shall take up the second route in this paper.

The Frobenius groups

Let $G = NH$ be a Frobenius group. That means, H is a (non-normal) subgroup such that $H \cap H^x = (1)$ and N is normal defined by

$$N = \left\{ G - \bigcup_{x \notin H} H^x \right\} \cup \{1\}.$$

As examples we have S_3, A_4, the dihedral group D_{2m}, m odd. For properties and character theory of G, we refer to [4].

The set $IrrG$ is the disjoin union $A \cup B$, when

$A = \{\phi | Ker\phi \supset N\}$,

$B = \{\phi | Ker\phi \not\supset N\}$.

Theorem 5.1.[3] (i) The graph $\Gamma(G, H)$ is connected, $|V| = a + b$ where $a = |A| = |IrrH|$ and $b = |B| = t/h(t+1) = |IrrH|$ and $|H| = h$.

(ii) $\Gamma(G, H)$ contains K_b as a complete subgraph and the remaining a vertices are such that each one is adjacent to every vertex in K_b.

(iii) None of these a vertices (which include I_G) are adjacent among themselves.

Theorem 5.2. Let $G = NH$ be Frobenius, such that $|B| > 1$. Then the eigen values of the adjacency matrix of $\Gamma(G, H)$ constitute the set: $\{0$ (repeater) a -1 times) -1 (repeated b times) and $b\}$.

Proof. The vertices can be arranged in such a way that the first a vertices are taken from A (in some order) and the next b vertices are taken from B (in some order).

Then the adjacency matrix A is of the form:

(1) The top left corner has $a \times a$ zero matrix block.

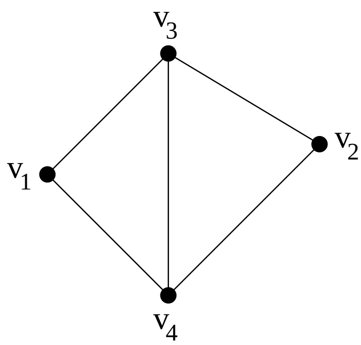
(2) The remaining diagonal entries are 0's.

(3) All other entries are 1's.

By simple matrix manipulations, the eigen values, as stated in the theorem, can be easily found.

Example 5.3. Let $G = D_{10} = C_5 \cdot C_2$, where D_{10} is the Dihedral group of 10 elements, C_2 and are cyclic subgroups of orders 2 and 5 respectively, with normal (Frobenius Kernel) and non-normal (Frobenius complement). We take $H = C_2$.

Then is the following graph: $A = \{V_1, V_2\}$, $B = \{V_3, V_4\}$.



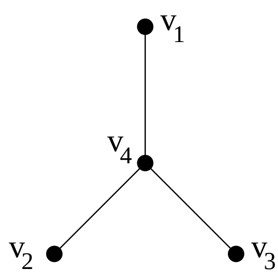
The matrix $A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$.

The eigen values are $\{0, -1, -1, 2\}$. There are some special Frobenius groups for which $\Gamma(G, H)$ become trees.

Theorem 5.4. Let $G = NH$ be Frobenius so that $|B| = 1$. Then $\Gamma(G, H)$ is a star and the eigen values of A are : $\{0$ (repeated a - 1 times and $\pm\sqrt{d}$ where d is the degree of the middle vertex$\}$.

Proof. In this case, all rows of A, except the last have 0's everywhere except 1 at the last column. The last row has 1's everywhere, except a 0 at the last place. Then $|\lambda I - A| = \lambda^{a-1}(\lambda^2 - d)$. Thus the eigen values are: 0 (repeated a -1 times and $\pi\sqrt{d}$).

Example 5.5. Let $G = A_4 = V_4 \cdot C_3$, where V is the Klein-four groups. Take $H = C_3$. Then $\Gamma(G, H)$ is the following star.



$A = \{v_1, v_2, v_3\}$, $B = \{v_4\}$.

Here $v_1 v_2$ and v_3 have character degrees 1 and v_4 has character degree 3. The eigen values are $\{0, 0, \pm\sqrt{3}\}$.

It is remarkable that the only RC-graphs which are trees are stars!

It is indeed a difficult job to fix the eigen values for arbitrary $\Gamma(G, H)$.

However we get some reasonable bounds for a well-known class of groups and subgroups.

Lemma 5.6. Let G be an arbitrary group and H, a nontrivial subgroup. Let the right action of G on G/H be doubly transitive. Then

(i) $\Gamma(G, H)$ consists of a subgraph T together with the trivial character adjacent to a unique vertex $\phi \in T$.

(ii) The eigen values $\Gamma(G, H)$ are caught up in the following inequalities:

$$\lambda_i(\Gamma) \geq \lambda_i(\Gamma - \{v\}) \geq \lambda_{i+1}(\Gamma) \quad (1 \leq i \leq n-1).$$

Proof. The first part of the statement is already well known (see eg. [3]). For (ii) we use the corresponding result in [7].

Theorem 5.7. Let $G = PSLC(2, q)$, $q = p^n$, p a prime, $H =$ Borel subgroup B of order $q(q-1)$. Then we get the following bounds for the eigen values of the adjacency matrix A of $\Gamma(G, H)$,

$$\lambda_1 \geq n-1 \geq \lambda_2 \geq -1,$$

$$\lambda_3 \geq \lambda_4 \geq \cdots \geq \lambda_{n-1} = 1, \ \lambda_n = -1.$$

Proof. The graph $\Gamma(G, H)$ consists of the complete subgraph K_{n-1} together with the vertex 1_G adjacent to a unique vertex of K_{n-1}.

Then taking v as the vertex corresponding to 1_G and using the result of [8], and Lemma 5.6, we immediately get our results.

§6. Laplace graphs

The Laplace Graph $L = D - A$, where D is the diagonal matrix whose $(i,i)^{th}$ entry is the vertex degree d_i.

Theorem 6.1. The cofactors of L have a common value k which also equals the number of spanning trees of L (this is the famous Matrix - Tree Theorem).

From this result, we can also derive the following results.

Corollary 6.2.

(i) $nk = \mu_1 \mu_2 \ldots \mu_{n-1}$ where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ are the eigen values of L.

(ii) L is connected if and only if $\mu_{n-1} > 0$.

There is a special case where in our $\Gamma(G, H)$ graph, the graph degree of each vertex is equal to the degree of that vertex as an irreducible character. This occurs for instance when $\Gamma(G, H)$ is a tree (star), with an additional property on H.

Theorem 6.3. Let $G = NH$ be a Frobenius group such that

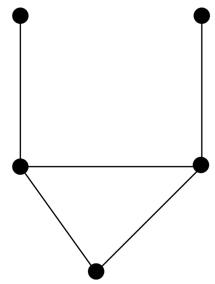
(i) N is elementary abelian of order p^m.

(ii) $O(H) = p^m - 1$.

(iii) H is abelian. Then for every vertex of $\Gamma(G, H)$, the character degree is the same as the graph degree.

Proof. First recall that the graph is a star and hence-except the middle vertex, all order vertices have graph degree one. But since H is abelian and hence every irreducible character ϕ_i has degree one, by the property of Frobenius groups, all these ϕ_i can be 'lifted' as irreducible characters of G as well. Hence the character degree of each pendent vertex v_i = graph degree of v_i. Finally let ψ denote the middle vertex of $\Gamma(G, H)$. Then, graph degree ψ = number of pendant vertices = order of H = character degree of ψ. Hence the theorem.

Remark 6.4. There are other cases where these two degrees precisely coincide. For instand let $G = S_4$ and $H = S_3$ (sitting inside S_4). Then $\Gamma(G, H)$, is the following graph.



Now the vertex degree of $\Gamma(G, H)$ are $\{1, 1, 2, 3, 3\}$. It is remarkable that the corresponding character degrees are also 1, 1, 2, 3 and 3.

Theorem 6.5. For all the above cases, we can replace the graph degrees in L by the corresponding character degrees and still maintain the same properties and get the same results.

§7. Future directions

We propose the following directions in which this study of eigen value problem for RC-graphs can be extended.

1. Put $L^* = D^* - A$ where D^* is the diagonal matrix when $(i, i)^{th}$-entry is equal to the character degree of the vertex v_i (corresponding to the i^{th} irreducible character ϕ_i); study the eigen value problem for L^*.

2. The group G acts on the set $IrrG$ by conjugation (If $\phi \in IrrG$, $\phi \in IrrG$ where $\phi^g(x) = \phi(g \times g^{-1})$). This action also preserves the adjacency property: ϕ is adjacent to ψ if and only if ϕ^g is adjacent to ψ^g. In this sense, our RC-graph becomes a pseudo-homogenous graphs, generalizing the classical definition of homogeneous graphs. One can initiate a study of eigen value problem in the context of (Pseudo-homogenous) RC-graphs, following the works of F. R. K. Chang [1], [2] and others.

3. One can use QR-Factorization to obtain deeper and finer bounds for the eigen values of general RC-graphs, in particular, when G is non-abelian simple.

References

[1] F. R. K. Chung, Eigen values of graphs, proceeding of ICM, Zurich, 1994, 1333-1342.

[2] Lectures on Spectral Graph Theory, CBMS Lecture Notes, AMS Publications, 1995.

[3] T. Gnanaseelan, Studies in algebraic groups and representations, Ph. D. Thesis, Madurai Kamaraj University, Madurai (India), 2000.

[4] F. Harary, Graph Theory, Addison-Wesley Publ. Co., Reading, Massachusetts, 1969.

[5] M. Isaacs, Character theory of Finite graphs, Academic Press, New York, 1969.

[6] A. V. Jeyakumar, Construction of some new finite graphs using group representations, In: Graphs, Combinatorics, Algorithms and Applications (Eds: S. Arumugam, B. D. Acharya and S. B. Rao), 49-58. Narosa Publishing House, New Delhi, 2004.

[7] H. Yuan, Bounds of Eigen values of a graph, Acta Mathematics Applicatae Sinica, Vol. 4, 165-168.

Scientia Magna
Vol. 7 (2011), No. 1, 61-68

A result about Young's inequality and several applications

Nicuşor Minculete

"Dimitrie Cantemir" University, Braşov 500068, România
E-mail: minculeten@yahoo.com

Abstract The main aim of this paper is to show a refinement of Young's inequality. We also give several applications of itself. Among these, we have a refinement of the following inequalities: Bernoulli's inequality, Hölder's inequality, Cauchy's inequality and Minkowski's inequality.

Keywords Young's inequality, Hölder's inequality, Cauchy's inequality, Minkowski's inequality.

§1. Introduction

A series of the inequalities played an important role in various fields of mathematics. Among these we found the famous Young inequality

$$\lambda a + (1-\lambda)b \geq a^{\lambda} b^{1-\lambda}, \tag{1}$$

for nonnegative real numbers a, b and $\lambda \in [0,1]$.

The Young inequality was refined by F. Kittaneh and Y. Manasrah in [6], thus:

$$\lambda a + (1-\lambda)b \geq a^{\lambda} b^{1-\lambda} + r(\sqrt{a} - \sqrt{b})^2, \tag{2}$$

where $r = \min\{\lambda, 1-\lambda\}$.

This inequality was generalized by S. Furuichi in [4], thus

$$\sum_{i=1}^{n} p_i a_i - \prod_{i=1}^{n} a_i^{p_i} \geq n p_{\min} \left(\frac{1}{n} \sum_{i=1}^{n} a_i - \prod_{i=1}^{n} a_i^{1/n} \right), \tag{3}$$

for $a_1, \ldots, a_n \geq 0$ and $p_1, \ldots, p_n \geq 0$ with $p_1 + \cdots + p_n = 1$, where $p_{\min} = \min\{p_1, \ldots, p_n\}$.

Another generalizations can be found by J. M. Aldaz in [1] and [2].

In [9], M. Tominaga, showed the reverse inequality for Young's inequality, using Specth's ratio, thus

$$S\left(\frac{a}{b}\right) a^{\lambda} b^{1-\lambda} \geq \lambda a + (1-\lambda)b, \tag{4}$$

where the Specht's ratio [8] was defined by

$$S(h) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}, \ (h \neq 1)$$

for a positive real number h.

S. Furuichi, in [5] given another type of the improvement of the classical Young inequality by Specht's ratios, thus

$$\lambda a + (1-\lambda)b \geq S\left(\left(\frac{a}{b}\right)^r\right) a^\lambda b^{1-\lambda}. \tag{5}$$

In fact Young's inequality is a special case of the Jensen inequality. Therefore, we seek some improvements of this inequality in many papers and books.

A main result given by S. Dragomir [3], in general form, is studied by F. C. Mitroi [7] in a particular case, thus

$$\begin{aligned} np_{\min}\left(\frac{1}{n}\sum_{i=1}^n f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^n x_i\right)\right) &\leq \sum_{i=1}^n \alpha_i f(x_i) - f\left(\sum_{i=1}^n \alpha_i x_i\right) \\ &\leq np_{\max}\left(\frac{1}{n}\sum_{i=1}^n f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^n x_i\right)\right) \end{aligned} \tag{6}$$

where f is a convex function, $p_i > 0$ for all $i = 1, ..., n$ and $\sum_{i=1}^n p_i = 1$.

§2. Main results

Theorem 2.1. For $a, b > 0$ and $\lambda \in (0,1)$, we have

$$a^\lambda b^{1-\lambda}\left(\frac{a+b}{2\sqrt{ab}}\right)^{2r} \leq \lambda a + (1-\lambda)b \leq a^\lambda b^{1-\lambda}\left(\frac{a+b}{2\sqrt{ab}}\right)^{2(1-r)}, \tag{7}$$

where $r = \min\{\lambda, 1-\lambda\}$.

Proof. In inequality (6) for $n = 2,\ p_1 = \lambda,\ p_2 = 1-\lambda$, with $\lambda \in (0,1),\ x_1 = a,\ x_2 = b,\ f(x) = -\log x$ and taking account that $1 - r = \max\{\lambda, 1-\lambda\}$ when $r = \min\{\lambda, 1-\lambda\}$, we deduce the inequality of the statement.

Remark 2.1.

a) Because $\frac{a+b}{2} \geq \sqrt{ab}$, it follows that $\frac{a+b}{2\sqrt{ab}} \geq 1$ and using inequality (7) we obtain the Young inequality.

b) In relation (7) we have equality if only if $a = b$.

Theorem 2.2. For $x > -1$ and $\lambda \in (0,1)$, we have the inequality

$$(x+1)^\lambda\left[\frac{(x+1)^2+1}{2(x+1)}\right]^{2r} \leq \lambda x + 1 \leq (x+1)^\lambda\left[\frac{(x+1)^2+1}{2(x+1)}\right]^{2(1-r)}, \tag{8}$$

where $r = \min\{\lambda, 1-\lambda\}$.

Proof. If we take $\frac{a}{b} = t$ in inequality (7), then we have the following inequality

$$t^\lambda\left(\frac{t^2+1}{2t}\right)^{2r} \leq \lambda t + (1-\lambda) \leq t^\lambda\left(\frac{t^2+1}{2t}\right)^{2(1-r)}. \tag{9}$$

But, making the substitution $t = x + 1$ in relation (9) we have inequality (8).

Remark 2.2. Taking into account that $\dfrac{(x+1)^2+1}{2(x+1)} \geq 1$, it is easy to see that inequality (8) is an improvement of the Bernoulli inequality (in the case $\lambda \in (0,1)$). The equality holds when $x = 1$.

Theorem 2.3. Let $p, q > 1$ be real numbers satisfying $\dfrac{1}{p} + \dfrac{1}{q} = 1$. If $a_i, b_i > 0$ for all $i = 1, \ldots, n$ then there is the inequality

$$m^{2r} \sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^p \right)^{1/p} \left(\sum_{i=1}^{n} b_i^q \right)^{1/q} \leq M^{2(1-r)} \sum_{i=1}^{n} a_i b_i, \tag{10}$$

where $r = \min\left\{ \dfrac{1}{p}, \dfrac{1}{q} \right\}$, $A_i = \dfrac{a_i^p \sum\limits_{i=1}^{n} b_i^q + b_i^q \sum\limits_{i=1}^{n} a_i^p}{2\sqrt{a_i^p b_i^q \sum\limits_{i=1}^{n} a_i^p \sum\limits_{i=1}^{n} b_i^q}}$, $m = \min\limits_{1 \leq i \leq n} A_i$ and $M = \max\limits_{1 \leq i \leq n} A_i$.

Proof. In Theorem 2.1 we take $\lambda = \dfrac{1}{p}$, which implies $1 - \lambda = \dfrac{1}{q}$ and $a = \dfrac{a_i^p}{\sum\limits_{i=1}^{n} a_i^p}$, $b = \dfrac{b_i^q}{\sum\limits_{i=1}^{n} b_i^q}$, thus we obtain

$$\begin{aligned}
m^{2r} \cdot \frac{a_i b_i}{\left(\sum\limits_{i=1}^{n} a_i^p \right)^{1/p} \left(\sum\limits_{i=1}^{n} b_i^q \right)^{1/q}} &\leq \frac{a_i b_i}{\left(\sum\limits_{i=1}^{n} a_i^p \right)^{1/p} \left(\sum\limits_{i=1}^{n} b_i^q \right)^{1/q}} \cdot A_i^{2r} \\
&\leq \frac{a_i^p}{p \sum\limits_{i=1}^{n} a_i^p} + \frac{b_i^q}{q \sum\limits_{i=1}^{n} b_i^q} \\
&\leq \frac{a_i b_i}{\left(\sum\limits_{i=1}^{n} a_i^p \right)^{1/p} \left(\sum\limits_{i=1}^{n} b_i^q \right)^{1/q}} A_i^{2(1-r)} \\
&\leq M^{2(1-r)} \cdot \frac{a_i b_i}{\left(\sum\limits_{i=1}^{n} a_i^p \right)^{1/p} \left(\sum\limits_{i=1}^{n} b_i^q \right)^{1/q}}.
\end{aligned} \tag{11}$$

Making the sum for $i = 1, ..., n$ we deduce inequality (10).

Remark 2.3.

a) It is easy to see that $m \geq 1$ and using inequality (10) we have a refinement of Hölder's inequality.

b) In relation (10) the equality holds when $a_1 = \cdots = a_n$ and $b_1 = \cdots = b_n$.

c) For $p = q = 2$ in inequality (10), we obtain a refinement of Cauchy's inequality

$$\left(\sum_{i=1}^{n} a_i b_i \right)^2 \leq m \left(\sum_{i=1}^{n} a_i b_i \right)^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \leq M \left(\sum_{i=1}^{n} a_i b_i \right)^2, \tag{12}$$

where $A_i = \dfrac{a_i^2 \sum_{i=1}^{n} b_i^2 + b_i^2 \sum_{i=1}^{n} a_i^2}{2a_i b_i \sqrt{\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)}}$, $m = \min_{1\le i\le n} A_i$ and $M = \max_{1\le i\le n} A_i$.

Theorem 2.4. For any real numbers $a_i,\ b_i > 0$, for all $i = 1, ..., n$ and $p > 0$, we have

$$m^{2r}\left[\sum_{i=1}^{n}(a_i+b_i)^p\right]^{1/p} \le \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} + \left(\sum_{i=1}^{n} b_i^p\right)^{1/p} \le M^{2(1-r)}\left[\sum_{i=1}^{n}(a_i+b_i)^p\right]^{1/p} \tag{13}$$

where $r = \min\left\{\dfrac{1}{p}, 1-\dfrac{1}{p}\right\}$,

$$A_{i1} = \frac{a_i^p \sum_{i=1}^{n}(a_i+b_i)^p + (a_i+b_i)^p \sum_{i=1}^{n} a_i^p}{2\sqrt{a_i^p(a_i+b_i)^p\left(\sum_{i=1}^{n} a_i^p\right)\left(\sum_{i=1}^{n}(a_i+b_i)^p\right)}},$$

$$A_{i2} = \frac{b_i^p \sum_{i=1}^{m}(a_i+b_i)^p + (a_i+b_i)^p \sum_{i=1}^{n} b_i^p}{2\sqrt{(a_i+b_i)^p b_i^p\left(\sum_{i=1}^{n}(a_i+b_i)^p\right)\left(\sum_{i=1}^{n} b_i^p\right)}},$$

$m = \min_{1\le i\le n}\{A_{i1}, A_{i2}\}$ and $M = \max_{1\le i\le n}\{A_{i1}, A_{i2}\}$.

Proof. To prove this inequality, we will use the improvement of Hölder's inequality from relation (10). We write

$$(a_i+b_i)^p = a_i(a_i+b_i)^{p-1} + b_i(a_i+b_i)^{p-1},$$

so

$$\sum_{i=1}^{n}(a_i+b_i)^p = \sum_{i=1}^{n} a_i(a_i+b_i)^{p-1} + \sum_{i=1}^{n} b_i(a_i+b_i)^{p-1}.$$

Right now we apply inequality (10), in the following way,

$$\begin{aligned}(\min_{1\le i\le n} A_{i1})^{2r}\sum_{i=1}^{n} a_i(a_i+b_i)^{p-1} &\le \left(\sum_{i=1}^{n} a_i^p\right)^{1/p}\left(\sum_{i=1}^{n}(a_i+b_i)^{(p-1)q}\right)^{1/q} \\ &\le (\max_{1\le i\le n} A_{i1})^{2(1-r)}\sum_{i=1}^{n} a_i(a_i+b_i)^{p-1},\end{aligned} \tag{14}$$

$$\begin{aligned}(\min_{1\le i\le n} A_{2i})^{2r}\sum_{i=1}^{n} b_i(a_i+b_i)^{p-1} &\le \left(\sum_{i=1}^{n} b_i^p\right)^{1/p}\left(\sum_{i=1}^{n}(a_i+b_i)^{(p-1)q}\right)^{1/q} \\ &\le (\max_{1\le i\le n} A_{2i})^{2(1-r)}\sum_{i=1}^{n} b_i(a_i+b_i)^{p-1}.\end{aligned} \tag{15}$$

But $(p-1)q = p$, because $\frac{1}{p} + \frac{1}{q} = 1$. Adding relations (14) and (15), and taking into account that $m = \min_{1\le i\le n}\{A_{i1}, A_{i2}\}$ and $M = \max_{1\le i\le n}\{A_{i1}, A_{i2}\}$, we deduce the inequality

$$\begin{aligned} m^{2r}\sum_{i=1}^{n}(a_i+b_i)^p &\le \left[\left(\sum_{i=1}^{n} a_i^p\right)^{1/p} + \left(\sum_{i=1}^{p} b_i^p\right)^{1/p}\right]\left[\sum_{i=1}^{n}(a_i+b_i)^p\right]^{1/q} \\ &\le M^{2(1-r)}\sum_{i=1}^{n}(a_i+b_i)^p \qquad (16) \end{aligned}$$

Dividing by $\sum_{i=1}^{n}(a_i+b_i)^p$ in relation (14), we obtain the inequality required.

Remark 2.4.

a) Since $m \ge 1$, we have an improvement of Minkowski's inequality.

b) The equality holds in relation (13) for $a_1 = \cdots = a_n$ and $b_1 = \cdots = b_n$. The integral versions of these inequality can be formulated as follows.

Theorem 2.5. Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions $f, g \neq 0$ defined on $[a, b]$ such that $|f|^p$ and $|g|^p$ are integrable functions on $[a, b]$, then

$$\begin{aligned} m^{2r}\left(\int_a^b |f(x)g(x)|dx\right) &\le \left(\int_a^b |f(x)|^p dx\right)^{1/p}\left(\int_a^b |g(x)|^q dx\right)^{1/q} \\ &\le M^{2(1-r)}\left(\int_a^b |f(x)g(x)|dx\right), \qquad (17) \end{aligned}$$

where $r = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$,

$$A(x) = \frac{|f(x)|^p \int_a^b |g(x)|^q dx + |g(x)|^q \int_a^b |f(x)|^p dx}{2\sqrt{|f(x)|^p|g(x)|^q \int_a^b |f(x)|^p dx \int_a^b |g(x)|^q dx}},$$

$m = \min_{x\in[a,b]} A(x)$ and $M = \max_{x\in[a,b]} A(x)$.

Equality holds iff $|f(x)|^p = |g(x)|^q$.

Proof. We consider in Theorem 2.1 that $\lambda = \frac{1}{p}$ and $a = \dfrac{|f(x)|^p}{\int_a^b |f(x)|^p dx}$, $b = \dfrac{|g(x)|^q}{\int_a^b |g(x)|^q dx}$.

Therefore, we obtain

$$\begin{aligned} &m^{2r}\cdot\frac{|f(x)g(x)|}{\left(\int_a^b |f(x)|^p dx\right)^{1/p}\left(\int_a^b |g(x)|^q dx\right)^{1/q}} \le \frac{|f(x)g(x)|\cdot A^{2r}(x)}{\left(\int_a^b |f(x)|^p dx\right)^{1/p}\left(\int_a^b |g(x)|^q dx\right)^{1/q}} \\ &\le \frac{|f(x)|^p}{p\left(\int_a^b |f(x)|^p dx\right)^{1/p}} + \frac{|g(x)|^q}{q\left(\int_a^b |g(x)|^q dx\right)^{1/q}} \le \frac{|f(x)g(x)|\cdot A^{2(1-r)}(x)}{\left(\int_a^b |f(x)|^p dx\right)^{1/p}\left(\int_a^b |g(x)|^q dx\right)^{1/q}} \end{aligned}$$

$$\leq \frac{|f(x)g(x)|}{\left(\int_a^b |f(x)|^p dx\right)^{1/p} \left(\int_a^b |g(x)|^q dx\right)^{1/q}} \cdot M^{2(1-r)}.$$

By integrates from a to b in above inequality and by simple calculations, we deduce the inequality of statement. For $|f(x)|^p = |g(x)|^q$ it is obvious that the equality holds.

Remark 2.5.

a) Because $m \geq 1$ and according to inequality (17), we find a refinement for the integrated version of the Hölder inequality.

b) For $p = q = 2$, we deduce a refinement for the integral version of the Cauchy inequality can be formulated as follows:

$$\begin{aligned} \left(\int_a^b |f(x)g(x)|dx\right)^2 &\leq m\left(\int_a^b |f(x)g(x)|dx\right)^2 \\ &\leq \left(\int_a^b f^2(x)dx\right)\left(\int_a^b g^2(x)dx\right) \\ &\leq M\left(\int_a^b |f(x)g(x)|dx\right)^2, \end{aligned} \tag{18}$$

where $A(x) = \dfrac{f^2(x)\displaystyle\int_a^b g^2(x)dx + g^2(x)\int_a^b f^2(x)dx}{2|f(x)g(x)|\sqrt{\displaystyle\int_a^b f^2(x)dx \cdot \int_a^b g^2(x)dx}}$, $m = \min\limits_{x\in[a,b]} A(x)$, $M = \max\limits_{x\in[a,b]} A(x)$ and $f(x),\ g(x) \neq 0$ for any $x \in [a,b]$.

Theorem 2.6. Let $p > 1$ and $f, g \neq 0$, two real functions defined on $[a,b]$ such that $|f|^p$ and $|g|^p$ are integrable functions on $[a,b]$, then

$$\begin{aligned} m^{2r}\left(\int_a^b |f(x)+g(x)|^p dx\right)^{1/p} &\leq \left(\int_a^b |f(x)|^p dx\right)^{1/p} + \left(\int_a^b |g(x)|^p dx\right)^{1/p} \\ &\leq M^{2(1-r)} \frac{\displaystyle\int_a^b (|f(x)|+|g(x)|)|f(x)+g(x)|^{p-1}dx}{\left(\displaystyle\int_a^b |f(x)+g(x)|^p dx\right)^{1-\frac{1}{p}}}, \end{aligned} \tag{19}$$

where $r = \min\{p, 1 - \dfrac{1}{p}\}$,

$$A_1(x) = \frac{|f(x)|^p \displaystyle\int_a^b |f(x)+g(x)|^p dx + |f(x)+g(x)|^p \int_a^b |f(x)|^p dx}{2\sqrt{|f(x)(f(x)+g(x))|^p \displaystyle\int_a^b |f(x)|^p dx \cdot \int_a^b |f(x)+g(x)|^p dx}},$$

$$A_2(x) = \frac{|g(x)|^p \displaystyle\int_a^b |f(x)+g(x)|^p dx + |f(x)+g(x)|^p \int_a^b |g(x)|^p dx}{2\sqrt{|g(x)(f(x)+g(x))|^p \displaystyle\int_a^b |(f(x)+g(x))|^p dx \cdot \int_a^b |g(x)|^p dx}},$$

$m = \min\limits_{x\in[a,b]}\{A_1(x), A_2(x)\}$ and $M = \max\limits_{x\in[a,b]}\{A_1(x), A_2(x)\}$.

Proof. Since the Hölder inequality is used to prove the Minkowski inequality, then we use Theorem 2.5, which is refinement of Hölder's inequality, for to prove inequality (19). Therefore

$$|f(x)+g(x)|^p \le |f(x)||f(x)+g(x)|^{p-1} + |g(x)|\cdot|f(x)+g(x)|^{p-1},$$

it follows that

$$\int_a^b |f(x)+g(x)|^p dx \le \int_a^b |f(x)||f(x)+g(x)|^{p-1}dx + \int_a^b |g(x)||f(x)+g(x)|^{p-1}dx.$$

We apply Theorem 2.5 in the following way:

$$\begin{aligned} m_1^{2r}\left(\int_a^b |f(x)|\cdot|f(x)+g(x)|^{p-1}dx\right) &\le \left(\int_a^b |f(x)|^p dx\right)^p \left(\int_a^b |f(x)+g(x)|^{(p-1)q}dx\right)^{1/q} \\ &\le M_1^{2(1-r)}\left(\int_a^b |f(x)|\cdot|f(x)+g(x)|^{p-1}dx\right), \qquad (20) \end{aligned}$$

where $m_1 = \min\limits_{x\in[a,b]} A_1(x),\ M_1 = \max\limits_{x\in[a,b]} A_2(x)$ and $r = \min\{\frac{1}{p}, \frac{1}{q}\}$.

In analogous way, we have

$$\begin{aligned} m_2^{2r}(\int_a^b |g(x)|\cdot|f(x)+g(x)|^{p-1}dx) &\le \left(\int_a^b |g(x)|^p dx\right)^{1/p} \left(\int_a^b |f(x)+g(x)|^{(p-1)q}dx\right)^{1/q} \\ &\le M_2^{2(1-r)}\left(\int_a^b |g(x)||f(x)+g(x)|^{p-1}dx\right), \qquad (21) \end{aligned}$$

where $m_2 = \min\limits_{x\in[a,b]} A_2(x)$ and $M_2 = \max\limits_{x\in[a,b]} A_2(x)$.

But $(p-1)q = p$. Therefore, adding inequalities (20) and (21), and taking into account that $m = \min\limits_{x\in[a,b]}\{A_1(x), A_2(x)\},\ M = \max\limits_{x\in[a,b]}\{A_1(x), A_2(x)\}$ we deduce

$$\begin{aligned} & m^{2r}\left(\textstyle\int_a^b |f(x)+g(x)|^p dx\right) \\ \le\ & \left[\left(\int_a^b |f(x)|^p dx\right)^{1/p} + \left(\int_a^b |g(x)|^p dx\right)^{1/p}\right]\left(\int_a^b |f(x)+g(x)|^p dx\right)^{1/q} \\ \le\ & M^{2(1-r)}\int_a^b (|f(x)|+|g(x)|)\,(|f(x)+g(x)|^{p-1})dx. \qquad (22) \end{aligned}$$

Dividing the above inequality by $\left(\int_a^b |f(x)+g(x)|^p dx\right)^{1/q}$, we obtain the inequality desired.

References

[1] J. M. Aldaz, Comparison of differences between arithmetic and geometric means, arXiv 1001. 5055v2, 2010.

[2] J. M. Aldaz, Self-improvement of the inequality between arithmetic and geometric means, Journal of Mathematical Inequalities, **3**(2009), No. 2, 213-216.

[3] S. S. Dragomir, Bounds for the Normalized Jensen Functional, Bull. Austral. Math. Soc., **74**(2006), No. 3, 471-478.

[4] S. Furuichi, A refinement of the arithmetic-geometric mean inequality, arXiv: 0912. 52 27v1, 2009.

[5] S. Furuichi, Refined Young inequalities with Specht's ratio, arXiv: 1004. 0581v2, 2010.

[6] F. Kittaneh and Y. Manasrah, Improved Young and Heinz inequalities for matrices, J. Math. Anal. Appl., **36**(2010), 262-269.

[7] F. C. Mitroi, About the precision in Jensen-Steffensen inequality, Annals of the University of Craiova, Mathematics and Computer Science Series, **37**(2010), No. 3, 73-84.

[8] W. Specht, Zer Theorie der elementaren Mittel, Math. Z., **74**(1960), 91-98.

[9] M. Tominaga, Specht'ratio in the Young inequality, Sci. Math. Japon, **55**(2002), 538-588.

Scientia Magna
Vol. 7 (2011), No. 1, 69-73

Composition operators of k-paranormal operators

S. Panayappan[†], D. Sumathi[‡] and N. Jayanthi[♯]

Post Graduate and Research Department of Mathematics,
Government Arts College, Coimbatore-18, India
E-mail: Jayanthipadmanaban@yahoo.in

Abstract In this article, composition operators and weighted composition operators of k-paranormal operators and their adjoints are characterized in L^2 spaces.

Keywords K-paranormal operators, composition operators, weighted composition operators, Aluthge transformation.

§1. Introduction

Let $B(H)$ be the Banach Algebra of all bounded linear operators on a non-zero complex Hilbert space H. By an operator, we mean an element from $B(H)$. If T lies in $B(H)$, then T^* denotes the adjoint of T in $B(H)$. For $0 < p < 1$, an operator T is said to be p-hyponormal if $(T^*T)^p \geq (TT^*)^p$. If $p = 1$, T is called hyponormal. If $p = \frac{1}{2}$, T is called semi-hyponormal. An operator T is called paranormal, if $\|Tx\|^2 \leq \|T^2x\| \|x\|$, for every $x \in H$. In general, hyponormal $\Rightarrow$ p-hyponormal $\Rightarrow$paranormal $\Rightarrow$ k-paranormal. Composition operators on hyponormal operators are studied by Alan Lambert [1]. Paranormal composition operators are studied by T. Veluchamy and S. Panayappan [10]. In this paper we characterise k-paranormal composition operators.

§2. Preliminaries

Let (X, Σ, λ) be a sigma-finite measure space. The relation of being almost everywhere, denoted by a.e, is an equivalence relation in $L^2(X, \Sigma, \lambda)$ and this equivalence relation splits $L^2(X, \Sigma, \lambda)$ into equivalence classes. Let T be a measurable transformation from X into itself. $L^2(X, \Sigma, \lambda)$ is denoted as $L^2(\lambda)$. The equation $C_Tf = f \circ T$, $f \in L^2(\lambda)$ defines a composition transformation on $L^2(\lambda)$. T induces a composition operator C_T on $L^2(\lambda)$ if (i) the measure $\lambda \circ T^{-1}$ is absolutely continuous with respect to λ and (ii) the Radon-Nikodym derivative $\frac{\mathrm{d}(\lambda T^{-1})}{\mathrm{d}\lambda}$ is essentially bounded (Nordgren). Harrington and Whitley have shown that if $C_T \in B(L^2(\lambda))$, then $C_T^*C_Tf = f_0f$ and $C_TC_T^*f = (f_0 \circ T)Pf$ for all $f \in L^2(\lambda)$ where P denotes the projection of $L^2(\lambda)$ onto $\overline{ran(C_T)}$. Thus it follows that C_T has dense range if and only if $C_TC_T^*$ is the operator of multiplication by $f_0 \circ T$, where f_0 denotes $\frac{\mathrm{d}(\lambda T^{-1})}{\mathrm{d}\lambda}$. Every essentially bounded complex valued measurable function f_0 induces a bounded operator M_{f_0} on $L^2(\lambda)$, which is defined by $M_{f_0} f = f_0 f$, for every $f \in L^2(\lambda)$. Further $C_T^* C_T = M_{f_0}$ and ${C_T^*}^2 {C_T}^2 = M_{h_0}$. Let us denote $\frac{\mathrm{d}(\lambda T^{-1})}{\mathrm{d}\lambda}$ by h, i.e. f_0 by h and $\frac{\mathrm{d}(\lambda T^{-k})}{\mathrm{d}\lambda}$ by h_k, where k is a positive integer greater than or equal to one. Then $C_T^* C_T = M_h$ and ${C_T^*}^2 {C_T}^2 = M_{h_2}$. In general, ${C_T^*}^k {C_T}^k = M_{h_k}$, where M_{h_k} is the multiplication operator on $L^2(\lambda)$ induced by the complex valued measurable function h_k.

§3. k-paranormal composition operators

Definition 3.1. An operator T is called k-paranormal if $\left\|T^{k+1}x\right\| \|x\|^k \geq \|Tx\|^{k+1}$, for some positive integer $k \geq 1$ and for every $x \in H$. Equivalently, T is called k-paranormal if $\left\|T^{k+1}x\right\| \geq \|Tx\|^{k+1}$ for some integer $k \geq 1$ and for every unit vector $x \in H$. A paranormal operator is simply a 1-paranormal operator. Also a paranormal operator is k-paranormal, for every $k \geq 1$.

Ando [4] has characterized paranormal operators as follows:

Theorem 3.2. An operator $T \in B(H)$ is paranormal if and only if ${T^*}^2 T^2 - 2kT^*T + k^2 \geq 0$, for every $k \in R$.

Generalising this, Yuan and Gao [11] has characterised k-paranormal operators as follows:

Theorem 3.3. For each positive integer k, an operator $T \in B(H)$ is k-paranormal if and only if ${T^*}^{1+k} T^{1+k} - (1+k)\mu^k T^*T + k\mu^{1+k} I \geq 0$, for every $\mu > 0$.

Using this theorem, we characterize the composition operators induced by k-paranormal operators.

Theorem 3.4. For each positive integer k, $C_T \in B(L^2(\lambda))$ is k-paranormal if and only if $h_{1+k} - (1+k)\mu^k h + k\mu^{1+k} \geq 0$ a.e, for every $\mu > 0$.

Proof. By Theorem 3.3, C_T is k-paranormal if and only if

$${C_T^*}^{1+k} C_T^{1+k} - (1+k)\mu^k C_T^* C_T + k\mu^{1+k} I \geq 0, \text{ for every } \mu > 0.$$

This is true if and only if for every $f \in L^2(\lambda)$ and $\mu > 0$,

$$\left\langle M_{h_{1+k}} f,\, f\right\rangle - (1+k)\mu^k \left\langle M_h f\, f\right\rangle + k\mu^{1+k} \left\langle f,\, f\right\rangle \geq 0,$$

if and only if

$$\langle h_{1+k} f,\, f\rangle - (1+k)\mu^k \langle hf\, f\rangle + k\mu^{1+k} \langle f,\, f\rangle \geq 0,$$

if and only if

$$\langle h_{1+k}\chi_E,\, \chi_E\rangle - (1+k)\mu^k \langle h\chi_E\, \chi_E\rangle + k\mu^{1+k} \langle \chi_E,\, \chi_E\rangle \geq 0,$$

for every characteristic function χ_E of E in Σ,

if and only if

$$\int_E (h_{1+k} - (1+k)\mu^k h + k\mu^{1+k})\, \mathrm{d}\lambda \geq 0, \text{ for every } E \text{ in } \Sigma,$$

if and only if

$$h_{1+k} - (1+k)\mu^k h + k\mu^{1+k} \geq 0 \text{ a.e, for every } \mu > 0.$$

Corollary 3.5. For each positive integer k, $C_T \in B(L^2(\lambda))$ is k-paranormal if and only if $h_{1+k} \geq h^{1+k}$, a. e.

Example 3.6. Let $X = N$ the set of all natural numbers and λ be the counting measure on it. Define $T : N \to N$ by $T(1) = T(2) = 1, T(3) = 2,\ T(4n + m - 1) = n + 2,$ for $m = 1, 2, 3, 4$ and $n \in N$. Then for each $k \geq 3$,

$$h_{1+k}(n) \geq h^{1+k}(n), \text{ for every } n \in N.$$

Hence T is k-paranormal, for each $k = 3, 4, 5, \ldots$.

Theorem 3.7. For each positive integer k, C_T^* is k-paranormal if and only if $h^{1+k} \circ TP_1 \leq (h_{1+k} \circ T^{1+k} P_{1+k})$, a. e, where $P_i's$ are the projections of $L^2(\lambda)$ onto $\overline{ran(C_T^i)}$.

Proof. By Theorem 3.3, C_T^* is k-paranormal if and only if

$$\left\langle ({C_T}^{1+k} {C_T^*}^{1+k} - (1 + k)\mu^k C_T C_T^* + k\mu^{1+k})g,\, g \right\rangle \geq 0, \quad \text{for all } g \in L^2(\lambda),$$

if and only if

$$h_{1+k} \circ T^{1+k} P_{1+k} - (1 + k)\mu^k h \circ TP_1 + k\mu^{1+k} \geq 0,\ a.\ e, \quad \text{for all } \mu \geq 0,$$

if and only if

$$h^{1+k} \circ TP_1 \leq (h_{1+k} \circ T^{1+k} P_{1+k}),\ a.\ e.$$

Corollary 3.8. If $C_T \in B(L^2(\lambda))$ has dense range, then C_T^* is k-paranormal if and only if $h_{1+k} \circ T^{1+k} \geq h^{1+k} \circ T$, a. e.

§4. Weighted composition operators and Aluthge transformation of k-paranormal operators

A weighted composition operator induced by T is defined as $Wf = w(f \circ T)$, is a complex valued function Σ measurable function. Let w_k denote $w(w \circ T)(w \circ T^2) \cdots (w \circ T^{k-1})$. Then $W^k f = w_k(f \circ T)^k$ [9]. To examine the weighted composition operators effectively Alan Lambert [1] associated conditional expectation operator E with T as $E(\cdot/T^{-1}\Sigma) = E(\cdot)$. $E(f)$ is defined for each non-negative measurable function $f \in L^p (p \geq 1)$ and is uniquely determined by the conditions

1. $E(f)$ is $T^{-1}\Sigma$ measurable,
2. if B is any $T^{-1}\Sigma$ measurable set for which $\int_B f \mathrm{d}\lambda$ converges, we have $\int_B f \mathrm{d}\lambda = \int_B E(f) \mathrm{d}\lambda$.

As an operator on L^p, E is the projection onto the closure of range of T and E is the identity operator on L^p if and only if $T^{-1}\Sigma = \Sigma$. Detailed discussion of E is found in [5], [7], [8].

The following proposition due to Campbell and Jamison is well-known.

Proposition 4.1.[5] For $w \geq 0$,

1. $W^*Wf = h[E(w^2)] \circ T^{-1} f$.
2. $WW^*f = w(h \circ T)E(wf)$.

Since $W^k f = w_k(f \circ T^k)$ and $W^{*k} f = h_k E(w_k f) \circ T^{-k}$, we have $W^{*k} W^k = h_k E(w_k^2) \circ T^{-k} f$, for $f \in L^2(\lambda)$.

Now we are ready to characterize k-paranormal weighted composition operators.

Theorem 4.2. Let $W \in B(L^2(\lambda))$. Then W is k-paranormal if and only if $h_{k+1}E(w_{k+1}^2) \circ T^{-(k+1)} - (1+k)M^k f_0 E(w^2) \circ T^{-1} + k\mu^{1+k} \geq 0$, a. e, for every $\mu > 0$.

Proof. Since W is k-paranormal ,

$$W^{*1+k} W^{1+k} - (1+k)\mu^k W^* W + k\mu^{1+k} I \geq 0, \text{ for every} \mu > 0.$$

Hence

$$\int_E h_{k+1}E(w_{k+1}^2) \circ T^{-(k+1)} - (1+k)\mu^k hE(w^2) \circ T^{-1} + k\mu^{1+k} \mathrm{d}\,\lambda \geq 0,$$

for every $E \in \Sigma$ and so

$$h_{k+1}E(w_{k+1}^2) \circ T^{-(k+1)} - (1+k)\mu^k hE(w^2) \circ T^{-1} + k\mu^{1+k} \geq 0, \text{ a. e. for every } \mu > 0.$$

Corollary 4.3. Let $T^{-1}\Sigma = \Sigma$. Then W is k-paranormal if and only if $h_{k+1}w_{k+1}^2 \circ T^{-(k+1)} - (1+k)\mu^k hw^2 \circ T^{-1} + k\mu^{1+k} \geq 0$, a. e. for every $\mu > 0$.

The Alugthe transformation of T is the operator $\widetilde{T}$ given by $\widetilde{T} = |T|^{1/2} U |T|^{1/2}$ was introduced by Alugthe [2]. More generally we may form the family of operators $T_r : 0 < r \leq 1$ where $T_r = |T|^r U |T|^{1-r}$ [2]. For a composition operator C, the polar decomposition is given by $C = U|C|$ where $|C| = \sqrt{h} f$ and $Uf = \frac{1}{\sqrt{h \circ T}} h \circ T$. Lambert [6] has given a more general Alugthe transformation for composition operators as $C_r = |C|^r U |C|^{1-r}$ as $C_r f = \left(\frac{h}{h \circ T}\right)^{r/2} f \circ T$. i. e. C_r is weighted composition with weight $\pi = \left(\frac{h}{h \circ T}\right)^{r/2}$.

Corollary 4.4. Let $C_r \in B(L^2(\lambda))$. Then C_r is of k-paranormal if and only if

$$h_{k+1}E(\pi_{k+1}^2) \circ T^{-(k+1)} - (1+k)\mu^k hE(\pi^2) \circ T^{-1} + k\mu^{k+1} \geq 0, \ a.\ e. \text{ for every } \ \mu > 0.$$

References

[1] Alan Lambert, Hyponormal Composition operators, Bull. London. Math Soc, **18**(1986), 395-400.

[2] A. Alugthe, On p-hyponormal operators for $0 < p < 1$, Integral Equations and Operator theory, **13**(1990), 307-315.

[3] A. Alugthe, Some generalized theorems on p-hyponormal operators for $0 < p < 1$, Integral Equations and Operator theory, **24**(1996), 497-502.

[4] T. Ando, Operators with a norm condition, Acta Sci. Math.(Szeged), **33**(1972), 169-178.

[5] J. Campbell and J. Jamison, On some classes of weighted composition operators, Glasgow Math. J, **32**(1990), 82-94.

[6] Charles Burnap, K. Bong Jung and Alan Lambert, Separating partial normality classes with composition operators, J. Operator theory, **53**(2005), 381-397.

[7] M. Embry Wardrop and A. Lambert, Measurable transformations and centred composition operators, Proc. Royal, Irish acad, **90A**(1990), 165-172.

[8] S. R. Foguel, Selected topics in the study of markov operators, Carolina Lectures Series, Dept. Math., UNC-CH, Chapel Hill, NC. 27514, No. 9, 1980.

[9] S. Panayappan, Non-hyponormal weighted composition operators, Indian J. of pure. App. Maths, **27**(1996), 979-983.

[10] T. Veluchamy and S. Panayappan, Paranormal composition operators, Indian J. of pure and applied math, **24**(2004), 257-262.

[11] J. Yuan and Z. Gao, Weyl spectrum of class A(n) and n-paranormal operators, Integral Equations and Operator theory, **55**(2006), 291-298.

Scientia Magna
Vol. 7 (2011), No. 1, 74-78

Inequalities between the sides and angles of an acute triangle

M. Bencze[†], N. Minculete[‡] and O. T. Pop[♯]

† Áprily Lajos National College, Braşov 500026, Romania
‡ Dimitrie Cantemir University, Braşov 500026, Romania
♯ Mihai Eminescu National College, Satu Mare 440014, Romania
E-mail: benczemihaly@gmail.com minculeten@yahoo.com ovidiutiberiu@yahoo.com

Abstract The objective of this paper is to present several inequalities between the angles of a triangle and the sides based on Jordan's inequality, $\frac{2\pi}{x} < \sin x < x$, for every $x \in \left(0, \frac{\pi}{2}\right)$.
Keywords Geometric inequality, triangle.
Mathematics Subject Classification (2010): 26D15, 51M16.

§1. Introduction

In this paper, we will study the inequalities of type

$$f(a, b, c, A, B, C, r, s, R, \Delta) \geq 0,$$

in an acute triangle ABC, where a, b, c are the lenghts of sides BC, CA, AB; A, B, C are the measures of the angles $\widehat{BAC}, \widehat{ABC}, \widehat{BCA}$ calculated in radians, r is the radius of incircle; s is the semi-perimeter; R is the radius of circumcircle and Δ is the area.

In many books of the Mathematical Analysis [3,4] can be found the following inequality of Jordan

$$\frac{2x}{\pi} < \sin x < x, \tag{1}$$

for all $x \in \left(0, \frac{\pi}{2}\right)$.

We will apply this inequality in the triangle ABC in different forms.

§2. Main results

Theorem 1. In any acute triangle, there are the following

$$2 < \frac{s}{R} < \pi, \tag{2}$$

$$\frac{s^2 + r^2 + 4Rr}{4R^2} < AB + BC + CA < \frac{\pi^2(s^2 + r^2 + 4Rr)}{16R^2}, \tag{3}$$

$$\frac{sr}{2R^2} < ABC < \frac{\pi^3 sr}{16R^2}, \tag{4}$$

$$\frac{s^2 - r(4R+3)}{2R^2} < A^2 + B^2 + C^2 < \frac{\pi^2(s^2 - r(4R+r))}{8R^2} \tag{5}$$

and

$$\frac{s(s^2 - 3r^2 - 6Rr)}{4R^3} < A^3 + B^3 + C^3 < \frac{\pi^3 s(s^2 - 3r^2 - 6Rr)}{32R^3}. \tag{6}$$

Proof. In inequality (1) for $x \in \{A, B, C\}$ we obtain

$$\frac{2A}{\pi} < \sin A < A, \ \frac{2B}{\pi} < \sin B < B, \ \frac{2C}{\pi} < \sin C < C. \tag{7}$$

Adding the above inequalities we deduce inequality (2). Multiplying two by two and taking their sum, we give over inequality (3). Multiplying the inequalities from (7) we obtain inequality (4). Squaring the inequalities from (7) and making their sum implies inequality (5). Using the inequalities from (7) thus

$$\left(\frac{2}{\pi}\right)^3 A^3 < \sin^3 A < A^3, \ \left(\frac{2}{\pi}\right) B^3 < \sin^3 B < B^3,$$

$\left(\frac{2}{\pi}\right)^3 C^3 < \sin^3 C < C^3$ and adding them we find inequality (6). We also use the following equalities (see [1,2]):

$$\prod_{cyclic} \sin A \sin B = \frac{s^2 + r^2 + 4Rr}{4R^2},$$

$$\sin A \sin B \sin C = \frac{sr}{2R^2},$$

$$\sum_{cyclic} \sin^2 A = \frac{s^2 - r(4R+r)}{2R^2}$$

and

$$\sum_{cyclic} \sin^3 A = \frac{s(s^2 - 3r^2 - 6Rr)}{4R^3}.$$

Theorem 2. In any acute triangle, there are the following inequalities

$$2\left(2 - \frac{r}{R}\right) < A^2 + B^2 + C^2 < \pi\left(2 - \frac{r}{R}\right), \tag{8}$$

$$4\left(\frac{s^2 + r^2 - 2Rr}{R^2} - 3\right) < A^2B^2 + B^2C^2 + C^2A^2 < \pi^2\left(\frac{s^2 + r^2 - 2Rr}{R^2} - 3\right) \tag{9}$$

and

$$8\left(\frac{3s^2 + 5r^2}{4R^2} - 3\right) < A^2B^2C^2 < \pi^3\left(\frac{3s^2 + 5r^2}{4R^2} - 3\right). \tag{10}$$

Proof. From inequality (1), we have

$$\frac{1}{\pi}\sum_{cyclic} A^2 = \sum_{cyclic}\int_0^A \frac{2x}{\pi}dx < \sum_{cyclic}\int_0^A \sin x dx$$

$$= 3 - \sum_{cyclic} \cos A = 2 - \frac{r}{R} < \sum_{cyclic}\int_0^A x dx = \frac{1}{2}\sum_{cyclic} A^2,$$

so, we deduce inequality (8).

Using double integrals in relation (1), we obtain

$$\frac{1}{\pi^2}\cdot\sum_{cyclic}A^2B^2=\sum_{cyclic}\int_0^A\int_0^B\frac{4xy}{\pi^2}dxdy$$

$$<\sum_{cyclic}\int_0^A\int_0^B\sin x\sin ydxdy=\sum_{cyclic}(1-\cos A)(1-\cos B)$$

$$=\frac{s^2+r^2-2Rr}{R^2}-3<\sum_{cyclic}\int_0^A\int_0^B xydxdy=\frac{1}{4}\sum_{cyclic}A^2B^2,$$

so it follows inequality (9).

Now, using triple integrals, we have

$$\frac{A^2B^2C^2}{\pi^3}=\int_0^A\int_0^B\int_0^C\frac{8xyz}{\pi^3}dxdydz$$

$$\leq\int_0^A\int_0^B\int_0^C\sin x\sin y\sin zdxdydz=\prod(1-\cos A)$$

$$=\frac{3s^2+5r^2}{4R^2}-3<\int_0^A\int_0^B\int_0^C xyzdxdydz=\frac{A^2B^2C^2}{8},$$

thus, the proof of inequality (10) is complete.

Theorem 3. In any acute triangle ABC, there are the following inequalities:

$$\pi\left(\pi-\frac{s}{R}\right)<A^2+B^2+C^2\leq 2\left(\frac{\pi^2}{2}-\frac{s}{R}\right)\tag{11}$$

and

$$\frac{s^2+r^2+4Rr}{R^2}-3\pi ABC<A^2B^2+B^2C^2+C^2A^2<\frac{s^2+r^2+4Rr}{R^2}-\frac{12ABC}{\pi}.\tag{12}$$

Proof. By replacing x with $\frac{\pi}{2}-x$ in inequality (1) we find the relation

$$1-\frac{2x}{\pi}<\cos x<\frac{\pi}{2}-x,\tag{13}$$

where $x\in\left(0,\frac{\pi}{2}\right)$.

By integrating, in relation (13) we obtain

$$\sum_{cyclic}\int_0^A\left(1-\frac{2x}{\pi}\right)dx<\sum_{cyclic}\int_0^A\cos xdx<\sum_{cyclic}\int_0^A\left(\frac{\pi}{2}-x\right)dx,$$

so

$$\pi-\frac{1}{\pi}\sum_{cyclic}A^2<\sum_{cyclic}\sin A=\frac{s}{R}<\frac{\pi^2}{2}-\frac{1}{2}\sum_{cyclic}A^2,$$

which means that we proved inequality (11).
Using double integrals, we have

$$\begin{aligned}\sum_{cyclic}\int_0^A\int_0^B\left(1-\frac{2x}{\pi}\right)\left(1-\frac{2y}{\pi}\right)dxdy &< \sum_{cyclic}\int_0^A\int_0^B \cos x\cos y dxdy\\ &< \sum_{cyclic}\int_0^A\int_0^B\left(\frac{\pi}{2}-x\right)\left(\frac{\pi}{2}-y\right)dxdy\end{aligned}$$

which implies inequality (12).

Theorem 4. In any acute triangle ABC, there are the following inequalities

$$\frac{2r}{R} < ABC < \frac{\sqrt{2\pi}}{2}\frac{r}{R}, \tag{14}$$

$$\frac{s}{R} < \sum_{cyclic} A\cos\frac{A}{2} < \sqrt{\frac{A}{2}}\frac{s}{R} \tag{15}$$

and

$$\sqrt{\frac{\pi}{2}} < \sum\sin\frac{A}{2} < \frac{\pi}{2}. \tag{16}$$

Proof. From inequality (1), by integrating, we deduce

$$\frac{A^2}{\pi} < 1-\cos A < \frac{A^2}{2}$$

which means that

$$\frac{A^2}{\pi} < 2\sin^2\frac{A}{2} < \frac{A^2}{2}.$$

Therefore, we have

$$\frac{A}{\sqrt{2\pi}} < \sin\frac{A}{2} < \frac{A}{2}. \tag{17}$$

Writing and similar inequalities, by multiplying, we find the inequality

$$\frac{ABC}{2\pi\sqrt{2\pi}} < \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} < \frac{ABC}{8}.$$

But, we know that $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{r}{4R}$, from [1,2], which means that inequality (14) is proved.

Multiplying in inequality (17) by $\cos\frac{A}{2}$, we obtain

$$\frac{A}{\sqrt{2\pi}}\cos\frac{A}{2} < \frac{1}{2}\sin A < \frac{A}{2}\cos\frac{A}{2},$$

which is equivalent to

$$\sqrt{\frac{2}{\pi}}\sum_{cyclic} A\cos\frac{A}{2} < \frac{s}{R} < \sum_{cyclic} A\cos\frac{A}{2}$$

from where, we find inequality (15).
Making the cyclic sum in relation (17), we have inequality (16).

Theorem 5. In any acute triangle ABC, we have the inequality

$$\left(3-\frac{s}{R}\right)\frac{\pi^3}{4} < A^3+B^3+C^3 < \frac{\pi^3}{4(\pi-2)}\left(\pi-\frac{s}{R}\right). \tag{18}$$

Proof. In [5] is proved the following inequality

$$\frac{2x}{\pi}+\frac{\pi^2x-4x^3}{\pi^3} < \sin x < \frac{2x}{\pi}+\frac{\pi-2}{\pi^3}(\pi^2x-4x^3) \tag{19}$$

which implies, working the sum for $x \in \{A, B, C\}$, the inequality of the statement.

References

[1] O. Bottema, R. Z. Djordjević, R. R. Janić, D. S. Mitrinović, P. M. Vasić, Geometric inequalities, Groöningen, 1969.

[2] N. Minculete, Egalităţi şi inegalităţi geometrice în triunghi, Editura Eurocarpatica, Sf. Gheorghe, 2003.

[3] C. P. Niculescu, L. E. Persson, Convex Functions and Their Applications, CMS Books in Mathematics, Springer-Vertag, New York, **23**(2006).

[4] Gh. Sireţchi, Calcul diferenţial şi integral, Editura Ştiinţifică şi Enciclopedică, 1985.

[5] L. Zhu, Sharpening Jordan's inequality and Yang Le inequality, Applied Mathematics Letters, **19**(2006), No. 3, 240-243.

[6] Octogon Mathematical Magazine, 1993-2011.

Scientia Magna
Vol. 7 (2011), No. 1, 79-81

The Estrada index of random graphs

Yilun Shang[†‡]

† Institute for Cyber Security, University of Texas at San Antonio,
San Antonio, Texas 78249, USA
‡ Department of Mathematics, Shanghai Jiao Tong University,
Shanghai 200240, China
E-mail: shylmath@hotmail.com

Abstract Let G be a graph of size n, and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be its eigenvalues. The Estrada index of G is defined as $EE(G) = \sum_{i=1}^{n} e^{\lambda_i}$. In this paper, we establish almost sure results on $EE(G(n,p))$ of Erdős-Rényi random graph $G(n,p)$ in the superconnectivity regime.

Keywords Graph spectrum, Estrada index, random graph.

§1. Introduction

Generally, a simple graph $G = (V, E)$ is defined by its vertex set V and edge set $E \subseteq V \times V$. Let n be the number of vertices of G. The eigenvalues of the adjacency matrix A are called the eigenvalues of G and form the spectrum [4] of G. Since A is a real symmetric matrix, its eigenvalues are real number. We then order the eigenvalues of G in a non-increasing manner as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Some basic properties of graph are reviewed in [4].

A recently introduced [5,8] spectrum-based graph invariant is

$$EE(G) = \sum_{i=1}^{n} e^{\lambda_i}, \tag{1}$$

which is called the Estrada index of G. The Estrada index has found numerous applications in biochemistry [5,6,9], physics and complex networks [7,8,13]. Some lower and upper bounds for $EE(G)$ of fixed graphs are deduced in [2,10,12].

An intriguing question is the random graph setting. Let $G = G(n,p)$ denote, as usual, the Erdős-Rényi random graph [3] with n vertices and edge probability p. In this brief paper, by using spectral theory we obtain the Estrada index $EE(G(n,p))$ in the regime that $G(n,p)$ is almost surely connected.

§2. Estimating $EE(G(n,p))$

Our main contribution in this section is the following concise result.

Theorem 2.1. For random graph $G(n,p)$ with

$$\frac{\ln n}{n} \ll p < 1 - \frac{\ln n}{n}, \tag{2}$$

the Estrada index is

$$EE(G(n,p)) = (1+o(1))e^{np}, \tag{3}$$

almost surely, as $n \to \infty$.

A key technique in the proof is a spectral density representation of random graph $G(n,p)$. It is shown [11] that the largest eigenvalue λ_1 of $G(n,p)$ is almost surely $(1+o(1))np$ provided that $np \gg \ln n$. Furthermore, Wigner's semicircular law [14,15] says that the spectral density of $G(n,p)$ converges to the semicircular distribution

$$\rho(\lambda) = \begin{cases} \frac{2\sqrt{r^2-\lambda^2}}{\pi r^2}, & |\lambda| \le r, \\ 0, & |\lambda| > r, \end{cases} \tag{4}$$

as $n \to \infty$, where $r = 2\sqrt{np(1-p)}$ is the radius of the bulk part of the spectrum.

For $p \gg (\ln n)/n$ and $n \to \infty$, by continuous approximation for $EE(G)$ in (1), the Estrada index of $G(n,p)$ can be reformulated in the spectral density form

$$\begin{aligned} \frac{EE(G(n,p))}{n} &= \int_{-r}^{r} \rho(\lambda) e^{\lambda} \mathrm{d}\lambda + \frac{e^{\lambda_1}}{n} \\ &= \psi(1) + \frac{e^{np}}{n} \\ &= \frac{e^{np}}{n}\Big(1 + \frac{n\psi(1)}{e^{np}}\Big), \end{aligned} \tag{5}$$

where $\psi(t)$ is the moment generating function of density $\rho(\lambda)$ and

$$\psi(1) = \int_{-r}^{r} \frac{2\sqrt{r^2-\lambda^2}}{\pi r^2} e^{\lambda} \mathrm{d}\lambda = \frac{2}{\pi} \int_0^{\pi} e^{r\cos\theta} \sin^2\theta \mathrm{d}\theta. \tag{6}$$

The following lemma can be proved by involving a modified Bessel function [1].

Lemma 2.2.[16] The function

$$g(p) = n\psi(1)/e^{np} \sim n\sqrt{\frac{2}{\pi}} \frac{e^{r-np}}{r^{3/2}} \tag{7}$$

is monotonically decreasing for $(\ln n)/n < p < 1 - (\ln n)/n$ as $n \to \infty$, where $\psi(1)$ is given by (6) and r is defined in (4).

Now we are on the stage to prove our main result.

Proof of Theorem 2.1. Let $p = p_c = (\ln n)/n$. Therefore, $1 - p_c \to 1$ as $n \to \infty$, and $r \sim 2\sqrt{\ln n}$ from the definition in (4). By (7) we get

$$\begin{aligned} g(p_c) &\sim n\sqrt{\frac{2}{\pi}} \frac{e^{r-np_c}}{r^{3/2}} \sim n\sqrt{\frac{2}{\pi}} \cdot \frac{e^{2\sqrt{\ln n} - \ln n}}{(2\sqrt{\ln n})^{3/2}} \\ &= \frac{n}{2\sqrt{\pi}} \cdot \frac{e^{-\ln n + 2\sqrt{\ln n}}}{(\ln n)^{3/4}} = \frac{e^{2\sqrt{\ln n}}}{2\sqrt{\pi}(\ln n)^{3/4}}, \end{aligned} \tag{8}$$

which approaches 0 as $n \to \infty$.

By Lemma 2.2, for $p_c \le p \le 1 - p_c$, we obtain $g(p) \le g(p_c) \to 0$ as $n \to \infty$. Combining this with (5) and (7), we then conclude the proof of Theorem 2.1.

References

[1] M. Abramowitz, I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover Publications, New York, 1972.

[2] J. Antonio de la Peña, I. Gutman, J. Rada, Estimating the Estrada index, Lin. Alg. Appl., **427**(2007), 70-76.

[3] B. Bollobás, Random Graphs, Cambridge University Press, New York, 2001.

[4] D. Cvetković, M. Doob, H. Sachs, Spectra of Graphs–Theory and Application, Johann Ambrosius Barth Verlag, Heidelberg, Leipzig, 1995.

[5] E. Estrada, Characterization of 3D molecular structure, Chem. Phys. Lett., **319**(2000), 713-718.

[6] E. Estrada, Characterization of the folding degree of proteins, Bioinformatics, **18**(2002), 697-704.

[7] E. Estrada, J. A. Rodríguez-Velázquez, Spectral measures of bipartivity in complex networks, Phys. Rev. E, **72**(2005), 046105.

[8] E. Estrada, J. A. Rodríguez-Velázquez, Subgraph centrality in complex networks, Phys. Rev. E, **71**(2005), 056103.

[9] E. Estrada, J. A. Rodríguez-Velázquez, M. Randić, Atomic branching in molecules, Int. J. Quantum Chem, **106**(2006), 823-832.

[10] I. Gutman, Lower bounds for Estrada index, Publications de L'Institut Mathématique, **83**(2008), 1-7.

[11] M. Krivelevich, B. Sudakov, The largest eigenvalue of sparse random graphs, Combin. Probab. Comput., **12**(2003), 61-72.

[12] J. P. Liu, B. L. Liu, Bounds of the Estrada index of graphs, Appl. Math. J. Chinese Univ., **25**(2010), 325-330.

[13] Y. Shang, Perturbation results for the Estrada index in weighted networks, J. Phys. A: Math. Theor., **44**(2011), 075003.

[14] E. Wigner, Characteristic vectors of bordered matrices with infinite dimensions, Ann. Math., **62**(1955), 548-564.

[15] E. Wigner, On the distribution of the roots of certain symmetric matrices, Ann. Math., **67**(1958), 325-328.

[16] J. Wu, M. Barahona, Y. J. Tan, H. Z. Deng, Robustness of random graphs based on natural connectivity, arXiv: 1009. 3430, 2010.

Scientia Magna
Vol. 7 (2011), No. 1, 82-93

Smarandache isotopy of second Smarandache Bol loops

Tèmítọ́pẹ́ Gbọ́láhàn Jaíyéọlá

Department of Mathematics, Obafemi Awolowo University,
Ile Ife 220005, Nigeria
E-mail: jaiyeolatemitope@yahoo.com tjayeola@oauife.edu.ng

Abstract The pair $(G_H, \cdot)$ is called a special loop if $(G, \cdot)$ is a loop with an arbitrary subloop $(H, \cdot)$ called its special subloop. A special loop $(G_H, \cdot)$ is called a second Smarandache Bol loop ($S_{2^{\rm nd}}BL$) if and only if it obeys the second Smarandache Bol identity $(xs \cdot z)s = x(sz \cdot s)$ for all x, z in G and s in H. The popularly known and well studied class of loops called Bol loops fall into this class and so $S_{2^{\rm nd}}BLs$ generalize Bol loops. The Smarandache isotopy of $\mathrm{S}_{2^{\rm nd}}$BLs is introduced and studied for the first time. It is shown that every Smarandache isotope ($S-isotope$) of a special loop is Smarandache isomorphic ($S-isomorphic$) to a S-principal isotope of the special loop. It is established that every special loop that is S-isotopic to a $S_{2^{\rm nd}}BL$ is itself a $\mathrm{S}_{2^{\rm nd}}$BL. A special loop is called a Smarandache G-special loop ($SGS-loop$) if and only if every special loop that is S-isotopic to it is S-isomorphic to it. A $\mathrm{S}_{2^{\rm nd}}$BL is shown to be a SGS-loop if and only if each element of its special subloop is a $S_{1^{\rm st}}$ companion for a $S_{1^{\rm st}}$ pseudo-automorphism of the $S_{2^{\rm nd}}BL$. The results in this work generalize the results on the isotopy of Bol loops as can be found in the Ph. D. thesis of D. A. Robinson.

Keywords Special loop, second Smarandache Bol loop, Smarandache princiapl isotope, Sm -arandache isotopy.

§1. Introduction

The study of the Smarandache concept in groupoids was initiated by W. B. Vasantha Kandasamy in [24]. In her book [22] and first paper [23] on Smarandache concept in loops, she defined a Smarandache loop ($S-loop$) as a loop with at least a subloop which forms a subgroup under the binary operation of the loop. The present author has contributed to the study of S-quasigroups and S-loops in [5]-[12] by introducing some new concepts immediately after the works of Muktibodh [15]-[16]. His recent monograph [14] gives inter-relationships and connections between and among the various Smarandache concepts and notions that have been developed in the aforementioned papers.

But in the quest of developing the concept of Smarandache quasigroups and loops into a theory of its own just as in quasigroups and loop theory (see [1]-[4], [17], [22]), there is the need to introduce identities for types and varieties of Smarandache quasigroups and loops. This led Jaíyéọlá [13] to the introduction of second Smarandache Bol loop ($\mathrm{S}_{2^{\rm nd}}$BL) described by the second Smarandache Bol identity $(xs \cdot z)s = x(sz \cdot s)$ for all x, z in G and s in H where the pair $(G_H, \cdot)$ is called a special loop if $(G, \cdot)$ is a loop with an arbitrary subloop $(H, \cdot)$. For now, a Smarandache loop or Smarandache quasigroup will be called a first Smarandache loop ($S_{1^{st}} - loop$) or first Smarandache quasigroup ($S_{1^{st}} - quasigroup$).

Let L be a non-empty set. Define a binary operation $(\cdot)$ on L: if $x \cdot y \in L$ for all $x, y \in L$, $(L, \cdot)$ is called a groupoid. If the equations; $a \cdot x = b$ and $y \cdot a = b$ have unique solutions for x and y respectively, then $(L, \cdot)$ is called a quasigroup. For each $x \in L$, the elements $x^\rho = xJ_\rho$, $x^\lambda = xJ_\lambda \in L$ such that $xx^\rho = e^\rho$ and $x^\lambda x = e^\lambda$ are called the right, left inverses of x respectively. Furthermore, if there exists a unique element $e = e_\rho = e_\lambda$ in L called the identity element such that for all x in L, $x \cdot e = e \cdot x = x$, $(L, \cdot)$ is called a loop. We write xy instead of $x \cdot y$, and stipulate that $\cdot$ has lower priority than juxtaposition among factors to be multiplied. For instance, $x \cdot yz$ stands for $x(yz)$. A loop is called a right Bol loop (Bol loop in short) if and only if it obeys the identity

$$(xy \cdot z)y = x(yz \cdot y).$$

This class of loops was the first to catch the attention of loop theorists and the first comprehensive study of this class of loops was carried out by Robinson [19].

The popularly known and well studied class of loops called Bol loops fall into the class of $S_{2^{nd}}BLs$ and so $S_{2^{nd}}BLs$ generalize Bol loops. The aim of this work is to introduce and study for the first time, the Smarandache isotopy of $\mathrm{S}_{2^{nd}}\mathrm{BLs}$. It is shown that every Smarandache isotope (S-isotope) of a special loop is Smarandache isomorphic (S-isomorphic) to a S-principal isotope of the special loop. It is established that every special loop that is S-isotopic to a $S_{2^{nd}}BL$ is itself a $S_{2^{nd}}BL$. A $S_{2^{nd}}BL$ is shown to be a Smarandache G-special loop if and only if each element of its special subloop is a $S_{1^{st}}$ companion for a $S_{1^{st}}$ pseudo-automorphism of the $\mathrm{S}_{2^{nd}}\mathrm{BL}$. The results in this work generalize the results on the isotopy of Bol loops as can be found in the Ph. D. thesis of D. A. Robinson.

§2. Preliminaries

Definition 1. Let $(G, \cdot)$ be a quasigroup with an arbitrary non-trivial subquasigroup $(H, \cdot)$. Then, $(G_H, \cdot)$ is called a special quasigroup with special subquasigroup $(H, \cdot)$. If $(G, \cdot)$ is a loop with an arbitrary non-trivial subloop $(H, \cdot)$. Then, $(G_H, \cdot)$ is called a special loop with special subloop $(H, \cdot)$. If $(H, \cdot)$ is of exponent 2, then $(G_H, \cdot)$ is called a special loop of Smarandache exponent 2.

A special quasigroup $(G_H, \cdot)$ is called a second Smarandache right Bol quasigroup ($\mathrm{S}_{2^{nd}}$-right Bol quasigroup) or simply a second Smarandache Bol quasigroup ($\mathrm{S}_{2^{nd}}$-Bol quasigroup) and abbreviated $\mathrm{S}_{2^{nd}}\mathrm{RBQ}$ or $\mathrm{S}_{2^{nd}}\mathrm{BQ}$ if and only if it obeys the second Smarandache Bol identity ($\mathrm{S}_{2^{nd}}$-Bol identity) i.e $\mathrm{S}_{2^{nd}}\mathrm{BI}$

$$(xs \cdot z)s = x(sz \cdot s) \text{ for all } x, z \in G \text{ and } s \in H. \quad (1)$$

Hence, if $(G_H, \cdot)$ is a special loop, and it obeys the $\mathrm{S}_{2^{nd}}\mathrm{BI}$, it is called a second Smarandache Bol loop($\mathrm{S}_{2^{nd}}$-Bol loop) and abbreviated $S_{2^{nd}}BL$.

Remark 1. A Smarandache Bol loop (i.e a loop with at least a non-trivial subloop that is a Bol loop) will now be called a first Smarandache Bol loop ($S_{1^{st}}$-Bol loop). It is easy to see that a $S_{2^{nd}}$BL is a $S_{1^{st}}$BL. But the converse is not generally true. So $S_{2^{nd}}$BLs are particular types of $S_{1^{st}}$BL. Their study can be used to generalise existing results in the theory of Bol loops by simply forcing H to be equal to G.

Definition 2. Let $(G,\cdot)$ be a quasigroup (loop). It is called a right inverse property quasigroup (loop) [RIPQ (RIPL)] if and only if it obeys the right inverse property (RIP) $yx{\cdot}x^\rho = y$ for all $x,y\in G$. Similarly, it is called a left inverse property quasigroup (loop) [LIPQ (LIPL)] if and only if it obeys the left inverse property (LIP) $x^\lambda\cdot xy = y$ for all $x,y\in G$. Hence, it is called an inverse property quasigroup (loop) [IPQ (IPL)] if and only if it obeys both the RIP and LIP.

$(G,\cdot)$ is called a right alternative property quasigroup (loop) [RAPQ (RAPL)] if and only if it obeys the right alternative property (RAP) $y\cdot xx = yx\cdot x$ for all $x,y\in G$. Similarly, it is called a left alternative property quasigroup (loop) [LAPQ (LAPL)] if and only if it obeys the left alternative property (LAP) $xx\cdot y = x\cdot xy$ for all $x,y\in G$. Hence, it is called an alternative property quasigroup (loop) [APQ (APL)] if and only if it obeys both the RAP and LAP.

The bijection $L_x : G\to G$ defined as $yL_x = x\cdot y$ for all $x,y\in G$ is called a left translation (multiplication) of G while the bijection $R_x : G\to G$ defined as $yR_x = y\cdot x$ for all $x,y\in G$ is called a right translation (multiplication) of G. Let

$$x\backslash y = yL_x^{-1} = y\mathbb{L}_x \qquad \text{and} \qquad x/y = xR_y^{-1} = x\mathbb{R}_y,$$

and note that

$$x\backslash y = z \Longleftrightarrow x\cdot z = y \qquad \text{and} \qquad x/y = z \Longleftrightarrow z\cdot y = x.$$

The operations $\backslash$ and $/$ are called the left and right divisions respectively. We stipulate that $/$ and $\backslash$ have higher priority than $\cdot$ among factors to be multiplied. For instance, $x\cdot y/z$ and $x\cdot y\backslash z$ stand for $x(y/z)$ and $x\cdot(y\backslash z)$ respectively.

$(G,\cdot)$ is said to be a right power alternative property loop (RPAPL) if and only if it obeys the right power alternative property (RPAP)

$$xy^n = \underbrace{(((xy)y)y)y\cdots y}_{n\text{-times}} \text{ i.e. } R_{y^n} = R_y^n \text{ for all } x,y\in G \text{ and } n\in\mathbb{Z}.$$

The right nucleus of G denoted by $N_\rho(G,\cdot) = N_\rho(G) = \{a\in G : y\cdot xa = yx\cdot a\ \forall\ x,y\in G\}$.

Let $(G_H,\cdot)$ be a special quasigroup (loop). It is called a second Smarandache right inverse property quasigroup (loop) [$S_{2^{nd}}$RIPQ ($S_{2^{nd}}$RIPL)] if and only if it obeys the second Smarandache right inverse property ($S_{2^{nd}}$RIP) $ys\cdot s^\rho = y$ for all $y\in G$ and $s\in H$. Similarly, it is called a second Smarandache left inverse property quasigroup (loop) [$S_{2^{nd}}$LIPQ ($S_{2^{nd}}$LIPL)] if and only if it obeys the second Smarandache left inverse property ($S_{2^{nd}}$LIP) $s^\lambda\cdot sy = y$ for all $y\in G$ and $s\in H$. Hence, it is called a second Smarandache inverse property quasigroup (loop) [$S_{2^{nd}}$IPQ ($S_{2^{nd}}$IPL)] if and only if it obeys both the $S_{2^{nd}}$RIP and $S_{2^{nd}}$LIP.

$(G_H,\cdot)$ is called a third Smarandache right inverse property quasigroup (loop) [$S_{3^{rd}}$RIPQ ($S_{3^{rd}}$RIPL)] if and only if it obeys the third Smarandache right inverse property ($S_{3^{rd}}$RIP) $sy\cdot y^\rho = s$ for all $y\in G$ and $s\in H$.

$(G_H, \cdot)$ is called a second Smarandache right alternative property quasigroup (loop) [$\mathrm{S_{2^{nd}}}$RA PQ($\mathrm{S_{2^{nd}}}$RAPL)] if and only if it obeys the second Smarandache right alternative property ($\mathrm{S_{2^{nd}}}$RAP) $y \cdot ss = ys \cdot s$ for all $y \in G$ and $s \in H$. Similarly, it is called a second Smarandache left alternative property quasigroup (loop) [$\mathrm{S_{2^{nd}}}$LAPQ ($\mathrm{S_{2^{nd}}}$LAPL)] if and only if it obeys the second Smarandache left alternative property ($\mathrm{S_{2^{nd}}}$LAP) $ss \cdot y = s \cdot sy$ for all $y \in G$ and $s \in H$. Hence, it is called an second Smarandache alternative property quasigroup (loop) [$\mathrm{S_{2^{nd}}}$APQ ($\mathrm{S_{2^{nd}}}$APL)] if and only if it obeys both the $\mathrm{S_{2^{nd}}}$RAP and $\mathrm{S_{2^{nd}}}$LAP.

$(G_H, \cdot)$ is said to be a Smarandache right power alternative property loop (SRPAPL) if and only if it obeys the Smarandache right power alternative property (SRPAP)

$$xs^n = \underbrace{(((xs)s)s)s \cdots s}_{n\text{-times}} \text{ i.e. } R_{s^n} = R_s^n \text{ for all } x \in G,\ s \in H \text{ and } n \in \mathbb{Z}.$$

The Smarandache right nucleus of G_H denoted by $SN_\rho(G_H, \cdot) = SN_\rho(G_H) = N_\rho(G) \cap H$. G_H is called a Smarandache right nuclear square special loop if and only if $s^2 \in SN_\rho(G_H)$ for all $s \in H$.

Remark 2. A Smarandache; RIPQ or LIPQ or IPQ (i.e a loop with at least a non-trivial subquasigroup that is a RIPQ or LIPQ or IPQ) will now be called a first Smarandache; RIPQ or LIPQ or IPQ ($\mathrm{S_{1^{st}}}$RIPQ or $\mathrm{S_{1^{st}}}$LIPQ or $\mathrm{S_{1^{st}}}$IPQ). It is easy to see that a $\mathrm{S_{2^{nd}}}$RIPQ or $\mathrm{S_{2^{nd}}}$LIPQ or $\mathrm{S_{2^{nd}}}$IPQ is a $\mathrm{S_{1^{st}}}$RIPQ or $\mathrm{S_{1^{st}}}$LIPQ or $\mathrm{S_{1^{st}}}$IPQ respectively. But the converse is not generally true.

Definition 3. Let $(G, \cdot)$ be a quasigroup (loop). The set $SYM(G, \cdot) = SYM(G)$ of all bijections in G forms a group called the permutation (symmetric) group of G. The triple (U, V, W) such that $U, V, W \in SYM(G, \cdot)$ is called an autotopism of G if and only if

$$xU \cdot yV = (x \cdot y)W \ \forall\ x, y \in G.$$

The group of autotopisms of G is denoted by $AUT(G, \cdot) = AUT(G)$.

Let $(G_H, \cdot)$ be a special quasigroup (loop). The set $SSYM(G_H, \cdot) = SSYM(G_H)$ of all Smarandache bijections (S-bijections) in G_H i.e $A \in SYM(G_H)$ such that $A : H \to H$ forms a group called the Smarandache permutation (symmetric) group [S-permutation group] of G_H. The triple (U, V, W) such that $U, V, W \in SSYM(G_H, \cdot)$ is called a first Smarandache autotopism ($\mathrm{S_{1^{st}}}$ autotopism) of G_H if and only if

$$xU \cdot yV = (x \cdot y)W \ \forall\ x, y \in G_H.$$

If their set forms a group under componentwise multiplication, it is called the first Smarandache autotopism group ($\mathrm{S_{1^{st}}}$ autotopism group) of G_H and is denoted by $\mathrm{S_{1^{st}}}AUT(G_H, \cdot) = \mathrm{S_{1^{st}}}AUT(G_H)$.

The triple (U, V, W) such that $U, W \in SYM(G, \cdot)$ and $V \in SSYM(G_H, \cdot)$ is called a second right Smarandache autotopism ($\mathrm{S_{2^{nd}}}$ right autotopism) of G_H if and only if

$$xU \cdot sV = (x \cdot s)W \ \forall\ x \in G \text{ and } s \in H.$$

If their set forms a group under componentwise multiplication, it is called the second right Smarandache autotopism group ($\mathrm{S_{2^{nd}}}$ right autotopism group) of G_H and is denoted by $\mathrm{S_{2^{nd}}}RAU$ $T(G_H, \cdot) = \mathrm{S_{2^{nd}}}RAUT(G_H)$.

The triple (U, V, W) such that $V, W \in SYM(G, \cdot)$ and $U \in SSYM(G_H, \cdot)$ is called a second left Smarandache autotopism ($\mathrm{S}_{2^{\mathrm{nd}}}$ left autotopism) of G_H if and only if

$$sU \cdot yV = (s \cdot y)W \ \forall \ y \in G \text{ and } s \in H.$$

If their set forms a group under componentwise multiplication, it is called the second left Smarandache autotopism group ($\mathrm{S}_{2^{\mathrm{nd}}}$ left autotopism group) of G_H and is denoted by $\mathrm{S}_{2^{\mathrm{nd}}}LAUT(G_H, \cdot) = \mathrm{S}_{2^{\mathrm{nd}}}LAUT(G_H)$.

Let $(G_H, \cdot)$ be a special quasigroup (loop) with identity element e. A mapping $T \in SSYM(G_H)$ is called a first Smarandache semi-automorphism ($\mathrm{S}_{1^{\mathrm{st}}}$ semi-automorphism) if and only if $eT = e$ and

$$(xy \cdot x)T = (xT \cdot yT)xT \text{ for all } x, y \in G.$$

A mapping $T \in SSYM(G_H)$ is called a second Smarandache semi-automorphism ($\mathrm{S}_{2^{\mathrm{nd}}}$ semi-automorphism) if and only if $eT = e$ and

$$(sy \cdot s)T = (sT \cdot yT)sT \text{ for all } y \in G \text{ and all } s \in H.$$

A special loop $(G_H, \cdot)$ is called a first Smarandache semi-automorphic inverse property loop ($\mathrm{S}_{1^{\mathrm{st}}}$SAIPL) if and only if J_ρ is a $\mathrm{S}_{1^{\mathrm{st}}}$ semi-automorphism.

A special loop $(G_H, \cdot)$ is called a second Smarandache semi-automorphic inverse property loop ($\mathrm{S}_{2^{\mathrm{nd}}}$SAIPL) if and only if J_ρ is a $\mathrm{S}_{2^{\mathrm{nd}}}$ semi-automorphism. Let $(G_H, \cdot)$ be a special quasigroup (loop). A mapping $A \in SSYM(G_H)$ is a

1. First Smarandache pseudo-automorphism ($\mathrm{S}_{1^{\mathrm{st}}}$ pseudo-automorphism) of G_H if and only if there exists a $c \in H$ such that $(A, AR_c, AR_c) \in \mathrm{S}_{1^{\mathrm{st}}}AUT(G_H)$. c is reffered to as the first Smarandache companion ($\mathrm{S}_{1^{\mathrm{st}}}$ companion) of A. The set of such A's is denoted by $\mathrm{S}_{1^{\mathrm{st}}}PAUT(G_H, \cdot) = \mathrm{S}_{1^{\mathrm{st}}}PAUT(G_H)$.

2. Second right Smarandache pseudo-automorphism ($\mathrm{S}_{2^{\mathrm{nd}}}$ right pseudo-automorphism) of G_H if and only if there exists a $c \in H$ such that $(A, AR_c, AR_c) \in \mathrm{S}_{2^{\mathrm{nd}}}RAUT(G_H)$. c is reffered to as the second right Smarandache companion ($\mathrm{S}_{2^{\mathrm{nd}}}$ right companion) of A. The set of such A's is denoted by $\mathrm{S}_{2^{\mathrm{nd}}}RPAUT(G_H, \cdot) = \mathrm{S}_{2^{\mathrm{nd}}}RPAUT(G_H)$.

3. Second left Smarandache pseudo-automorphism ($\mathrm{S}_{2^{\mathrm{nd}}}$ left pseudo-automorphism) of G_H if and only if there exists a $c \in H$ such that $(A, AR_c, AR_c) \in \mathrm{S}_{2^{\mathrm{nd}}}LAUT(G_H)$. c is reffered to as the second left Smarandache companion ($\mathrm{S}_{2^{\mathrm{nd}}}$ left companion) of A. The set of such A's is denoted by $\mathrm{S}_{2^{\mathrm{nd}}}LPAUT(G_H, \cdot) = \mathrm{S}_{2^{\mathrm{nd}}}LPAUT(G_H)$.

Let $(G_H, \cdot)$ be a special loop. A mapping $A \in SSYM(G_H)$ is a

1. First Smarandache automorphism ($\mathrm{S}_{1^{\mathrm{st}}}$ automorphism) of G_H if and only if $A \in \mathrm{S}_{1^{\mathrm{st}}}PAUT(G_H)$ such that $c = e$. Their set is denoted by $\mathrm{S}_{1^{\mathrm{st}}}AUM(G_H, \cdot) = \mathrm{S}_{1^{\mathrm{st}}}AUM(G_H)$.

2. Second right Smarandache automorphism ($\mathrm{S}_{2^{\mathrm{nd}}}$ right automorphism) of G_H if and only if $A \in \mathrm{S}_{2^{\mathrm{nd}}}RPAUT(G_H)$ such that $c = e$. Their set is denoted by $\mathrm{S}_{2^{\mathrm{nd}}}RAUM(G_H, \cdot) = \mathrm{S}_{2^{\mathrm{nd}}}RAUM(G_H)$.

3. Second left Smarandache automorphism ($S_{2^{nd}}$ left automorphism) of G_H if and only if $A \in S_{2^{nd}}LPAUT(G_H)$ such that $c = e$. Their set is denoted by $S_{2^{nd}}LAUM(G_H, \cdot) = S_{2^{nd}}LAUM(G_H)$.

A special loop $(G_H, \cdot)$ is called a first Smarandache automorphism inverse property loop ($S_{1^{st}}$AIPL) if and only if $(J_\rho, J_\rho, J_\rho) \in AUT(H, \cdot)$.

A special loop $(G_H, \cdot)$ is called a second Smarandache right automorphic inverse property loop ($S_{2^{nd}}$RAIPL) if and only if J_ρ is a $S_{2^{nd}}$ right automorphism.

A special loop $(G_H, \cdot)$ is called a second Smarandache left automorphic inverse property loop ($S_{2^{nd}}$LAIPL) if and only if J_ρ is a $S_{2^{nd}}$ left automorphism.

Definition 4. Let $(G, \cdot)$ and $(L, \circ)$ be quasigroups (loops). The triple (U, V, W) such that $U, V, W \ : G \to L$ are bijections is called an isotopism of G onto L if and only if

$$xU \circ yV = (x \cdot y)W \ \forall \ x, y \in G. \tag{2}$$

Let $(G_H, \cdot)$ and $(L_M, \circ)$ be special groupoids. G_H and L_M are Smarandache isotopic (S-isotopic) [and we say $(L_M, \circ)$ is a Smarandache isotope of $(G_H, \cdot)$] if and only if there exist bijections $U, V, W \ : H \to M$ such that the triple $(U, V, W) : (G_H, \cdot) \to (L_M, \circ)$ is an isotopism. In addition, if $U = V = W$, then $(G_H, \cdot)$ and $(L_M, \circ)$ are said to be Smarandache isomorphic (S-isomorphic) [and we say $(L_M, \circ)$ is a Smarandache isomorph of $(G_H, \cdot)$ and thus write $(G_H, \cdot) \succsim (L_M, \circ)$.].

$(G_H, \cdot)$ is called a Smarandache G-special loop (SGS-loop) if and only if every special loop that is S-isotopic to $(G_H, \cdot)$ is S-isomorphic to $(G_H, \cdot)$.

Theorem 1. (Jaíyéọlá [13]) Let the special loop $(G_H, \cdot)$ be a $S_{2^{nd}}$BL. Then it is both a $S_{2^{nd}}$RIPL and a $S_{2^{nd}}$RAPL.

Theorem 2. (Jaíyéọlá [13]) Let $(G_H, \cdot)$ be a special loop. $(G_H, \cdot)$ is a $S_{2^{nd}}$BL if and only if $(R_s^{-1}, L_sR_s, R_s) \in S_{1^{st}}AUT(G_H, \cdot)$.

§3. Main results

Lemma 1. Let $(G_H, \cdot)$ be a special quasigroup and let $s, t \in H$. For all $x, y \in G$, let

$$x \circ y = xR_t^{-1} \cdot yL_s^{-1}. \tag{3}$$

Then, $(G_H, \circ)$ is a special loop and so $(G_H, \cdot)$ and $(G_H, \circ)$ are S-isotopic.

Proof. It is easy to show that $(G_H, \circ)$ is a quasigroup with a subquasigroup $(H, \circ)$ since $(G_H, \cdot)$ is a special quasigroup. So, $(G_H, \circ)$ is a special quasigroup. It is also easy to see that $s \cdot t \in H$ is the identity element of $(G_H, \circ)$. Thus, $(G_H, \circ)$ is a special loop. With $U = R_t$, $V = L_s$ and $W = I$, the triple $(U, V, W) \ : \ (G_H, \cdot) \to (G_H, \circ)$ is an S-isotopism.

Remark 3. $(G_H, \circ)$ will be called a Smarandache principal isotopism (S-principal isotopism) of $(G_H, \cdot)$.

Theorem 3. If the special quasigroup $(G_H, \cdot)$ and special loop $(L_M, \circ)$ are S-isotopic, then $(L_M, \circ)$ is S-isomorphic to a S-principal isotope of $(G_H, \cdot)$.

Proof. Let e be the identity element of the special loop $(L_M, \circ)$. Let U, V and W be 1-1 S-mappings of G_H onto L_M such that

$$xU \circ yV = (x \cdot y)W \ \forall \ x, y \in G_H.$$

Let $t = eV^{-1}$ and $s = eU^{-1}$. Define $x * y$ for all $x, y \in G_H$ by

$$x * y = (xW \circ yW)W^{-1}. \tag{4}$$

From (2), with x and y replaced by xWU^{-1} and yWV^{-1} respectively, we get

$$(xW \circ yW)W^{-1} = xWU^{-1} \cdot yWV^{-1} \ \forall \ x, y \in G_H. \tag{5}$$

In (5), with $x = eW^{-1}$, we get $WV^{-1} = L_s^{-1}$ and with $y = eW^{-1}$, we get $WU^{-1} = R_t^{-1}$. Hence, from (4) and (5),

$$x * y = xR_t^{-1} \cdot yL_s^{-1} \text{ and } (x * y)W = xW \circ yW \ \forall \ x, y \in G_H.$$

That is, $(G_H, *)$ is a S-principal isotope of $(G_H, \cdot)$ and is S-isomorphic to $(L_M, \circ)$.

Theorem 4. Let $(G_H, \cdot)$ be a $\mathrm{S_{2^{nd}}}$RIPL. Let $f, g \in H$ and let $(G_H, \circ)$ be a S-principal isotope of $(G_H, \cdot)$. $(G_H, \circ)$ is a $\mathrm{S_{2^{nd}}}$RIPL if and only if $\alpha(f, g) = (R_g, L_f R_g^{-1} L_{f \cdot g}^{-1}, R_g^{-1}) \in \mathrm{S_{2^{nd}}}RAUT(G_H, \cdot)$ for all $f, g \in H$.

Proof. Let $(G_H, \cdot)$ be a special loop that has the $\mathrm{S_{2^{nd}}}$RIP and let $f, g \in H$. For all $x, y \in G$, define $x \circ y = xR_g^{-1} \cdot yL_f^{-1}$ as in (3). Recall that $f \cdot g$ is the identity in $(G_H, \circ)$, so $x \circ x^{\rho'} = f \cdot g$ where $xJ'_\rho = x^{\rho'}$ i.e the right identity element of x in $(G_H, \circ)$. Then, for all $x \in G$, $x \circ x^{\rho'} = xR_g^{-1} \cdot xJ'_\rho L_f^{-1} = f \cdot g$ and by the $\mathrm{S_{2^{nd}}}$RIP of $(G_H, \cdot)$, since $sR_g^{-1} \cdot sJ'_\rho L_f^{-1} = f \cdot g$ for all $s \in H$, then $sR_g^{-1} = (f \cdot g) \cdot (sJ'_\rho L_f^{-1})J_\rho$ because $(H, \cdot)$ has the RIP. Thus,

$$sR_g^{-1} = sJ'_\rho L_f^{-1} J_\rho L_{f \cdot g} \Rightarrow sJ'_\rho = sR_g^{-1} L_{f \cdot g}^{-1} J_\lambda L_f. \tag{6}$$

$(G_H, \circ)$ has the $\mathrm{S_{2^{nd}}}$RIP iff $(x \circ s) \circ sJ'_\rho = s$ for all $s \in H$, $x \in G_H$ iff $(xR_g^{-1} \cdot sL_f^{-1})R_g^{-1} \cdot sJ'_\rho L_f^{-1} = x$, for all $s \in H$, $x \in G_H$. Replace x by $x \cdot g$ and s by $f \cdot s$, then $(x \cdot s)R_g^{-1} \cdot (f \cdot s)J'_\rho L_f^{-1} = x \cdot g$ iff $(x \cdot s)R_g^{-1} = (x \cdot g) \cdot (f \cdot s)J'_\rho L_f^{-1} J_\rho$ for all $s \in H$, $x \in G_H$ since $(G_H, \cdot)$ has the $\mathrm{S_{2^{nd}}}$RIP. Using (6),

$$(x \cdot s)R_g^{-1} = xR_g \cdot (f \cdot s)R_g^{-1} L_{f \cdot g}^{-1} \Leftrightarrow (x \cdot s)R_g^{-1} = xR_g \cdot sL_f R_g^{-1} L_{f \cdot g}^{-1} \Leftrightarrow$$

$$\alpha(f, g) = (R_g, L_f R_g^{-1} L_{f \cdot g}^{-1}, R_g^{-1}) \in \mathrm{S_{2^{nd}}}RAUT(G_H, \cdot) \text{ for all } f, g \in H.$$

Theorem 5. If a special loop $(G_H, \cdot)$ is a $\mathrm{S_{2^{nd}}}$BL, then any of its S-isotopes is a $\mathrm{S_{2^{nd}}}$RIPL.

Proof. By virtue of theorem 3, we need only to concern ourselves with the S-principal isotopes of $(G_H, \cdot)$. $(G_H, \cdot)$ is a $\mathrm{S_{2^{nd}}}$BL iff it obeys the $\mathrm{S_{2^{nd}}}$BI iff $(xs \cdot z)s = x(sz \cdot s)$ for all $x, z \in G$ and $s \in H$ iff $L_{xs}R_s = L_s R_s L_x$ for all $x \in G$ and $s \in H$ iff $R_s^{-1} L_{xs}^{-1} = L_x^{-1} R_s^{-1} L_s^{-1}$ for all $x \in G$ and $s \in H$ iff

$$R_s^{-1} L_s^{-1} = L_x R_s^{-1} L_{xs}^{-1} \text{ for all } x \in G \text{ and } s \in H. \tag{7}$$

Assume that $(G_H, \cdot)$ is a $\mathrm{S_{2^{nd}}}$BL. Then, by theorem 2,

$$(R_s^{-1}, L_s R_s, R_s) \in \mathrm{S_{1^{st}}}AUT(G_H, \cdot) \Rightarrow (R_s^{-1}, L_s R_s, R_s) \in \mathrm{S_{2^{nd}}}RAUT(G_H, \cdot) \Rightarrow$$

$$(R_s^{-1}, L_sR_s, R_s)^{-1} = (R_s, R_s^{-1}L_s^{-1}, R_s^{-1}) \in \mathrm{S}_{2^{\mathrm{nd}}}RAUT(G_H, \cdot).$$

By (7), $\alpha(x,s) = (R_s, L_xR_s^{-1}L_{xs}^{-1}, R_s^{-1}) \in \mathrm{S}_{2^{\mathrm{nd}}}RAUT(G_H, \cdot)$ for all $f, g \in H$. But $(G_H, \cdot)$ has the $\mathrm{S}_{2^{\mathrm{nd}}}$RIP by theorem 1. So, following theorem 4, all special loops that are S-isotopic to $(G_H, \cdot)$ are $\mathrm{S}_{2^{\mathrm{nd}}}$RIPLs.

Theorem 6. Suppose that each special loop that is S-isotopic to $(G_H, \cdot)$ is a $\mathrm{S}_{2^{\mathrm{nd}}}$RIPL, then the identities:

1. $(fg)\backslash f = (xg)\backslash x$;
2. $g\backslash(sg^{-1}) = (fg)\backslash[(fs)g^{-1}]$

are satisfied for all $f, g, s \in H$ and $x \in G$.

Proof. In particular, $(G_H, \cdot)$ has the $\mathrm{S}_{2^{\mathrm{nd}}}$RIP. Then by theorem 3, $\alpha(f,g) = (R_g, L_fR_g^{-1}L_{f\cdot g}^{-1}, R_g^{-1}) \in \mathrm{S}_{2^{\mathrm{nd}}}RAUT(G_H, \cdot)$ for all $f, g \in H$. Let

$$Y = L_fR_g^{-1}L_{f\cdot g}^{-1}. \tag{8}$$

Then,

$$xg \cdot sY = (xs)R_g^{-1}. \tag{9}$$

Put $s = g$ in (9), then $xg \cdot gY = (xg)R_g^{-1} = x$. But, $gY = gL_fR_g^{-1}L_{f\cdot g}^{-1} = (fg)\backslash[(fg)g^{-1}] = (fg)\backslash f$. So, $xg \cdot (fg)\backslash f = x \Rightarrow (fg)\backslash f = (xg)\backslash x$.

Put $x = e$ in (9), then $sYL_g = sR_g^{-1} \Rightarrow sY = sR_g^{-1}L_g^{-1}$. So, combining this with (8), $sR_g^{-1}L_g^{-1} = sL_fR_g^{-1}L_{f\cdot g}^{-1} \Rightarrow g\backslash(sg^{-1}) = (fg)\backslash[(fs)g^{-1}]$.

Theorem 7. Every special loop that is S-isotopic to a $\mathrm{S}_{2^{\mathrm{nd}}}$BL is itself a $\mathrm{S}_{2^{\mathrm{nd}}}$BL.

Proof. Let $(G_H, \circ)$ be a special loop that is S-isotopic to an $\mathrm{S}_{2^{\mathrm{nd}}}$BL $(G_H, \cdot)$. Assume that $x \cdot y = x\alpha \circ y\beta$ where $\alpha, \beta : H \to H$. Then the $\mathrm{S}_{2^{\mathrm{nd}}}$BI can be written in terms of $(\circ)$ as follows. $(xs \cdot z)s = x(sz \cdot s)$ for all $x, z \in G$ and $s \in H$.

$$[(x\alpha \circ s\beta)\alpha \circ z\beta]\alpha \circ s\beta = x\alpha \circ [(s\alpha \circ z\beta)\alpha \circ s\beta]\beta. \tag{10}$$

Replace $x\alpha$ by $\overline{x}$, $s\beta$ by $\overline{s}$ and $z\beta$ by $\overline{z}$, then

$$[(\overline{x} \circ \overline{s})\alpha \circ \overline{z}]\alpha \circ \overline{s} = \overline{x} \circ [(\overline{s}\beta^{-1}\alpha \circ \overline{z})\alpha \circ \overline{s}]\beta. \tag{11}$$

If $\overline{x} = e$, then

$$(\overline{s}\alpha \circ \overline{z})\alpha \circ \overline{s} = [(\overline{s}\beta^{-1}\alpha \circ \overline{z})\alpha \circ \overline{s}]\beta. \tag{12}$$

Substituting (12) into the RHS of (11) and replacing $\overline{x}$, $\overline{s}$ and $\overline{z}$ by x, s and z respectively, we have

$$[(x \circ s)\alpha \circ z]\alpha \circ s = x \circ [(s\alpha \circ z)\alpha \circ s]. \tag{13}$$

With $s = e$, $(x\alpha \circ z)\alpha = x \circ (e\alpha \circ z)\alpha$. Let $(e\alpha \circ z)\alpha = z\delta$, where $\delta \in SSYM(G_H)$. Then,

$$(x\alpha \circ z)\alpha = x \circ z\delta. \tag{14}$$

Applying (14), then (13) to the expression $[(x \circ s) \circ z\delta] \circ s$, that is

$$[(x \circ s) \circ z\delta] \circ s = [(x \circ s)\alpha \circ z]\alpha \circ s = x \circ [(s\alpha \circ z)\alpha \circ s] = x \circ [(s \circ z\delta) \circ s].$$

implies

$$[(x \circ s) \circ z\delta] \circ s = x \circ [(s \circ z\delta) \circ s].$$

Replace $z\delta$ by z, then

$$[(x \circ s) \circ z] \circ s = x \circ [(s \circ z) \circ s].$$

Theorem 8. Let $(G_H, \cdot)$ be a $\mathrm{S_{2^{nd}}BL}$. Each special loop that is S-isotopic to $(G_H, \cdot)$ is S-isomorphic to a S-principal isotope $(G_H, \circ)$ where $x \circ y = xR_f \cdot yL_f^{-1}$ for all $x, y \in G$ and some $f \in H$.

Proof. Let e be the identity element of $(G_H, \cdot)$. Let $(G_H, *)$ be any S-principal isotope of $(G_H, \cdot)$ say $x * y = xR_v^{-1} \cdot yL_u^{-1}$ for all $x, y \in G$ and some $u, v \in H$. Let e' be the identity element of $(G_H, *)$. That is, $e' = u \cdot v$. Now, define $x * y$ by

$$x \circ y = [(xe') * (ye')]e'^{-1} \text{ for all } x, y \in G.$$

Then $R_{e'}$ is an S-isomorphism of $(G_H, \circ)$ onto $(G_H, *)$. Observe that e is also the identity element for $(G_H, \circ)$ and since $(G_H, \cdot)$ is a $\mathrm{S_{2^{nd}}BL}$,

$$(pe')(e'^{-1}q \cdot e'^{-1}) = pq \cdot e'^{-1} \text{ for all } p, q \in G. \tag{15}$$

So, using (15),

$$x \circ y = [(xe') * (ye')]e'^{-1} = [xR_{e'}R_v^{-1} \cdot yR_{e'}L_u^{-1}]e'^{-1} = xR_{e'}R_v^{-1}R_{e'} \cdot yR_{e'}L_u^{-1}L_{e'^{-1}}R_{e'^{-1}}$$

implies that

$$x \circ y = xA \cdot yB, \ A = R_{e'}R_v^{-1}R_{e'} \text{ and } B = R_{e'}L_u^{-1}L_{e'^{-1}}R_{e'^{-1}}. \tag{16}$$

Let $f = eA$. then, $y = e \circ y = eA \cdot yB = f \cdot yB$ for all $y \in G$. So, $B = L_f^{-1}$. In fact, $eB = f^\rho = f^{-1}$. Then, $x = x \circ e = xA \cdot eB = xA \cdot f^{-1}$ for all $x \in G$ implies $xf = (xA \cdot f^{-1})f$ implies $xf = xA$ ($\mathrm{S_{2^{nd}}RIP}$) implies $A = R_f$. Now, (16) becomes $x \circ y = xR_f \cdot yL_f^{-1}$.

Theorem 9. Let $(G_H, \cdot)$ be a $\mathrm{S_{2^{nd}}BL}$ with the $\mathrm{S_{2^{nd}}RAIP}$ or $\mathrm{S_{2^{nd}}LAIP}$, let $f \in H$ and let $x \circ y = xR_f \cdot yL_f^{-1}$ for all $x, y \in G$. Then $(G_H, \circ)$ is a $\mathrm{S_{1^{st}}AIPL}$ if and only if $f \in N_\lambda(H, \cdot)$.

Proof. Since $(G_H, \cdot)$ is a $\mathrm{S_{2^{nd}}BL}$, $J = J_\lambda = J_\rho$ in $(H, \cdot)$. Using (6) with $g = f^{-1}$,

$$sJ'_\rho = sR_fJL_f. \tag{17}$$

$(G_H, \circ)$ is a $\mathrm{S_{1^{st}}AIPL}$ iff $(x \circ y)J'_\rho = xJ'_\rho \circ yJ'_\rho$ for all $x, y \in H$ iff

$$(xR_f \cdot yL_f^{-1})J'_\rho = xJ'_\rho R_f \cdot yJ'_\rho L_f^{-1}. \tag{18}$$

Let $x = uR_f^{-1}$ and $y = vL_f$ and use (16), then (18) becomes $(uv)R_fJL_f = uJL_fR_f \cdot vL_fR_fJ$ iff $\alpha = (JL_fR_f, L_fR_fJ, R_fJL_f) \in AUT(H, \cdot)$. Since $(G_H, \cdot)$ is a $\mathrm{S_{1^{st}}AIPL}$, so $(J, J, J) \in AUT(H, \cdot)$. So, $\alpha \in AUT(H, \cdot) \Leftrightarrow \beta = \alpha(J, J, J)(R_{f^{-1}}^{-1}, L_{f^{-1}}R_{f^{-1}}, R_{f^{-1}}) \in AUT(H, \cdot)$. Since $(G_H, \cdot)$ is a $\mathrm{S_{2^{nd}}BL}$,

$xL_fR_fL_{f^{-1}}R_{f^{-1}} = [f^{-1}(fx \cdot f)]f^{-1} = [(f^{-1}f \cdot x)f]f^{-1} = x$ for all $x \in G$. That is, $L_fR_fL_{f^{-1}}R_{f^{-1}} = I$ in $(G_H, \cdot)$. Also, since $J \in AUM(H, \cdot)$, then $R_fJ = JR_{f^{-1}}$ and $L_fJ =$

$JL_{f^{-1}}$ in $(H,\cdot)$. So,

$$\begin{aligned}\beta &= (JL_fR_fJR_{f^{-1}}^{-1}, L_fR_fJ^2L_{f^{-1}}R_{f^{-1}}, R_fJL_fJR_{f^{-1}})\\ &= (JL_fJR_{f^{-1}}R_{f^{-1}}^{-1}, L_fR_fL_{f^{-1}}R_{f^{-1}}, R_fL_{f^{-1}}R_{f^{-1}})\\ &= (L_{f^{-1}}, I, R_fL_{f^{-1}}R_{f^{-1}}).\end{aligned}$$

Hence, $(G_H,\circ)$ is a $S_{1^{st}}$AIPL iff $\beta \in AUT(H,\cdot)$.

Now, assume that $\beta \in AUT(H,\cdot)$. Then, $xL_{f^{-1}} \cdot y = (xy)R_fL_{f^{-1}}R_{f^{-1}}$ for all $x,y \in H$. For $y = e$, $L_{f^{-1}} = R_fL_{f^{-1}}R_{f^{-1}}$ in $(H,\cdot)$. so, $\beta = (L_{f^{-1}}, I, L_{f^{-1}}) \in AUT(H,\cdot) \Rightarrow f^{-1} \in N_\lambda(H,\cdot) \Rightarrow f \in N_\lambda(H,\cdot)$.

On the other hand, if $f \in N_\lambda(H,\cdot)$, then, $\gamma = (L_f, I, L_f) \in AUT(H,\cdot)$. But $f \in N_\lambda(H,\cdot) \Rightarrow L_f^{-1} = L_{f^{-1}} = R_fL_{f^{-1}}R_{f^{-1}}$ in $(H,\cdot)$. Hence, $\beta = \gamma^{-1}$ and $\beta \in AUT(H,\cdot)$.

Corollary 1. Let $(G_H,\cdot)$ be a $S_{2^{nd}}$BL and a $S_{1^{st}}$AIPL. Then, for any special loop $(G_H,\circ)$ that is S-isotopic to $(G_H,\cdot)$, $(G_H,\circ)$ is a $S_{1^{st}}$AIPL iff $(G_H,\cdot)$ is a $S_{1^{st}}$-loop and a $S_{1^{st}}$ commutative loop.

Proof. Suppose every special loop that is S-isotopic to $(G_H,\cdot)$ is a $S_{1^{st}}$AIPL. Then, $f \in N_\lambda(H,\cdot)$ for all $f \in H$ by theorem 9. So, $(G_H,\cdot)$ is a $S_{1^{st}}$-loop. Then, $y^{-1}x^{-1} = (xy)^{-1} = x^{-1}y^{-1}$ for all $x,y \in H$. So, $(G_H,\cdot)$ is a $S_{1^{st}}$ commutative loop.

The proof of the converse is as follows. If $(G_H,\cdot)$ is a $S_{1^{st}}$-loop and a $S_{1^{st}}$ commutative loop, then for all $x,y \in H$ such that $x \circ y = xR_f \cdot yL_f^{-1}$,

$$(x \circ y) \circ z = (xR_f \cdot yL_f^{-1})R_f \cdot zL_f^{-1} = (xf \cdot f^{-1}y)f \cdot f^{-1}z.$$

$$x \circ (y \circ z) = xR_f \cdot (yR_f \cdot zL_f^{-1})L_f^{-1} = xf \cdot f^{-1}(yf \cdot f^{-1}z).$$

So, $(x \circ y) \circ z = x \circ (y \circ z)$. Thus, $(H,\circ)$ is a group. Furthermore,

$$x \circ y = xR_f \cdot yL_f^{-1} = xf \cdot f^{-1}y = x \cdot y = y \cdot x = yf \cdot f^{-1}x = y \circ x.$$

So, $(H,\circ)$ is commutative and so has the AIP. Therefore, $(G_H,\circ)$ is a $S_{1^{st}}$AIPL.

Lemma 2. Let $(G_H,\cdot)$ be a $S_{2^{nd}}$BL. Then, every special loop that is S-isotopic to $(G_H,\cdot)$ is S-isomorphic to $(G_H,\cdot)$ if and only if $(G_H,\cdot)$ obeys the identity $(x\cdot fg)g^{-1}\cdot f\backslash(y\cdot fg) = (xy)\cdot(fg)$ for all $x,y \in G_H$ and $f,g \in H$.

Proof. Let $(G_H,\circ)$ be an arbitrary S-principal isotope of $(G_H,\cdot)$. It is claimed that $(G_H,\cdot) \overset{R_{fg}}{\succsim} (G_H,\circ)$ iff $xR_{fg} \circ yR_{fg} = (x\cdot y)R_{fg}$ iff $(x \cdot fg)R_g^{-1} \cdot (y \cdot fg)L_f^{-1} = (x \cdot y)R_{fg}$ iff $(x \cdot fg)g^{-1} \cdot f\backslash(y \cdot fg) = (xy) \cdot (fg)$ for all $x,y \in G_H$ and $f,g \in H$.

Theorem 10. Let $(G_H,\cdot)$ be a $S_{2^{nd}}$BL, let $f \in H$, and let $x \circ y = xR_f \cdot yL_f^{-1}$ for all $x,y \in G$. Then, $(G_H,\cdot) \succsim (G_H,\circ)$ if and only if there exists a $S_{1^{st}}$ pseudo-automorphism of $(G_H,\cdot)$ with $S_{1^{st}}$ companion f.

Proof. $(G_H,\cdot) \succsim (G_H,\circ)$ if and only if there exists $T \in SSYM(G_H,\cdot)$ such that $xT \circ yT = (x\cdot y)T$ for all $x,y \in G$ iff $xTR_f \cdot yTL_f^{-1} = (x\cdot y)T$ for all $x,y \in G$ iff $\alpha = (TR_f, TL_f^{-1}, T) \in S_{1^{st}}AUT(G_H)$.

Recall that by theorem 2, $(G_H,\cdot)$ is a $\mathrm{S}_{2^{\mathrm{nd}}}$BL iff $(R_f^{-1},L_fR_f,R_f)\in \mathrm{S}_{1^{\mathrm{st}}}AUT(G_H,\cdot)$ for each $f\in H$. So,

$$\alpha\in \mathrm{S}_{1^{\mathrm{st}}}AUT(G_H)\Leftrightarrow \beta=\alpha(R_f^{-1},L_fR_f,R_f)=$$
$$(T,TR_f,TR_f)\in \mathrm{S}_{1^{\mathrm{st}}}AUT(G_H,\cdot)\Leftrightarrow T\in \mathrm{S}_{1^{\mathrm{st}}}PAUT(G_H)$$

with $\mathrm{S}_{1^{\mathrm{st}}}$ companion f.

Corollary 2. Let $(G_H,\cdot)$ be a $\mathrm{S}_{2^{\mathrm{nd}}}$BL, let $f\in H$ and let $x\circ y=xR_f\cdot yL_f^{-1}$ for all $x,y\in G_H$. If $f\in N_\rho(H,\cdot)$, then, $(G_H,\cdot)\succsim(G_H,\circ)$.

Proof. Following theorem 10, $f\in N_\rho(H,\cdot)\Rightarrow T\mathrm{S}_{1^{\mathrm{st}}}PAUT(G_H)$ with $\mathrm{S}_{1^{\mathrm{st}}}$ companion f.

Corollary 3. Let $(G_H,\cdot)$ be a $\mathrm{S}_{2^{\mathrm{nd}}}$BL. Then, every special loop that is S-isotopic to $(G_H,\cdot)$ is S-isomorphic to $(G_H,\cdot)$ if and only if each element of H is a $\mathrm{S}_{1^{\mathrm{st}}}$ companion for a $\mathrm{S}_{1^{\mathrm{st}}}$ pseudo-automorphism of $(G_H,\cdot)$.

Proof. This follows from theorem 8 and theorem 10.

Corollary 4. Let $(G_H,\cdot)$ be a $\mathrm{S}_{2^{\mathrm{nd}}}$BL. Then, $(G_H,\cdot)$ is a SGS-loop if and only if each element of H is a $\mathrm{S}_{1^{\mathrm{st}}}$ companion for a $\mathrm{S}_{1^{\mathrm{st}}}$ pseudo-automorphism of $(G_H,\cdot)$.

Proof. This is an immediate consequence of corollary 4.

Remark 4. Every Bol loop is a $\mathrm{S}_{2^{\mathrm{nd}}}$BL. Most of the results on isotopy of Bol loops in chapter 3 of [19] can easily be deduced from the results in this paper by simply forcing H to be equal to G.

References

[1] R. H. Bruck, A survey of binary systems, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1966.

[2]O. Chein, H. O. Pflugfelder and J. D. H. Smith, Quasigroups and Loops: Theory and Applications, Heldermann Verlag, 1990.

[3] J. Dene and A. D. Keedwell, Latin squares and their applications, the English University press Lts, 1974.

[4] E. G. Goodaire, E. Jespers and C. P. Milies, Alternative loop rings, NHMS (184), Elsevier, 1996.

[5] T. G. Jaíyéọlá, An holomorphic study of the Smarandache concept in loops, Scientia Magna Journal, **2**(2006), No. 1, 1-8.

[6] T. G. Jaíyéọlá, Parastrophic invariance of Smarandache quasigroups, Scientia Magna Journal, **2**(2006), No. 3, 48-53.

[7] T. G. Jaíyéọlá, On the universality of some Smarandache loops of Bol-Moufang type, Scientia Magna Journal, **2**(2006), No. 4, 45-48.

[8] T. G. Jaíyéọlá, A Pair Of Smarandachely Isotopic Quasigroups And Loops Of The Same Variety, International Journal of Mathematical Combinatorics, **1**(2008), 36-44.

[9] T. G. Jaíyéọlá, An Holomorphic Study Of Smarandache Automorphic and Cross Inverse Property Loops, Proceedings of the 4^{th} International Conference on Number Theory and Smarandache Problems, Scientia Magna, **4**(2008), No. 1, 102–108.

[10] T. G. Jaíyéọlá, Smarandache isotopy theory of Smarandache: quasigroups and loops, Proceedings of the 4th International Conference on Number Theory and Smarandache Problems, Scientia Magna, **4**(2008), No. 1, 168-177.

[11] T. G. Jaíyéọlá, On Smarandache Bryant Schneider group of a Smarandache loop, International Journal of Mathematical Combinatorics, **2**(2008), 51-63.

[12] T. G. Jaíyéọlá, A Double Cryptography Using The Smarandache Keedwell Cross Inverse Quasigroup, International Journal of Mathematical Combinatorics, **3**(2008), 28-33.

[13] T. G. Jaíyéọlá, Basic properties of second Smarandache Bol loops, International Journal of Mathematical Combinatorics, **2**(2009), 11-20.

[14] T. G. Jaíyéọlá, A study of new concepts in Smarandache quasigroups and loops, ProQuest Information and Learning (ILQ), Ann Arbor, USA, 2009.

[15] A. S. Muktibodh, Smarandache quasigroups rings, Scientia Magna, **1**(2005), No. 2, 139-144.

[16] A. S. Muktibodh, Smarandache Quasigroups, Scientia Magna, **2**(2006), No. 2, 13-19.

[17] H. O. Pflugfelder, Quasigroups and Loops: Introduction, Sigma series in Pure Math. 7, Heldermann Verlag, Berlin, 1990.

[18] A. Sade, Quasigroupes parastrophiques, Math. Nachr, **20**(1959), 73-106.

[19] D. A. Robinson, Bol loops, Ph. D thesis, University of Wisconsin, Madison, Wisconsin, 1964.

[20] D. A. Robinson, Bol loops, Trans. Amer. Math. Soc., **123**(1966), No. 2, 341-354.

[21] A. R. T. Solarin and B. L. Sharma, On the construction of Bol loops, Al. I. Cuza. Univ., **27**(1981), No. 1, 23-28.

[22] W. B. Vasantha Kandasamy, Smarandache Loops, Department of Mathematics, Indian Institute of Technology, Madras, India, 2002.

[23] W. B. Vasantha Kandasamy, Smarandache Loops, Smarandache notions journal, 252-258.

[24] W. B. Vasantha Kandasamy, Smarandache groupoids, Scientia Magna, **1**(2005), No. 1, 27-35.

Scientia Magna
Vol. 7 (2011), No. 1, 94-98

Implicative filters in pocrims

Mahboobeh Mohamadhasani† and Masoud Haveshki‡

Department of Mathematics, Hormozgan University, Bandarabbas, Iran
E-mail: ma.mohamadhasani@gmail.com ma.haveshki@gmail.com

Abstract In this paper, we define the notion of implicative filters in pocrims. We give several characterizations about implicative filters and consider a relation between these filters and quotient algebra that constructed via these filters.

Keywords Pocrim, implicative filter, Brouwerian semilattice.

§1. Introduction

Bounded pocrims form a class of algebras containing as proper subclasses, among others, the class of algebras of some logics, e.g, the class of BL-algebras, i.e., algebras of the basic fuzzy logic [2] (and consequently the class of MV-algebras, i.e., algebras of the Lukasiewicz infinite valued logic), as well as the class of Heyting algebras, i.e., algebras of intuitionistic logic. Filters in pocrims are defined [1,3]. In this paper we define the notion of implicative filter. We show that $\{1\}$ is an implicative filter of pocrim A iff A is Brouwerian semilattice.

§2. Periliminiaries

Definition 2.1.[1,3] A pocrim (partially ordered commutative integral residuated monoid) is a algebra $(A, *, \rightarrow, 1)$ with binary operations $*$, $\rightarrow$ and a constant 1 such that:

(a) $(A, *, \leq)$ is a partially ordered commutative monoid with a greatest element 1 where $x \leq y$ if and only if $x \rightarrow y = 1$.

(b) $*$ and $\rightarrow$ are residuated, i.e., the following adjointness condition holds on A:

$$z \leq x \rightarrow y \text{ if and only if } \mathrm{z} * \mathrm{x} \leq \mathrm{y}.$$

If $(A, \leq)$ has a least element 0, a pocrim is called bounded.

A pocrim is called:

(1) Brouwerian semilattice if $x^2 = x$, for all $x \in A$, where $x^2 = x * x$.

(2) Generalized Boolean algebra if $(x \rightarrow y) \rightarrow x = x$, for all $x, y \in A$.

It is worth noticing that pocrims are closely related to BCK-algebras introduced by Iséki [4] as an algebraic semantics of BCK-implicational calculus. Namely, pocrims are just BCK-algebras satisfying the condition (P), i.e., BCK-algebras expanded by a binary oparation $*$ which satisfies the identity $(x * y) \rightarrow z = x \rightarrow (y \rightarrow z)$. On the other hand there are BCK-algebras which do not admit such a multiplication.

Lemma 2.2.[3] In any pocrim A, the following relations hold for all $x, y, z \in A$:

(1) $1 \to x = x$, $x \to x = x \to 1 = 1$.

(2) $x * (x \to y) \leq y$.

(3) $x \leq (y \to (x * y))$.

(4) $x \to (y \to z) = y \to (x \to z) = (x * y) \to z$.

(5) If $x \leq y$, then $y \to z \leq x \to z$ and $z \to x \leq z \to y$.

(6) $y \leq (y \to x) \to x$.

(7) $x \to y \leq (y \to z) \to (x \to z)$.

(8) $x \to y \leq (z \to x) \to (z \to y)$.

(9) $x * y \leq x, y$.

Definition 2.3.[1,3] A filter of a pocrim A is a nonempty subset F of A such that for all $a, b \in A$, we have

(1) $a * b \in F$, for all $a, b \in F$.

(2) $a \leq b$ and $a \in F$ imply $b \in F$.

Definition 2.4.[1,3] A nonempty subset D of pocrim A is called a deductive system of A if :

(1) $1 \in D$.

(2) If $x \in D$ and $x \to y \in D$, then $y \in D$.

Proposition 2.5.[1] A nonempty subset F of pocrim A is a deductive system if and only if is a filter.

Theorem 2.6.[1,3] Let F be filter of a pocrim A. Define

$$x \equiv_F y \text{ if and only if } \mathrm{x} \to \mathrm{y} \in \mathrm{F} \text{ and } \mathrm{y} \to \mathrm{x} \in \mathrm{F}.$$

Then $\equiv_F$ is a congruence relation on A. The set of all congruence class is denoted by A/F, i.e, $A/F = \{[x] \mid x \in A\}\}$, where $[x] = \{y \in A \mid y \equiv_F x\}$. Define $*$ and $\to$ on A/F as follow:

$$[x] * [y] = [x * y], \ [x] \to [y] = [x \to y]$$

and $(A/F, *, \to, [1])$ is a pocrim which is called the quotient pocrim with respect to F.

§3. Implicative filters in pocrim

From now on $(A, *, \to, 1)$ or simply A is a pocrim.

Definition 3.1. A non-empty subset F of A is called an implicative filter of A if it satisfies:

(1) $1 \in F$;

(2) $x \to (y \to z) \in F$ and $x \to y \in F$ imply $x \to z \in F$.

Theorem 3.2. Any implicative filter of A is a filter but the converse is not true.

Proof. Let F be an implicative filter and $x, x \to y \in F$. By Lemma 2.2,

$$1 \to (x \to y) = x \to y \in F \text{ and } 1 \to \mathrm{x} = \mathrm{x} \in \mathrm{F}.$$

Hence $y = 1 \to y \in F$. Therefore F is a filter.

Example 3.3. Let $B = \{0, a, b, 1\}$. Define $*$ and $\rightarrow$ as follows:

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

$*$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

Then $(B, *, \rightarrow, 1)$ is a pocrim and it is clear that $F = \{b, 1\}$, is a filter, while it is not an implicative filter since $a \rightarrow (a \rightarrow 0) \in F$ and $a \rightarrow a \in F$ but $a \rightarrow 0 \notin F$.

Example 3.4. Let $B = \{0, a, b, c, 1\}$. Define $*$ and $\rightarrow$ as follows:

$*$	1	0	a	b	c
1	1	0	a	b	c
0	0	0	0	0	0
a	a	0	a	a	a
b	b	0	a	b	a
c	c	0	a	a	c

$\rightarrow$	1	0	a	b	c
1	1	0	a	b	c
0	1	1	1	1	1
a	1	0	1	1	1
b	1	0	c	1	c
c	1	0	b	b	1

Easily we check that $(B, *, \rightarrow, 1)$ is a pocrim. Consider the filter $F = \{b, 1\}$. Then F is an implicative filter.

Theorem 3.5. Let F be a filter of A, then F is an implicative filter if and only if for any $a \in A$, $A_a = \{x \in A \mid a \rightarrow x \in F\}$ is a filter of A.

Proof. Let F be an implicative filter and $a \in A$. Since $a \rightarrow 1 = 1 \in F$, $1 \in A_a$. If $x, x \rightarrow y \in A_a$, then $a \rightarrow x \in F$ and $a \rightarrow (x \rightarrow y) \in F$. Since F is an implicative filter,

$a \to y \in F$ and so $y \in A_a$. Therefore A_a is a filter. Conversely, let for any $a \in A$, A_a is a filter of A and $x \to (y \to z) \in F$ and $x \to y \in F$. Then $y \to z \in A_x$ and $y \in A_x$. Since A_x is filter we get $z \in A_x$ and so $x \to z \in F$.

Hence F is an implicative filter.

Theorem 3.6. Given a non-empty subset F of A, the following conditions are equivalent:

(a) F is an implicative filter.

(b) F is a filter and $y \to (y \to x) \in F$ implies $y \to x \in F$, for all $x, y \in A$.

(c) F is a filter and $z \to (y \to x) \in F$ implies $(z \to y) \to (z \to x) \in F$, for all $x, y, z \in A$.

(d) $1 \in F$, $z \to (y \to (y \to x)) \in F$ and $z \in F$ imply $y \to x \in F$.

(e) $x \to x^2 \in F$, for all $x \in A$.

Proof. (a $\Rightarrow$ b): Let F be an implicative filter, by Theorem 3.2, F is a filter. If $y \to (y \to x) \in F$, since $y \to y = 1$ by hypothesis we get $y \to x \in F$.

(b $\Rightarrow$ c): Let $z \to (y \to x) \in F$, by Lemma 2.2, we have

$$z \to (z \to ((z \to y) \to x)) = z \to ((z \to y) \to (z \to x)) \geq z \to (y \to x).$$

Since F is filter and $z \to (y \to x) \in F$, we get $z \to (z \to ((z \to y) \to x)) \in F$. By hypothesis we conclude that $z \to (z \to (y \to x)) \in F$ and so $(z \to y) \to (z \to x) \in F$.

(c $\Rightarrow$ d): Let $z, z \to (y \to (y \to x)) \in F$, since F is a filter, $1 \in F$ and $y \to (y \to x) \in F$. Hence by hypothesis we get $(y \to y) \to (y \to x) \in F$. On the other hand,

$$y \to x = 1 \to (y \to x) = (y \to y) \to (y \to x).$$

Therefore, $y \to x \in F$.

(d $\Rightarrow$ a): Let $z \to y \in F$ and $z \to (y \to x) \in F$. By Lemma 2.2, we have,

$$z \to (y \to x) = y \to (z \to x) \leq (z \to y) \to (z \to (z \to x)).$$

Since F is filter and $z \to (y \to x) \in F$, we get $(z \to y) \to (z \to (z \to x))$. $z \to y \in F$ and (d) imply $z \to x \in F$.

(a $\Rightarrow$ e): Let $x \in A$, hence, by Lemma 2.2,

$$x \to (x \to x^2) = x^2 \to x^2 = 1 \text{ and x} \to \text{x} = 1 \in \text{F}.$$

Since F is implicative filter, we get $x \to x^2 \in F$.

(e $\Rightarrow$ a): Let $x, y, z \in A$ be such that $x \to (y \to z) \in F$ and $x \to y \in F$. By Lemma 2.2,

$$(x \to (y \to z)) * (x \to y) * x^2 = (x * (x \to (y \to z))) * (x * (x \to y)) \leq (y \to z) * y \leq z.$$

Then $(x \to (y \to z)) * (x \to y) \leq x^2 \to z$. Since $x \to (y \to z) \in F$ and $x \to y \in F$ we get

$$(x \to (y \to z)) * (x \to y) \in F,$$

and so $x^2 \to z \in F$. By Lemma 2.2, $x \to x^2 \leq (x^2 \to z) \to (x \to z)$. On the other hand $x^2 \to z \in F$ and $x \to x^2 \in F$, then $x \to z \in F$. Hence F is an implicative filter.

Theorem 3.7. In any pocrim A, the following conditions are equivalent:

(a) A is Brouwerian semilattice.

(b) Any filter of A is an implicative filter of A.

(c) $\{1\}$ is an implicative filter of A.

Proof. (a $\Rightarrow$ b): Let A be Brouwerian semilattice and F be an arbitrary filter of A then $x^2 = x$, for all $x \in A$. To show that F is an implicative filter we use (e) of Theorem 3.6. Since $x^2 = x$ then $x \to x^2 = 1 \in F$ and F is an implicative filter.

(b $\Rightarrow$ c): is clear.

(c $\Rightarrow$ a): Since $\{1\}$ is implicative filter by Theorem 3.6, we get $x \to x^2 = 1$, for all $x \in A$. Hence $x \leq x^2$. By Lemma 2.2, $x^2 \leq x$ and so $x^2 = x$. Therefore A is Brouwerian semilattice.

Theorem 3.8. Suppose F and G are filters of A and $F \subseteq G$. If F is an implicative filter then G is an implicative filter.

Proof. Let F and G are filters of A, $F \subseteq G$, F an implicative filter and $x \in A$. Since F is implicative filter, by Theorem 3.6 we get $x \to x^2 \in F$, for all $x \in A$ and since $F \subseteq G$, $x \to x^2 \in G$, for all $x \in A$. Therefore F is implicative filter.

Theorem 3.9. Let F be a filter of A. Then F is an implicative filter if and only if A/F is Brouwerian semilattice.

Proof. Let F be a implicative filter of A and $[x] \in A/F$. By Theorem 3.6, $x \to x^2 \in F$. Hence $[x] \to [x^2] = [x \to x^2] = [1]$ and so $[x] \leq [x^2] = [x]^2$. On the other hand by Lemma 2.2, $[x]^2 \leq [x]$. Then $[x] = [x]^2$, for all $x \in A$. Therefore A/F is Brouwerian semilattice. Conversely, suppose that A/F be Brouwerian semilattice and $x \in A$. Then

$$[x \to x^2] = [x] \to [x^2] = [x] \to [x]^2 = [1]$$

and so $x \to x^2 \in F$. Therefore by Theorem 3.6, F is implicative filter.

References

[1] W. J Blok, J. G Raftery, Varieties of commutative residuated integral pomonoids and their residuation subreducts. J Algebra, **190**(1997), 280-328.

[2] P. Hájek, Metamathematics of fuzzy logic. Kluwer, Dordrecht, 1998.

[3] R. Halas, M. Botur, On very true operators on pocrims. Soft comput, **13**(2009), 1063-1072.

[4] K. Iséki, An algebra related to a propositional calculus. Proc Jpn Acad, **42**(1996), 26-29,

Scientia Magna
Vol. 7 (2011), No. 1, 99-106

Magic graphoidal on special type of unicyclic graphs

A. Nellai Murugan† and A. Nagarajan‡

Department of Mathematics, V. O. Chidambaram College, Tuticorin, Tamilnadu
E-mail: anellai.vocc@gmail.com

Abstract B. D. Acharya and E. Sampathkumar [1] defined Graphoidal cover as partition of edge set of G into internally disjoint paths (not necessarily open). The minimum cardinality of such cover is known as graphoidal covering number of G. Let $G = \{V, E\}$ be a graph and let ψ be a graphoidal cover of G. Define $f : V \cup E \to \{1, 2, \ldots, p+q\}$ such that for every path $P = (v_0v_1v_2 \ldots v_n)$ in ψ with $f^*(P) = f(v_0) + f(v_n) + \sum\limits_1^n f(v_{i-1}v_i) = k$, a constant, where f^* is the induced labeling on ψ. Then, we say that G admits ψ-magic graphoidal total labeling of G. A graph G is called magic graphoidal if there exists a minimum graphoidal cover ψ of G such that G admits ψ-magic graphoidal total labeling.

Keywords Graphoidal cover, magic graphoidal, graphoidal constant.
2010 Mathematics Subject Classification Number: 05C78.

§1. Introduction

By a graph, we mean a finite simple and undirected graph. The vertex set and edge set of a graph G denoted are by $V(G)$ and $E(G)$ respectively. C_n^+ is a crown, $C_n \odot P_n$ is a Dragon and $C_m \odot P_n$ is a Armed Crown. Terms and notations not used here are as in [3].

§2. Preliminaries

Let $G = \{V, E\}$ be a graph with p vertices and q edges. A graphoidal cover ψ of G is a collection of (open) paths such that

[1] Every edge is in exactly one path of ψ.

[2] Every vertex is an interval vertex of atmost one path in ψ.

We define $\gamma(G) = \min\limits_{\psi \in \zeta} |\psi|$, where ζ is the collection of graphoidal covers ψ of G and γ is graphoidal covering number of G.

Let ψ be a graphoidal cover of G. Then we say that G admits ψ-magic graphoidal total labeling of G if there exists a bijection $f : V \cup E \to \{1, 2, \ldots, p+q\}$ such that for every path $P = (v_0v_1v_2 \ldots v_n)$ in ψ, then, $f^*(P) = f(v_0) + f(v_n) + \sum\limits_1^n f(v_{i-1}v_i) = k$, a constant, where f^* is the induced labeling of ψ. A graph G is called magic graphoidal if there exists a minimum graphoidal cover ψ of G such that G admits ψ- magic graphoidal total labeling. In this paper, we proved that Crown C_n^+, Dragan $C_n \odot P_n$ and Armed grown $C_m \odot P_n$ is graph in which a path of length n is joined at every vertex of the cycle C_m are magic graphoidal.

Result 2.1.[11] Let $G = (p, q)$ be a simple graph. If every vertex of G is an internal vertex in ψ then $\gamma(G) = q - p$.

Result 2.2.[11] If every vertex v of a simple graph G, where degree is more than one *ie* $d(v) > 1$, is an internal vertex of ψ is minimum graphoidal cover of G and $\gamma(G) = q - p + n$ where n is the number of vertices having degree one.

Result 2.3.[11] Let G be (p, q) a simple graph then $\gamma(G) = q - p + t$ where t is the number of vertices which are not internal.

Result 2.4.[11] For any tree G, $\gamma(G) = \Delta$ where Δ is the maximum degree of a vertex in G.

Result 2.5.[11] For any k-regular graph G, $k \geq 3$, $\gamma(G) = q - p$.

Result 2.6.[11] For any graph G with $\delta \geq 3$, $\gamma(G) = q - p$.

Result 2.7.[11] Let G be a connected unicyclic graph with n vertices of degree 1, Z be its unique cycle and let m be the number of vertices of degree at least 3 on Z. Then

$$\gamma(G) = \left\{ \begin{array}{ll} 1 & \text{if } m = 0; \\ n+1 & \text{if } m = 1 \text{ and } d(v) = 3 \text{ where } v \text{ is the unique vertex of} \\ & \text{degree} \geq 3 \text{ on } z; \\ \text{otherwise} & \end{array} \right\}.$$

§3. Magic graphoidal on special type of unicyclic graphs

Theorem 3.1. Crown C_n^+ is magic graphoidal.

Proof. Let $V(C_n^+) = \{u_i, v_i : 1 \leq i \leq n\}$,
$E(C_n^+) = \{[(u_i u_{i+1}) : 1 \leq i \leq n-1] \cup (u_1 u_n) \cup (u_i v_i) : 1 \leq i \leq n\}$.
Define $f : V \cup E \to \{1, 2, 3, \ldots, p+q\}$ by

$$\begin{aligned} f(u_i) &= i, \ 1 \leq i \leq n; \\ f(v_1) &= 3n+1; \\ f(v_{i+1}) &= 4n+1-i, \ 1 \leq i \leq n-1; \\ f(u_i u_{i+1}) &= 3n+1-i, \ 1 \leq i \leq n-1; \\ f(u_1 u_n) &= 2n+1; \\ f(u_1 v_1) &= 2n; \\ f(u_i v_i) &= n+(i-1), \ 2 \leq i \leq n. \end{aligned}$$

Let $\psi = \{[(u_i u_{i+1} v_{i+1}) : 1 \leq i \leq n-1] \cup (u_n u_1 v_1)\}$. Clearly, ψ is a minimum graphoidal cover.

$$\begin{aligned} f^*[(u_n u_1 v_1)] &= f(u_n) + f(v_1) + f(u_n u_1) + f(u_1 v_1) \\ &= n + 3n + 1 + 2n + 1 + 2n \\ &= 8n + 2. \end{aligned} \tag{1}$$

For $1 \leq i \leq n-1$,

$$\begin{aligned} f^*[(u_iu_{i+1}v_{i+1})] &= f(u_i) + f(v_{i+1}) + f(u_iu_{i+1}) + f(u_{i+1}v_{i+1}) \\ &= i + 4n + 1 - i + 3n + 1 - i + n + i \\ &= 8n + 2. \end{aligned} \tag{2}$$

From (1) and (2), we conclude that ψ is minimum magic graphoidal cover. Hence, Crown C_n^+ is magic graphoidal. For example, the magic graphoidal cover of C_4^+ is shown in Figure 1.

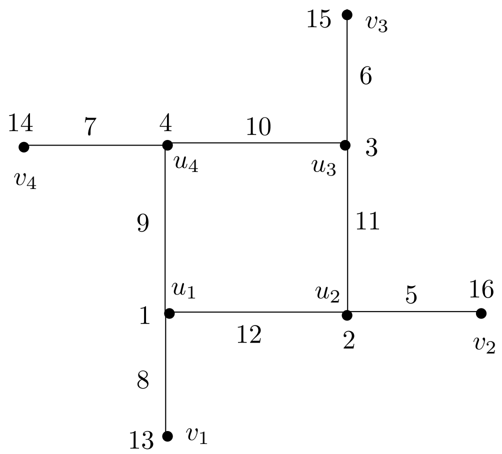


Figure 1. C_4^+

$\psi = \{(u_1u_2v_2), (u_2u_3v_3), (u_3u_4v_4), (u_4u_1v_1)\},\ \gamma = 4,\ K = 34.$

Theorem 3.2. Dragan $C_n \odot P_n$, (n -even) is magic graphoidal.

Proof. Let $G = C_n \odot Pn$.

$$\begin{aligned} \text{Let } V(G) &= \{[u_i, v_i : 1 \leq i \leq n-1], u_n\}; \\ E(G) &= \{[(u_iu_{i+1}) : 1 \leq i \leq n-1] \cup (u_1u_n) \cup (u_nv_{n-1}) \\ &\quad \cup [(v_iv_{i+1}) : 1 \leq i \leq n-2]\}. \end{aligned}$$

Define $f : V \cup E \to \{1, 2, 3, \ldots, p+q\}$ by

$$\begin{aligned} f(v_i) &= i,\ 1 \leq i \leq n; \\ f(u_1) &= 4n - 2; \\ f(u_i) &= n + (i-2),\ 2 \leq i \leq n-2; \\ f(u_{n-2}) &= 2n - 2; \\ f(v_n) = f(u_n) &= 2n - 1; \\ f(u_iu_{i+1}) &= \begin{cases} 4n - 4 - 2(i-1) & \text{if } i \equiv 1 \text{ mod } 2,\ 1 \leq \text{i} < \text{n}; \\ 4n - 5 - 2(i-2) & \text{if } i \equiv 0 \text{ mod } 2,\ 1 \leq \text{i} < \text{n}; \end{cases} \\ f(u_1u_n) &= 2n - 3; \\ f(v_iv_{i+1}) &= \begin{cases} 4n - 3 - 2(i-1) & \text{if } i \equiv 1 \text{ mod } 2,\ 1 \leq \text{i} < \text{n}; \\ 4n - 6 - 2(i-2) & \text{if } i \equiv 0 \text{ mod } 2,\ 1 \leq \text{i} < \text{n, with } v_n = u_n. \end{cases} \end{aligned}$$

Let $\psi = \{(u_1u_2\ldots u_n), (v_1v_2\ldots v_{n-1}u_nu_1)\}$. Clearly, $\gamma(G) = 2$.

$$\begin{aligned}
f^*[(u_1u_2\ldots u_n)] &= f(u_1) + f(u_n) + \sum_{i=1}^{n-1} f(u_iu_{i+1})\\
&= 4n-2+2n-1+\sum_{i=1,3}^{n-1}\left\{(4n-4)-4\left(\frac{i-1}{2}\right)\right\}\\
&\quad+\sum_{i=2,4}^{n-2}\left\{(4n-5)-4\left(\frac{i-2}{2}\right)\right\}\\
&= \frac{6n^2+3n-4}{2}; \qquad (3)\\
f^*[(v_1v_2\ldots v_{n-1}u_nu_1)] &= f(v_1) + f(u_1) + \sum_{i=1}^{n-1} f(v_iv_{i+1}) + f(u_nu_1)\\
&= 1+4n-2+2n-3+\sum_{i=1,3}^{n-1}\left\{(4n-3)-4\left(\frac{i-1}{2}\right)\right\}\\
&\quad+\sum_{i=2,4}^{n-2}\left\{(4n-6)-4\left(\frac{i-2}{2}\right)\right\}\\
&= \frac{6n^2+3n-4}{2}. \qquad (4)
\end{aligned}$$

From (3) and (4), we conclude that ψ is minimum magic graphoidal cover. Hence, $C_n \odot P_n$, (n-even) is magic graphoidal. For example, the magic graphoidal cover of $C_6 \odot P_6$ is shown in Figure 2.

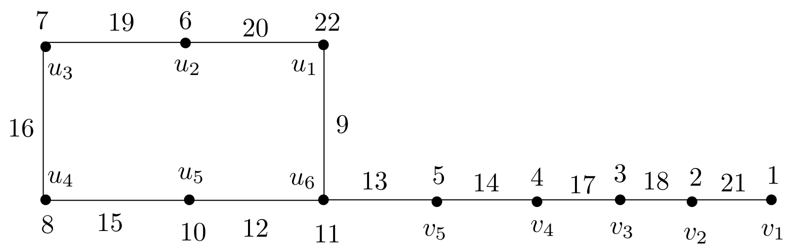


Figure 2. $C_6 \odot P_6$

$\psi = \{(u_1u_2u_3u_4u_5u_6), (v_1v_2v_3v_4v_5u_6u_1)\}, \gamma = 2,\ K = 115.$

Theorem 3.3. Dragan $C_n \odot P_n$, (n-odd) is magic graphoidal cover.

Proof. Let $G = C_n \odot P_n$.

$$\begin{aligned}
\text{Let } V(G) &= \{u_i, v_i : 1 \le i \le n-1, u_n\};\\
E(G) &= \{[(u_iu_{i+1}) : 1 \le i \le n-1] \cup (u_1u_n) \cup (u_nv_{n-1})\\
&\quad \cup [(v_iv_{i+1}) : 1 \le i \le n-2]\}.
\end{aligned}$$

Define $f : V \cup E \to \{1, 2, 3, \ldots, p+q\}$ by

$$
\begin{aligned}
f(v_i) &= i,\ 1 \le i \le n-1;\\
f(u_1) &= 4n-2;\\
f(u_i) &= n+(i-2),\ 2 \le i < n;\\
f(u_n) &= f(vn) = 2n-1;\\
f(u_iu_{i+1}) &= \begin{cases} 4n-4-2(i-1) & \text{if } i \equiv 1 \bmod 2,\ 1 \le \text{i} < \text{n};\\ 4n-5-2(i-2) & \text{if } i \equiv 0 \bmod 2,\ 1 \le \text{i} < \text{n};\end{cases}\\
f(u_1u_n) &= 2n-2;\\
f(v_iv_{i+1}) &= \begin{cases} 4n-3-2(i-1) & \text{if } i \equiv 1 \bmod 2,\ 1 \le \text{i} < \text{n};\\ 4n-6-2(i-2) & \text{if } i \equiv 0 \bmod 2,\ 1 \le \text{i} < \text{n, with } v_n = u_n.\end{cases}
\end{aligned}
$$

Let $\psi = \{(u_1u_2 \ldots u_n), (v_1v_2 \ldots v_{n-1}u_nu_1)\}$. Clearly, $\gamma(G) = 2$.

$$
\begin{aligned}
f^*[(u_1u_2 \ldots u_n)] &= f(u_1) + f(u_n) + \sum_{i=1}^{n-1} f(u_iu_{i+1})\\
&= 4n-2+2n-1+\sum_{i=1,3}^{n-2}\{(4n-4)-2(i-1)\}\\
&\quad + \sum_{i=2,4}^{n-1}\{(4n-5)-2(i-2)\}\\
&= (6n-3)\left(\frac{n+1}{2}\right); \qquad (5)
\end{aligned}
$$

$$
\begin{aligned}
f^*[(v_1v_2 \ldots v_{n-1}u_nu_1)] &= f(u_1) + f(u_n) + f(u_nu_1) + \sum_{i=1}^{n-1}(v_iv_{i+1})\\
&= f(u_1) + f(u_n) + f(u_nu_1)\\
&\quad + \sum_{i=1,3}^{n-2} f(v_iv_{i+1}) + \sum_{i=2,4}^{n-1} f(v_iv_{i+1})\\
&= 1+4n-2+2n-2+\sum_{i=1,3}^{n-2}\{(4n-3)-2(i-1)\}\\
&\quad + \sum_{i=2,4}^{n-1}\{(4n-6)-2(i-2)\}\\
&= (6n-3)\left(\frac{n+1}{2}\right). \qquad (6)
\end{aligned}
$$

From (5) and (6), we conclude that ψ is minimum magic graphoidal cover. Hence, $C_n \odot P_n$, (n-odd) is magic graphoidal. For example, the magic graphoidal cover of $C_5 \odot P_5$ is shown in Figure 3.

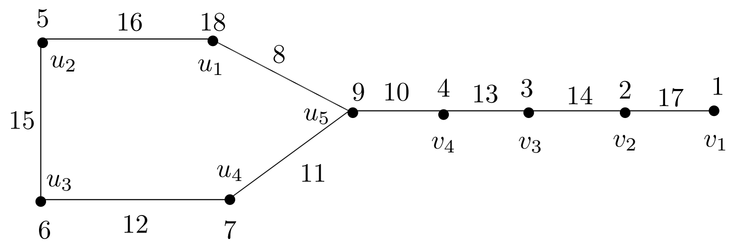


Figure 3. $C_5 \odot P_5$

$\psi = \{(u_1u_2u_3u_4u_5), (v_1v_2v_3v_4v_5u_1)\}, \gamma = 2,\ K = 81.$

Theorem 3.4. Armed grown $C_m\dot{P}_n$ is magic graphoidal.

Proof. Let $G = C_m \odot P_n$.

$$\begin{aligned} \text{Let } V(G) &= \{v_i : 1 \le i \le m, u_{ij} : 1 \le i \le m, 1 \le j \le n\}; \\ E(G) &= \{[(v_iv_{i+1}) : 1 \le i \le m-1] \cup (v_1v_m) \\ &\quad \cup (u_{ij}u_{ij+1}) : 1 \le i \le m, 1 \le j \le n-1\}. \end{aligned}$$

Let $v_1 = u_{m1}, v_i = u_{(i-1)1}, 2 \le i \le m$.

Let $\psi = \{(v_iv_{i+1}u_{i2}u_{i3}\dots u_{in}), 1 \le i \le m-1 \cup (v_mv_1u_{m2}u_{m3}\dots u_{mn})\}$.

Case (i) n is even.

Define $f : V \cup E \to \{1, 2, \dots, p+q\}$ by

$$\begin{aligned} f(v_i) &= i,\ 1 \le i \le m; \\ f(v_iv_{i+1}) &= 2m+1-i,\ 1 \le i \le m; \\ f(v_1v_m) &= m+1; \\ f(u_{ij}u_{ij+1}) &= (j+1)m+i,\ j \equiv 1 \text{ mod } 2,\ 1 \le \mathrm{i} \le \mathrm{m}, 1 \le \mathrm{j} \le \mathrm{n}-1; \\ f(u_{ij}u_{ij+1}) &= (j+2)m+1-i,\ j \equiv 0 \text{ mod } 2,\ 1 \le \mathrm{i} \le \mathrm{m},\ 1 \le \mathrm{j} \le \mathrm{n}-2; \\ f(u_{in}) &= (n+2)m+1-i,\ 1 \le i \le m; \\ f^*[(v_mv_1u_{m2}u_{m3}\dots u_{mn})] &= f(v_m) + f(v_{mn}) + f(v_mv_1) + f(v_1u_{m2}) \\ &\quad + f(u_{m2}u_{m3}) + \dots + f(u_{mn}u_{mn}) \\ &= m + (n+2)m + 1 - m + m + 1 \\ &\quad + \sum_{j=1,3}^{n-1}\{(j+1)m+m\} + \sum_{j=2,4}^{n-2}\{(j+2)m+1-m\} \\ &= nm + 2m + 1 + \frac{n}{2} + nm\left(\frac{n}{2}+1\right). \end{aligned} \tag{7}$$

For $1 \le i \le m-1$,

$$\begin{aligned} f^*[(v_iv_{i+1}u_{i2}u_{i3}\dots u_{in})] &= f(v_i) + f(u_{in}) + f(v_iv_{i+1}) + f(v_{i+1} = u_{i1}, u_{i2}) \\ &\quad + f(u_{i2}u_{i3}) + \dots + f(u_{in}u_{in+1}) \\ &= i + (n+2)m + 1 - i + 2m + 1 - i \\ &\quad + \sum_{j=1,3}^{n-1}\{(j+1)m+i\} + \sum_{j=2,4}^{n-2}\{(j+2)m+1-i\} \end{aligned}$$

$$= nm + 2m + 1 + \frac{n}{2} + nm\left(\frac{n}{2} + 1\right). \tag{8}$$

From (7) and (8), we conclude that ψ is minimum magic graphoidal cover. Hence, $C_m \odot P_n$ (n-even) is magic graphoidal. For example, the magic graphoidal cover of $C_4 \odot P_4$ is shown in Figure 4.

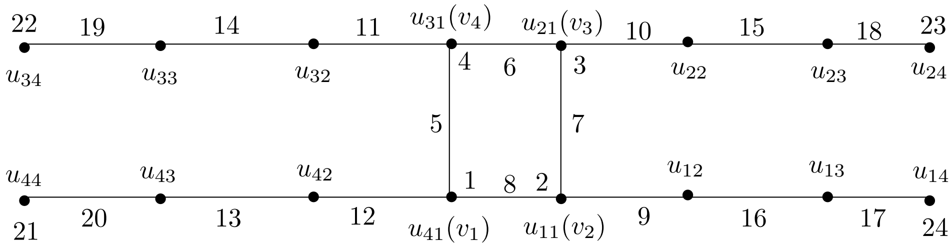


Figure 4. $C_4 \odot P_4$

$\psi = \{(v_1v_2u_{12}u_{13}u_{14}), (v_2v_3u_{22}u_{23}u_{24}), (v_3v_4u_{32}u_{33}u_{34}), (v_4v_1u_{42}u_{43}u_{44})\}, \gamma = 4,\ K = 75$

Case (ii) n is odd.

$$\begin{aligned}
f(v_i) &= i,\ 1 \le i \le m;\\
f(v_iv_{i+1}) &= m + i,\ 1 \le i \le m;\\
f(v_1v_m) &= 2m;\\
f(u_{i1}u_{i2}) &= 4m + 1 - 2i,\ 1 \le i \le m;\\
f(u_{ij}u_{ij+1}) &= (j+2)m + i - 1,\ j \equiv 0 \bmod 2,\ 1 \le \mathrm{i} \le \mathrm{m},\ 2 \le \mathrm{j} \le \mathrm{n} - 1;\\
f(u_{ij}u_{ij+1}) &= (j+3)m - i,\ j \equiv 1 \bmod 2,\ 1 \le \mathrm{i} \le \mathrm{m},\ 3 \le \mathrm{j} \le \mathrm{n} - 2;\\
f(u_{in}) &= (n+3)m - i,\ 1 \le i \le m;\\
f^*[(v_mv_1u_{m2}u_{m3}\dots u_{mn})] &= f(v_m) + f(u_{mn}) + f(v_mv_1) + f(u_{m1}u_{m2})\\
&\quad + f(u_{m2}u_{m3}) + \cdots + f(u_{mn}u_{mn})\\
&= m + (n+3)m - m + 2m + 4m + 1 - 2m\\
&\quad + \sum_{j=2,4}^{n-1}\{(j+2)m + m - 1\} + \sum_{j=3,5}^{n-2}\{(j+3)m - m\}\\
&= nm + 12m + 1 - \left(\frac{n-1}{2}\right) + 2m(6 + 8 + \cdots + n + 1).
\end{aligned} \tag{9}$$

For $1 \le i \le m - 1$,

$$\begin{aligned}
f^*[(v_iv_{i+1}u_{i2}u_{i3}\dots u_{in})] &= f(v_i) + f(u_{in}) + f(v_iv_{i+1}) + f(u_{i1}u_{i2}) + f(u_{i2}u_{i3})\\
&\quad + \cdots + f(u_{in}u_{in})\\
&= i + (n+3)m - i + m + i + 4m + 1 - 2i\\
&\quad + \sum_{j=2,4}^{n-1}\{(j+2)m + m - 1\} + \sum_{j=3,5}^{n-2}\{(j+3)m - m\}\\
&= nm + 12m + 1 - \left(\frac{n-1}{2}\right) + 2m(6 + 8 + \cdots + n + 1).
\end{aligned} \tag{10}$$

From (9) and (10), we conclude that ψ is minimum magic graphoidal cover. Hence, $C_m \odot P_n$ (n-odd) is magic graphoidal. For example, the magic graphoidal cover of $C_3 \odot P_5$ is shown in Figure 5.

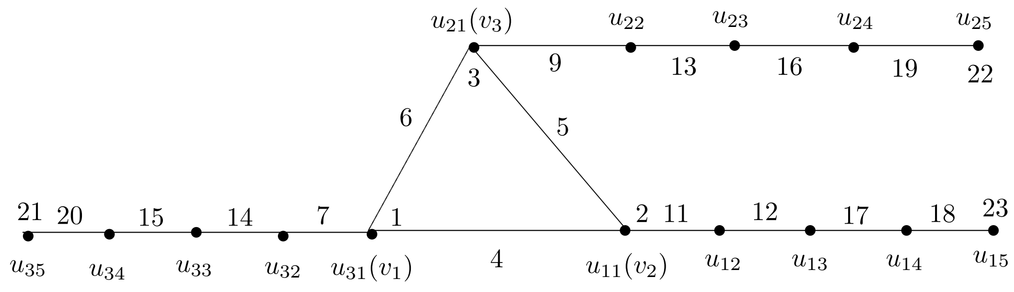


Figure 5. $C_3\dot{P}_5$

$\psi = \{(v_1v_2u_{22}u_{23}u_{24}u_{25}), (v_2v_3u_{32}u_{33}u_{34}u_{35}), (v_3v_1u_{12}u_{13}u_{14}u_{15})\}, \gamma = 3,\ K = 86.$

References

[1] B. D. Acarya and E. Sampath Kumar, Graphoidal covers and Graphoidal covering number of a Graph, Indian J. pure appl. Math., No. 10, **18**(1987), 882-890.

[2] J. A. Gallian, A Dynamic Survey of graph labeling, The Electronic journal of Coimbinotorics, **6**(2001), #DS6.

[3] F. Harary, Graph Theory, Addition - Wesley publishing company Inc, USA, 1969.

[4] A. Nellai Murugan and A. Nagarajan, Magic Graphoidal on Trees (Communicated).

[5] A. Nellai Murugan and A. Nagarajan, Magic Graphoidal on Join of two graphs (Communicated).

[6] A. Nellai Murugan and A. Nagarajan, Magic Graphoidal on Path related graphs (Communicated).

[7] A. Nellai Murugan and A. Nagarajan, On Magic Graphoidal graphs (Communicated).

[8] A. Nellai Murugan and A. Nagarajan, Magic Graphoidal on product graphs (Communicated).

[9] A. Nellai Murugan and A. Nagarajan, Magic Graphoidal on special types of graphs (Communicated).

[10] A. Nellai Murugan and A. Nagarajan, Magic Graphoidal on special class of graphs (Communicated).

[11] C. Packiam and S. Arumugam, On the Graphoidal covering number of a Graph, Indian J. pure. appl. Math., **20**(1989), 330-333.

Scientia Magna
Vol. 7 (2011), No. 1, 107-113

Minimal translation lightlike hypersurfaces of semi-euclidean spaces

D. Sağlam[†], E. Soytürk[‡] and A. Sabuncuoğlu[♯]

Afyon Kocatepe University, Faculty of Science-Literature,
Department of Mathematics, ANS Campus 03200, Afyonkarahisar, Turkey
E-mail: dryilmaz@aku.edu.tr

Abstract All lightlike hypersurfaces of a semi-Euclidean space that can locally be written as the sum of functions of one variable are parametrized and showed that they are hyperplanes. Also the only minimal translation lightlike hypersurfaces (zero mean curvature in all points) are hyperplanes.

Keywords Minimal, translation hypersurfaces, lightlike surfaces.

§1. Introduction and preliminaries

Semi-Riemannian geometry is the study of smooth manifolds with non-degenerate metric. The special cases are Riemannian geometry, with a positive definite metric and Lorentz geometry, the mathematical theory used in general relativity. Moreover, for any semi-Euclidean manifold there is a natural existence of null (lightlike) subspaces. The growing importance of lightlike hypersurfaces in mathematical physics, in particular their extensive use in relativity and very limited information available on the general theory of lightlike submanifolds, has attracted interest of many mathematicians [2].

Minimal surfaces are one of the most important surface classes in differential geometry. In previous studies, minimal surfaces have been studied in 3-dimensional and in higher dimensional Euclidean (or semi Euclidean) space by a number of differential geometricians. For instance, the minimal surfaces of revolution, ruled, translation and homothetical surfaces in the $\mathbb{R}^3_1$ are completely determined in [1,4,5,8,9,10,11]. Moreover the minimal surfaces of translation of a higher dimensional Euclidean space are obtained in [13] and of a semi-Euclidean space are investigated in [12]. In particular Lopez [6] proved that the only minimal translation surfaces in hyperbolic space are totally geodesic planes.

A hypersurface M^n in $(n+1)$-dimensional Euclidean (or semi-Euclidean) space determined by the transformation

$$\varphi = (\, x_1,\, x_2, \ldots, x_m, F\,)$$

is called translation, if the function F is the sum of the smooth functions $f_1,\, f_2, \ldots, f_m$ of one variable such that

$$F(x_1,\, x_2, \ldots, x_m) = f_1(x_1) + f_2(x_2) + \cdots + f_m(x_m)$$ [12].

We assume that f_i vanishes nowhere $(i = 1, 2, \ldots, n)$. Otherwise M^n is a hyperplane.

In this paper, we determine parametrization of the translation lightlike (degenerate) hypersurfaces of the semi-Euclidean space and show that they are hyperplanes. Also the translation lightlike (degenerate) hypersurfaces are minimal.

§2. Lightlike hypersurfaces of semi-riemannian manifolds

Let M be a hypersurface of a $(m+2)$-dimensional semi-Riemannian manifold $\overline{M}$ of index $q \in \{1, \ldots, m+1\}$, $m > 0$. Let $\overline{g}$ be the semi-Riemannian metric on $\overline{M}$. $\overline{g}$ induces on M a symmetric tensor field g of type $(0, 2)$.

The radical (null) space of T_uM is

$$Rad\, T_uM = \{\, \xi_u \in T_uM : \, g_u(\xi_u, X_u) = 0, \, \forall X_u \in T_uM\}.$$

Since

$$T_uM^{\perp} = \{\, V_u \in T_u\overline{M} : \, \overline{g}_u(V_u, W_u) = 0, \, \forall W_u \in T_uM\},$$

we have

$$Rad\, T_uM = T_uM \, \cap \, T_uM^{\perp}.$$

In this section, which follows almost entirely [2].

Definition 2.1. Let M be a hypersurface of an $(m+2)$-dimensional semi-Riemannian manifold $\overline{M}$, $m > 0$. If $Rad\, T_uM \neq \{0\}$ for any $u \in M$, M is called a **lightlike (degenerate) hypersurface of** $\overline{M}$.

If M is a lightlike hypersurface of $\overline{M}$, $T_uM^{\perp}$ is a one-dimensional vector subspace of the tangent space. Each m-dimensional subspace in T_uM that does not contain the subspace $T_uM^{\perp}$ is orthogonal to $T_uM^{\perp}$ and called a **screen space** at point u. The vector bundle that is constituted by choosing a screen space each point of M is said to be a **screen distribution** on M, denoted by $S(TM)$. Thus we have

$$TM = S(TM) \perp TM^{\perp}.$$

$T\overline{M}_{\diagup M}$ is a vector bundle that has M as base space and assigns $T_u\overline{M}$ to each point u of M. g_u is non-degenerate on $S(T_uM)$. If a subspace is non-degenerate, its complementary orthogonal subspace is also non-degenerate and is uniquely determined. Thus, the vector bundle that is determined by the complementary orthogonal subspace is called the **orthogonal complementary vector bundle** to $S(TM)$ in $T\overline{M}_{\diagup M}$, denoted by $S(TM)^{\perp}$. Also we have

$$T\overline{M}_{\diagup M} = S(TM) \perp S(TM)^{\perp}.$$

Theorem 2.2. Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then there exists a unique vector bundle $tr(TM)$ of rank 1 over M, such that for any non-zero $\xi \in \Gamma(TM^{\perp})$ on a coordinate neighbourhood $\mathcal{U} \in M$, there exists a unique section N of $tr(TM)$ on $\mathcal{U}$ with the following properties:

$$\overline{g}(N, \xi) = 1,$$

and

$$\overline{g}(N,N)=\overline{g}(N,W)=0,\ \forall W\in\Gamma(S(TM)_{\diagup U}).$$

The space that is the union of subspaces spanned by the vector N_u at each point $u\in M$ is a lightlike vector bundle and is called the **lightlike transversal vector bundle** of M with respect to $S(TM)$. It is denoted by $tr(TM)$. $tr(TM)_u$ is the subspace spanned by the vector N_u. Hence we have

$$T\overline{M}_{\diagup M}=S(TM)\perp(TM^{\perp}\oplus tr(TM))=TM\oplus tr(TM).$$

Definition 2.3.[1] Let $(M,\,g,\,S(TM))$ be a lightlike hypersurface of a $(m+2)-$ dimensional semi-Riemannian manifold $(\overline{M},\,\overline{g})$ and $\overline{\nabla}$ be the Levi-Civita connection on $\overline{M}$ with respect to $\overline{g}$. If $X,Y\in\Gamma(TM)$, then $\overline{\nabla}_XY\in\Gamma(T\overline{M})$. Using the decomposition $T\overline{M}_{\diagup M}=TM\oplus tr(TM)$, we obtain the formulas

$$\overline{\nabla}_XY=\nabla_XY+h(X,Y) \tag{2.1}$$

and

$$\overline{\nabla}_XV=-A_VX+\nabla^t_XV, \tag{2.2}$$

for any $X,Y\in\Gamma(TM)$ and $V\in\Gamma(tr(TM))$, where ∇_XY and A_VX belong to $\Gamma(TM)$ while $h(X,Y)$ and ∇^t_XV belong to $\Gamma(tr(TM))$. It is easy to check that ∇ is a torsion-free linear form on M, h is a symmetric $\mathcal{F}(M)$-bilinear form on $\Gamma(TM)$, which has range $\Gamma(tr(TM))$ and A_V is a $\mathcal{F}(M)$-linear operator on $\Gamma(TM)$ and ∇^t is a linear connection on the lightlike transversal vector bundle $tr(TM)$. We call ∇ and ∇^t the **induced connections** on M and $tr(TM)$ respectively. Consistent with the classical of Riemannian hypersurfaces we call h and A_V the **second fundamental form** and the **shape operator** respectively, of the lightlike immersion of M in $\overline{M}$. Also, we name (2.1) and (2.2) the **Gauss** and **Weingarten formulae**, respectively.

Definition 2.4. Let $(M,\,g,\,S(TM))$ be a lightlike hypersurface of a $(m+2)$-dimensional semi-Riemannian manifold $(\overline{M},\,\overline{g})$. Next, if P denotes the projection morphism of TM on $S(TM)$ with respect to the decomposition $TM=S(TM)\perp TM^{\perp}$ we obtain

$$\nabla_XPY=\overset{*}{\nabla}_XPY+\overset{*}{h}(X,PY) \tag{2.3}$$

and

$$\overline{\nabla}_XU=-\overset{*}{A}_UX+\overset{*}{\nabla}{}^t_XU, \tag{2.4}$$

where $\overset{*}{\nabla}_XPY$ and $\overset{*}{A}_UX$ belong to $\Gamma(S(TM))$ while $\overset{*}{h}(X,PY)$ and $\overset{*}{\nabla}{}^t_XU$ belong to $\Gamma(TM^{\perp})$. It follows that $\overset{*}{\nabla}$ and $\overset{*}{\nabla}{}^t$ are linear connections on vector bundles $S(TM)$ and $TM^{\perp}$ respectively, $\overset{*}{h}$ is a $\Gamma(TM^{\perp})$-valued $\mathcal{F}(M)$-bilinear form on $\Gamma(TM)\times\Gamma(S(TM))$ and $\overset{*}{A}_U$ is $\Gamma(S(TM))$-valued $\mathcal{F}(M)$-linear operator on $\Gamma(TM)$. We call $\overset{*}{h}$ and $\overset{*}{A}_U$ the **second fundamental form** and the **shape operator** of the screen distribution $S(TM)$, respectively. Also, equations (2.3) and (2.4) are the **Gauss** and **Weingarten equations** for the screen distribution $S(TM)$.

Proposition 2.5. On any lightlike Monge hypersurfaces M of R^{m+2}_q, the shape operators A_N and $\overset{*}{A}_\xi$ of M and of the naturel screen distribution are related by

$$A_{\overline{N}}=\frac{1}{2}\overset{*}{A}_\xi. \tag{2.5}$$

Definition 2.6. Let ξ be a normal null section. The trace of $-\overset{*}{A}_\xi$ is called the lightlike mean curvature H_ξ on M associated with ξ. Then

$$H_\xi = trace(-\overset{*}{A}_\xi) = -trace(\overset{*}{A}_\xi).$$

One of the good properties of the lightlike mean curvature is that it does not depend on the screen distribution chosen, but only of the local normal null section ξ [3].

§3. Mimimal translation lightlike (degenerate) hypersurfaces of semi-euclidean spaces

Let $y_0, y_1, \ldots, y_{m+1}$ be coordinate functions in $\mathbb{R}^{m+2}$ and $x_1, x_2, \ldots, x_{m+1}$ be coordinate functions in $\mathbb{R}^{m+1}$. If the coordinate axes are embedded in $\mathbb{R}^{m+2}$, then we have

$$y_k(0, a_1, a_2, \ldots, a_{m+1}) = x_k(a_1, a_2, \ldots, a_{m+1}),$$

where $1 \leq k \leq m+1$. Hence $y_k \diagup_{\{0\}\times\mathbb{R}^{m+1}} = x_k$.

Let M be a lightlike Monge hypersurface determined by the function $\varphi = (F, x_1, x_2, \ldots, x_{m+1})$, where F a smooth function $F : D \to R$ and D is an open subset of $\mathbb{R}^{m+1}$. We have

$$\partial_\alpha \circ \varphi = F_{x_\alpha} \frac{\partial}{\partial y_0} \circ \varphi + \frac{\partial}{\partial y_\alpha} \circ \varphi, \qquad 1 \leq \alpha \leq m+1$$

where $\partial_1, \partial_2, \ldots, \partial_{m+1}$ are coordinate frame fields on M. Since $g(\partial_\alpha \circ \varphi, \xi) = 0$ for each α, we have

$$\xi \circ \varphi = \frac{\partial}{\partial y_0} \circ \varphi - \sum_{j=1}^{q-1} F_{x_j} \frac{\partial}{\partial y_j} \circ \varphi + \sum_{\alpha=q}^{m+1} F_{x_\alpha} \frac{\partial}{\partial y_\alpha} \circ \varphi,$$

where ξ is the normal vector field on M.

Vector field $\overline{N}$ determined by equation

$$\overline{N} = -\frac{\partial}{\partial y_0} + \frac{1}{2}\xi \tag{3.1}$$

satisfies the conditions of Theorem 2.2. and spans vector bundle $tr(TM)$. $\overline{N}$ defined here, is named the **natural lightlike transversal vector bundle** of M [2].

Theorem 3.1.[2] The hypersurface $\varphi = (F, x_1, x_2, \ldots, x_{m+1})$ is lightlike if and only if

$$1 + \sum_{j=1}^{q-1} F_{x_j}^2 = \sum_{\alpha=q}^{m+1} F_{x^\alpha}^2. \tag{3.2}$$

Theorem 3.2.[7] Given an open subset $D \subset \mathbb{R}^{m+1}$ and a smooth transformation $F : D \to R$. Let M be the lightlike Monge hypersurface determined by $\varphi = (F, x_1, x_2, \ldots, x_{m+1})$. The matrix that corresponds to the shape operator of the lightlike hypersurface M in the semi-Euclidean space $\mathbb{R}_q^{m+2}$ is

$$A_{\overline{N}} = \frac{1}{2}\begin{bmatrix} F_{x_1x_1} & \cdots & F_{x_1x_{q-1}} & F_{x_1x_q} & F_{x_1x_{q+1}} & \cdots & F_{x_1x_{m+1}} \\ F_{x_2x_1} & \cdots & F_{x_2x_{q-1}} & F_{x_2x_q} & F_{x_2x_{q+1}} & \cdots & F_{x_2x_{m+1}} \\ F_{x_3x_1} & \cdots & F_{x_3x_{q-1}} & F_{x_3x_q} & F_{x_3x_{q+1}} & \cdots & F_{x_3x_{m+1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ F_{x_{q-1}x_1} & \cdots & F_{x_{q-1}x_{q-1}} & F_{x_{q-1}x_q} & F_{x_{q-1}x_{q+1}} & \cdots & F_{x_{q-1}x_{m+1}} \\ -F_{x_qx_1} & \cdots & -F_{x_qx_{q-1}} & -F_{x_qx_q} & -F_{x_qx_{q+1}} & \cdots & -F_{x_qx_{m+1}} \\ -F_{x_{q+1}x_1} & \cdots & -F_{x_{q+1}x_{q-1}} & -F_{x_{q+1}x_q} & -F_{x_{q+1}x_{q+1}} & \cdots & -F_{x_{q+1}x_{m+1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -F_{x_{m+1}x_1} & \cdots & -F_{x_{m+1}x_{q-1}} & -F_{x_{m+1}x_q} & -F_{x_{m+1}x_{q+1}} & \cdots & -F_{x_{m+1}x_{m+1}} \end{bmatrix}.$$

Proposition 3.3.[7] In the semi-Euclidean space $\mathbb{R}_q^{m+2}$ the lightlike mean curvature of the lightlike hypersurface represented by $\varphi = (F, x_1, x_2, \ldots, x_{m+1})$ respect to normal section ξ is determined by the following equation

$$H_\xi = -\sum_{j=1}^{q-1} F_{x_jx_j} + \sum_{\alpha=q}^{m+1} F_{x_\alpha x_\alpha}.$$

Corollary 3.4.[7] In the semi-Euclidean space $\mathbb{R}_q^{m+2}$, the lightlike hypersurface M determined by the transformation $\varphi = (F,\, x_1, x_2, \ldots, x_{m+1})$ is minimal if and only if

$$\sum_{j=1}^{q-1} F_{x_jx_j} = \sum_{\alpha=q}^{m+1} F_{x_\alpha x_\alpha}.$$

Remark 3.5. Let M be a translation hypersurface of $\mathbb{R}_q^{m+2}$. Then M can locally always be seen as the graph of a function $F : \mathbb{R}^{m+1} \to \mathbb{R}$. In what follows, we will assume that F is a function of the coordinates $x_1, x_2, \ldots, x_{m+1}$. This can easily be achieved possibly by rearranging the coordinates of $\mathbb{R}_q^{m+2}$. So M is locally given by

$$x_0 = F(x_1, x_2, \ldots, x_{m+1}) = f_1(x_1) + f_2(x_2) + \cdots + f_{m+1}(x_{m+1}).$$

Let $\varepsilon_j = \overline{g}(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_j}) = \begin{cases} -1, & j = 1, ...q-1 \\ +1, & j = q, ...m+1 \end{cases}.$

Theorem 3.6. In the semi Euclidean space $\mathbb{R}_q^{m+2}$, the $(m+1)-$ dimensional translation hypersurface given by

$$\varphi = (\sum_{j=1}^{m+1} f_j,\, x_1, x_2, \ldots, x_{m+1})$$

is lightlike if and only if

$$\sum_{j=1}^{m+1} \varepsilon_j (f_j^{'})^2 = 1 \tag{3.3}$$

($F \neq 0$ in any point).

Proof. Substitute $F = \sum_{j=1}^{m+1} f_j$ in the equation (3.2).

Theorem 3.6. In the semi-Euclidean space $\mathbb{R}_q^{m+2}$, if the $(m+1)-$ dimensional translation hypersurface given by

$$\varphi = (\sum_{j=1}^{m+1} f_j,\, x_1, x_2, \ldots, x_{m+1})$$

is lightlike, then

$$\varphi = (\sum_{j=1}^{m+1} a_j x_j + b,\, x_1, x_2, \ldots, x_{m+1}) \text{ with } \sum_{j=1}^{m+1} \varepsilon_j a_j^2 = 1,$$

on the corresponding domain where f_j is not constant and a_j, b is some constant.

Proof. Derivative of the equation (3.3) with respect to x_j for $j = 1, ..., m+1$:

$$f_j^{'} f_j^{''} = 0.$$

Since $f_j^{'} \neq 0$ on an interval, then

$$f_j^{''} = 0.$$

We twice integrate this equation

$$f_j = a_j x_j + b,$$

where $\sum_{j=1}^{m+1} \varepsilon_j a_j^2 = 1$, i.e. $(0, a_1, a_2, ...a_{m+1}) \in \mathbb{S}_q^{m+1}$.

Corollary 3.7. In the semi-Euclidean space $\mathbb{R}_q^{m+2}$, every translation lightlike hypersurface is minimal.

Remark 3.8. The minimality condition of translation lightlike hypersurface M given by $\varphi = (\sum_{j=1}^{m+1} f_j,\, x_1, x_2, \ldots, x_{m+1})$ can be developed as

$$\sum_{j=1}^{m+1} \varepsilon_j f_j^{''} = 0.$$

Clearly, every translation lightlike hypersurface is minimal.

References

[1] F. Dillen, L. Verstraelen and G. Zafindratafa, A generalization of the translation surfaces of Scherk, Diff. Geom. in honor of Radu Rosca (K.U.L.), 1991, 107-109.

[2] K. L. Duggal and A. Bejancu, Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications, Kluwer Academic Publishers, 1996.

[3] K. L. Duggal and A. Gimenez, Lightlike hypersurfaces of Lorentzian Manifolds and with distinguished screen, Journal of Geometry and Physics, **55**(2005), 107-122.

[4] L. Jiu, Translation surfaces with constant mean curvature in 3-dimensional spaces, J. Geom., **64**(1999), 141-149.

[5] O. Kobayashi, Maximal surfaces in the 3-dimensional Minkowski space L^3, Tokyo J. Math., **62**(1983), 7-309.

[6] R. Lopez, Minimal translation surfaces in hyperbolic space, To appear in Contributions to Algebra and Geometry.

[7] D. Sağlam and A. Sabuncuoğlu, Minimal homothetical lightlike hypersurfaces of semi-Euclidean spaces, To appear in Kuwait J. Science & Engineering.

[8] I. Van de Woestyne, Minimal surfaces of the 3-dimensional Minkowski space, Geometry and Topology of Submanifolds II. World Scientific Publ., 1990, 344-369.

[9] I. Van de Woestyne, Minimal S^1-symmetric surfaces, Geometry and Topology of Submanifolds III, World Scientific Publ., 1991, 294-303.

[10] I. Van de Woestyne, A new characterization of the helicoids, Geometry and Topology of Submanifolds V. World Scientific Publ., 1993, 267-273.

[11] I. Van de Woestyne, Minimal homothetical hypersurfaces of a semi-Euclidean space, Results in Mathematics, Birkhäuser Verlag, **27**(1995), 333-342.

[12] I. Van de Woestyne, L. Verstraelen, J. Walrave and Ş. Yaprak, Minimal surfaces of translation in $\mathbb{R}^n_1$, Geometry and Topology of Submanifolds, **6**(1991), 217-225.

[13] L. Verstraelen, J. Walrave and Ş. Yaprak, The minimal translation surfaces in Euclidean space, Soochow J. of Math., **201**(1994), 77-82.

Scientia Magna
Vol. 7 (2011), No. 1, 114-120

The generalized f-derivations of lattices

Sureeporn Harmaitree[†] and Utsanee Leerawat[‡]

Department of Mathematics, Faculty of Science,
Kasetsart University, Bangkok, Thailand
E-mail: fsciutl@ku.ac.th

Abstract In this paper, we study the notion of a generalized f-derivation for a lattice and investigate some related properties. We give some necessary and sufficient conditions under which a generalized f-derivation is an order-preserving for lattice with a greatest element, modular lattice, and distributive lattice.

Keywords Generalized f-derivations, lattices.

§1. Introduction

The concept of derivation for BCI-algebra was introduced by Y. B. Jun and X. L. Xin [3]. Further, in 2009, C. Prabprayak and U. Leerawat [7] also studied the derivation of BCC-algebra. In 2005, J. Zhan and Y. L. Liut [7] introduced the concept of a f-derivation for BCI-algebra and obtained some related properties. In 2008, L. X. Xin, T. Y. Li and J. H. Lu [6] studied derivation of lattice and investigated some of its properties. In 2010, N. O. Alshehri introduced the concept of a generalized derivation and investigated some of its properties. In 2011, S. Harmaitree and U. Leerawat studied the f-derivation of lattice and investigated some of its properties. The purpose of this paper, we applied the notion of a generalized f-derivation for a lattice and investigate some related properties.

§2. Preliminaries

We first recall some definitions and results which are essential in the development of this paper.

Definition 2.1.[5] An (algebraic) lattice $(L, \wedge, \vee)$ is a nonempty set L with two binary operation "$\wedge$"and "$\vee$"(read "*join*"and "*meet*", respectively) on L which satisfy the following condition for all $x, y, z \in L$:

(i) $x \wedge x = x$, $\quad x \vee x = x$;

(ii) $x \wedge y = y \wedge x$, $\quad x \vee y = y \vee x$;

(iii) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, $\quad x \vee (y \vee z) = (x \vee y) \vee z$;

(iv) $x = x \wedge (x \vee y)$, $\quad x = x \vee (x \wedge y)$.

We often abbreviate L is a lattice to $(L, \wedge, \vee)$ is an algebraic lattice.

Definition 2.2.[5] A poset $(L, \leq)$ is a *lattice ordered* if and only if for every pair x, y of elements of L both the $\sup\{x, y\}$ and the $\inf\{x, y\}$ exist.

Theorem 2.3.[5] In a lattice ordered set $(L, \leq)$ the following statements are equivalent for all $x, y \in L$:

(a) $x \leq y$; (b) $\sup\{x, y\} = y$; (c) $\inf\{x, y\} = x$.

Definition 2.4.[8] Let L be a lattice. A binary relation "$\leq$"is defined by $x \leq y$ if and only if $x \wedge y = x$ and $x \vee y = y$.

Lemma 2.5. Let L be a lattice. Then $x \wedge y = x$ if and only if $x \vee y = y$ for all $x, y \in L$.

Proof. Let $x, y \in L$ and assume $x \wedge y = x$. Then $x \vee y = (x \wedge y) \vee y = y$. Conversely, let $x \vee y = y$. So $x \wedge y = x \wedge (x \vee y) = x$.

Corollary 2.6. Let L be a lattice. Then $x \leq y$ if and only if either $x \wedge y = x$ or $x \vee y = y$.

Lemma 2.7.[8] Let L be a lattice. Define the binary relation "$\leq$"as Definition 2.3. Then $(L, \leq)$ is a poset and for any $x, y \in L$, $x \wedge y$ is the $\inf\{x, y\}$ and $x \vee y$ the $\sup\{x, y\}$.

Theorem 2.8.[5] Let L be a lattice. If we define $x \leq y$ if and only if $x \wedge y = x$ then $(L, \leq)$ is a lattice ordered set.

Definition 2.9.[5] If a lattice L contains a least (greatest) element with respect to $\leq$ then this uniquely determined element is called the zero element (one element), denoted by 0 (by 1).

Lemma 2.10.[5] Let L be a lattice. If $y \leq z$, then $x \wedge y \leq x \wedge z$ and $x \vee y \leq x \vee z$ for all $x, y, z \in L$.

Definition 2.11.[5] A nonempty subset S of a lattice L is called *sublattice* of L if S is a lattice with respect to the restriction of $\wedge$ and $\vee$ of L onto S.

Definition 2.12.[5] A lattice L is called *modular* if for any $x, y, z \in L$ if $x \leq z$, then $x \vee (y \wedge z) = (x \vee y) \wedge z$.

Definition 2.13.[5] A lattice L is called *distributive* if either of the following condition hold for all x, y, z in L: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ or $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Corollary 2.14.[5] Every distributive lattice is a modular lattice.

Definition 2.15.[5] Let $f : L \to M$ be a function from a lattice L to a lattice M.

(i) f is called a join-homomorphism if $f(x \vee y) = f(x) \vee f(y)$ for all $x, y \in L$.

(ii) f is called a meet-homomorphism if $f(x \wedge y) = f(x) \wedge f(y)$ for all $x, y \in L$.

(iii) f is called a lattice-homomorphism if f are both a join-homomorphism and a meet-homomorphism.

(iv) f is called an order-preserving if $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in L$.

Lemma 2.16.[5] Let $f : L \to M$ be a function from a lattice L to a lattice M. If f is a join-homomorphism (or a meet-homomorphism , or a lattice-homomorphism), then f is an order-preserving.

Definition 2.17.[5] An ideal is a nonempty subset I of a lattice L with the properties:

(i) if $x \leq y$ and $y \in I$, then $x \in I$ for all $x, y \in L$;

(ii) if $x, y \in I$, then $x \vee y \in I$.

Definition 2.18.[3] Let L be a lattice and $f : L \to L$ be a function. A function $d : L \to L$ is called a f-derivation on L if for any $x, y \in L$, $d(x \wedge y) = (dx \wedge f(y)) \vee (f(x) \wedge dy)$.

Proposition 2.19.[3] Let L be a lattice and d be a f-derivation on L where $f : L \to L$ is a function. Then the following conditions hold : for any element $x, y \in L$,

(1) $dx \leq f(x)$;

(2) $dx \wedge dy \leq d(x \wedge y) \leq dx \vee dy$;

(3) If L has a least element 0, then $f(0) = 0$ implies $d\,0 = 0$.

§3. The generalized f-derivations of lattices

The following definitions introduces the notion of a generalized f-derivation for lattices.

Definition 3.1. Let L be a lattice and $f : L \to L$ be a function. A function $D : L \to L$ is called a generalized f-derivation on L if there exists a f-derivation $d : L \to L$ such that $D(x \wedge y) = (D(x) \wedge f(y)) \vee (f(x) \wedge d(y))$ for all $x, y \in L$.

We often abbreviate $d(x)$ to dx and Dx to $D(x)$.

Remark. If $D = d$, then D is a f-derivation.

Now we give some examples and show some properties for a generalized f-derivation in lattices.

Example 3.2. Consider the lattice given by the following diagram of Fig. 1.

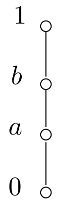


Fig. 1

Define, respectively, a function d, a function D and a function f by

$$dx = \begin{cases} 0, & \text{if } \mathrm{x = 1;} \\ b, & \text{if } \mathrm{x = b;} \\ a, & \text{if } \mathrm{x = 0, a.} \end{cases} \qquad Dx = \begin{cases} a, & \text{if } \mathrm{x = 0, a, 1;} \\ b, & \text{if } \mathrm{x = b.} \end{cases} \qquad f(x) = \begin{cases} a, & \text{if } \mathrm{x = 0, a;} \\ b, & \text{if } \mathrm{x = 1, b.} \end{cases}$$

Then it is easily checked that d is a f-derivation and D is a generalized f-derivation.

Example 3.3. Consider the lattice as show in Fig. 2.

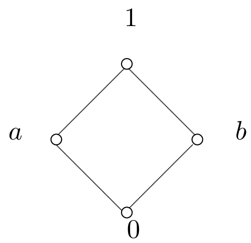


Fig. 2

Define, respectively, a function d, a function D and a function f by

$$dx = \begin{cases} 0, & \text{if } \mathrm{x = 0, b, 1;} \\ a, & \text{if } \mathrm{x = a.} \end{cases} \qquad Dx = \begin{cases} 0, & \text{if } \mathrm{x = 0, b;} \\ a, & \text{if } \mathrm{x = a, 1.} \end{cases} \qquad f(x) = \begin{cases} x & \text{if } \mathrm{x = 1, a;} \\ b & \text{if } \mathrm{x = 0, b.} \end{cases}$$

Then it is easily checked that d is a f-derivation and D is a generalized f-derivation.

Proposition 3.4. Let L be a lattice and D be a generalized f-derivation on L where $f : L \to L$ is a function. Then the following hold: for any element $x, y \in L$,

(1) $dx \leq Dx \leq f(x)$;

(2) $Dx \wedge Dy \leq D(x \wedge y) \leq Dx \vee Dy$.

Proof.(1) For all $x \in L$, we have $Dx = D(x \wedge x) = (Dx \wedge f(x)) \vee (f(x) \wedge dx) = (Dx \wedge f(x)) \vee dx$. Then $Dx \wedge dx = ((Dx \wedge f(x)) \vee dx) \wedge dx = dx$ and so $dx \leq Dx$. Also we get $Dx \vee f(x) = ((Dx \wedge f(x)) \vee dx) \vee f(x) = (Dx \wedge f(x)) \vee f(x) = f(x)$. So $Dx \leq f(x)$.

(2) Let $x, y \in L$, we have $D(x \wedge y) = (Dx \wedge f(y)) \vee (f(x) \wedge dy) \geq Dx \wedge f(y) \geq Dx \wedge Dy$. Moreover, we get $D(x \wedge y) = (Dx \wedge f(y)) \vee (f(x) \wedge dy) \leq Dx \vee dy \leq Dx \vee Dy$.

Proposition 3.5. Let L be a lattice and D be a generalized f-derivation on L where $f : L \to L$ is an order-preserving. Suppose $x, y \in L$ be such that $y \leq x$. If $Dx = f(x)$, then $Dy = f(y)$.

Proof. Since f is an order-preserving, $f(y) \leq f(x)$. Thus $Dy = D(x \wedge y) = (Dx \wedge f(y)) \vee (f(x) \wedge dy) = (f(x) \wedge f(y)) \vee (f(x) \wedge dy) = f(y) \vee dy = f(y)$.

Proposition 3.6. Let L be a lattice with a least element 0 and D be a generalized f-derivation on L where $f : L \to L$ is a function. Then

(1) if $f(0) = 0$, then $D0 = 0$;

(2) if $D0 = 0$, then $Dx \wedge f(0) = 0$ for all $x \in L$.

Proof. (1) By Proposition 3.4(1).

(2) Let $x \in L$. It is easily show that $d0 = 0$. Then

$$0 = D0 = D(x \wedge 0) = (Dx \wedge f(0)) \vee (f(x) \wedge d0) = Dx \wedge f(0).$$

The following result is immediately from Proposition 3.7(2).

Corollary 3.7. Let L be a lattice with a least element 0 and D be a generalized f-derivation on L where $f : L \to L$ is a function such that $D0 = 0$. Then we have,

(1) $Dx \leq f(0)$ if and only if $Dx = 0$ for all $x \in L$;

(2) $f(0) \leq Dx$ for all $x \in L$ if and only if $f(0) = 0$;

(3) if $f(0) \neq 0$ and there exist $x \in L$ such that $Dx \neq 0$, then $(L, \leq)$ is not a chain.

Proposition 3.8. Let L be a lattice with a greatest element 1 and D be a generalized f-derivation on L where $f : L \to L$ is a function. Then

(1) if $D1 = 1$, then $f(1) = 1$;

(2) if $f(1) = 1$, then $Dx = (D1 \wedge f(x)) \vee dx$ for all $x \in L$.

Proof. (1) By Proposition 3.4(1).

(2) Note that $Dx = D(1 \wedge x) = (D1 \wedge f(x)) \vee (f(1) \wedge dx) = (D1 \wedge f(x)) \vee (1 \wedge dx) = (D1 \wedge f(x)) \vee dx$.

Corollary 3.9. Let L be a lattice with a greatest element 1 and D be a generalized f-derivation on L where $f : L \to L$ is a function such that $f(1) = 1$. Then we have, for all $x \in L$,

(1) $D1 \leq f(x)$ if and only if $D1 \leq Dx$;

(2) if $D1 \leq f(x)$ and D is an order-preserving, then $Dx = D1$;

(3) $f(x) \leq D1$ if and only if $Dx = f(x)$;

(4) $D1 = 1$ if and only if $Dx = f(x)$.

Proposition 3.10. Let L be a lattice and D be a generalized f-derivation on L where $f : L \to L$ is a join-homomorphism. Then $D = f$ if and only if $D(x \vee y) = (Dx \vee f(y)) \wedge (f(x) \vee Dy)$ for all $x, y \in L$.

Proof. ($\Rightarrow$) Let $x, y \in L$. Then $D(x \vee y) = f(x \vee y) = f(x \vee y) \wedge f(x \vee y) = (f(x) \vee f(y)) \wedge (f(x) \vee f(y)) = (Dx \vee f(y)) \wedge (f(x) \vee Dy)$.

($\Leftarrow$) Assume that $D(x \vee y) = (Dx \vee f(y)) \wedge (f(x) \vee Dy)$. By putting $y = x$ in the assumption, we get $Dx = f(x)$ for all $x \in L$.

Proposition 3.11. Let L be a lattice and D be a generalized f-derivation on L where $f : L \to L$ is an order-preserving. Then $Dx = (D(x \vee y) \wedge f(x)) \vee dx$ for all $x, y \in L$.

Proof. Let $x, y \in L$. Then $dx \leq f(x) \leq f(x \vee y)$. So $Dx = D((x \vee y) \wedge x) = (D(x \vee y) \wedge f(x)) \vee (f(x \vee y) \wedge dx) = (D(x \vee y) \wedge f(x)) \vee dx$.

Proposition 3.12. Let L be a lattice and D be a generalized f-derivation on L where $f : L \to L$ is a function. If D is an order-preserving, then $Dx = D(x \vee y) \wedge f(x)$ for all $x \in L$.

Proof. Let $x, y \in L$. Then $dx \leq Dx \leq D(x \vee y) \leq f(x \vee y)$. So $Dx = D((x \vee y) \wedge x) = (D(x \vee y) \wedge f(x)) \vee (f(x \vee y) \wedge dx) = (D(x \vee y) \wedge f(x)) \vee dx$. Since $dx \leq D(x \vee y)$ and $dx \leq f(x)$, $dx \leq D(x \vee y) \wedge f(x)$. Hence $Dx = D(x \vee y) \wedge f(x)$.

Theorem 3.1.3 Let L be a lattice and D be a generalized f-derivation on L where $f : L \to L$ is a function. Then the following conditions are equivalent:

(1) D is an order-preserving;

(2) $Dx \vee Dy \leq D(x \vee y)$ for all $x, y \in L$;

(3) $D(x \wedge y) = Dx \wedge Dy$ for all $x, y \in L$.

Proof. (1) $\Rightarrow$ (2) Let $x, y \in L$. Then $Dx \leq D(x \vee y)$ and $Dy \leq D(x \vee y)$. Therefore $Dx \vee Dy \leq D(x \vee y)$.

(2) $\Rightarrow$ (1): Assume that (2) holds. Let $x, y \in L$ be such that $x \leq y$. Then $Dy = D(x \vee y) \geq Dx \vee Dy$ but we have $Dy \leq Dx \vee Dy$. So $Dy = Dx \vee Dy$, it follow that $Dx \leq Dy$.

(1) $\Rightarrow$ (3): Let $x, y \in L$. Then $D(x \wedge y) \leq Dx$ and $D(x \wedge y) \leq Dy$. Therefore $D(x \wedge y) \leq Dx \wedge Dy$. By Proposition 3.4(2), we have $D(x \wedge y) \geq Dx \wedge Dy$. Hence $D(x \wedge y) = Dx \wedge Dy$.

(3) $\Rightarrow$ (1): Assume that (3) holds. Let $x, y \in L$ be such that $x \leq y$. Then $Dx = D(x \wedge y) = Dx \wedge Dy$, it follow that $Dx \leq Dy$.

Theorem 3.14. Let L be a lattice with a greatest element 1 and D be a generalized f-derivation on L where $f : L \to L$ is a meet-homomorphism such that $f(1) = 1$. Then the following conditions are equivalent:

(1) D is an order-preserving;

(2) $Dx = f(x) \wedge D1$ for all $x \in L$;

(3) $D(x \wedge y) = Dx \wedge Dy$ for all $x, y \in L$;

(4) $Dx \vee Dy \leq D(x \vee y)$ for all $x, y \in L$.

Proof. By Theorem 3.13, we get the conditions (1) and (4) are equivalent.

(1)$\Rightarrow$(2): Assume that (1) holds. Let $x \in L$. Since $x \leq 1$, $Dx \leq D1$. We have $Dx \leq f(x)$. So we get $Dx \leq f(x) \wedge D1$. By Proposition 3.8(2), we have $Dx = dx \vee (f(x) \wedge D1)$. Thus $Dx = f(x) \wedge D1$.

(2)⇒(3): Assume that (2) holds. Then $Dx \wedge Dy = (f(x) \wedge D1) \wedge (f(y) \wedge D1) = f(x \wedge y) \wedge D1 = D(x \wedge y)$.

(3)⇒(1): Assume that (3) holds. Let $x, y \in L$ such that $x \leq y$. By (3), we get $Dx = D(x \wedge y) = Dx \wedge Dy$, it follows that $Dx \leq Dy$.

Theorem 3.15. Let L be a distributive lattice and D be a generalized f-derivation on L where $f : L \to L$ is a join-homomorphism. Then the following conditions are equivalent:

(1) D is an order-preserving;

(2) $D(x \wedge y) = Dx \wedge Dy$ for all $x, y \in L$;

(3) $D(x \vee y) = Dx \vee Dy$ for all $x, y \in L$.

Proof. By Theorem 3.13, we get the conditions (1) and (2) are equivalent.

(1)⇒(3): Assume that (1) holds and let $x, y \in L$. Then $dx \leq Dx \leq D(x \vee y) \leq f(x \vee y)$. By Proposition 3.11, we have $Dx = dx \vee (f(x) \wedge D(x \vee y))$. Then $Dx = (dx \vee f(x)) \wedge (dx \vee D(x \vee y)) = f(x) \wedge D(x \vee y)$. Similarly, we can prove $Dy = f(y) \wedge D(x \vee y)$. Thus $Dx \vee dy = (f(x) \wedge D(x \vee y)) \vee (f(y) \wedge D(x \vee y)) = f(x \vee y)) \wedge D(x \vee y) = D(x \vee y)$.

(3)⇒(1): Assume that (3) holds and let $x, y \in L$ be such that $x \leq y$. Then $Dy = D(x \vee y) = Dx \vee Dy$ by (3). It follows that $Dx \leq Dy$, this shows that D is an order-preseving.

Theorem 3.16. Let L be a modular lattice and D be a generalized f-derivation on L where $f : L \to L$ is a join-homomorphism. If there exist $a \in L$ such that $Da = f(a)$, then D is an order-preserving implies $D(x \vee a) = Dx \vee Da$ for all $x \in L$.

Proof. Let $x \in L$. Suppose that there exist $a \in L$ such that $Da = f(a)$ and D is an order-preserving. Then $Da \leq D(x \vee a)$. By Proposition 3.12, we get $Dx = D(x \vee y) \wedge f(x)$. So $Dx \vee Da = (D(x \vee a) \wedge f(x)) \vee Da = D(x \vee a) \wedge (Da \vee f(x)) = D(x \vee a) \wedge (f(a) \vee f(x)) = D(x \vee a) \wedge f(x \vee a) = D(x \vee a)$.

Let L be a lattice and D be a generalized f-derivation on L where $f : L \to L$ is a function. Denote $Fix_D(L) = \{x \in L | Dx = f(x)\}$.

In the following results, we assume that $Fix_D(L)$ is a nonempty proper subset of L.

Theorem 3.17. Let L be a lattice and D be a generalized f-derivation on L where $f : L \to L$ is a lattice-homomorphism. If D is an order-preserving, then $Fix_D(L)$ is a sublattice of L.

Proof. Let $x, y \in Fix_D(L)$. Then $Dx = f(x)$ and $Dy = f(y)$. Then $f(x \wedge y) = f(x) \wedge f(y) = Dx \wedge Dy \leq D(x \wedge y)$. So $D(x \wedge y) = f(x \wedge y)$, that is $x \wedge y \in Fix_D(L)$. Moreover, we get $f(x \vee y) = f(x) \vee f(y) = Dx \vee Dy \leq D(x \vee y)$ by Theorem 3.13. So $D(x \vee y) = f(x \vee y)$, this shows that $x \vee y \in Fix_D(L)$.

Theorem 3.18. Let L be a lattice and D be a generalized f-derivation on L where $f : L \to L$ is a lattice-homomorphism. If D is an order-serving, then $Fix_D(L)$ is an ideal of L.

Proof. The proof is by Proposition 3.5 and Theorem 3.17.

Let L be a lattice and D be a generalized f-derivation on L where $f : L \to L$ is a function. Denote $kerD = \{x \in L | Dx = 0\}$.

In the following results, we assume that $kerD$ is a nonempty proper subset of L.

Theorem 3.19. Let L be a distributive lattice and D be a generalized f-derivation on L where $f : L \to L$ is a lattice-homomorphism. If D is an order-preserving, then $kerD$ is a sublattice of L.

Proof. The proof is by Theorem 3.15.

Definition 3.20. Let L be a lattice and $f : L \to L$ be a function. A nonempty subset I of L is said to be a f-invariant if $f(I) \subseteq I$ where denote $f(I) = \{y \in L | y = f(x) \text{ for some } x \in I\}$.

Theorem 3.21. Let L be a lattice and D be a generalized f-derivation on L where $f : L \to L$ is a function. Let I be an ideal of L such that I is a f-invariant. Then I is a D-invariant.

Proof. Let $y \in DI$. Then there exist $x \in I$ such that $y = Dx$. Since I is a f-invariant, $f(x) \in I$. We have $y = Dx \leq f(x)$. Since I is an ideal and $f(x) \in I$, $y \in I$. Thus $dI \subseteq I$.

Acknowledgements

The author is greatly indebted to the referee for several useful suggestions and valuable comments which led to improvement the exposition. Moreover, this work was supported by a grant from Kasetsart University.

References

[1] Hamza, A. S. A. and N. O. Al-Shehri, Some results on derivations of BCI-algebras, Coden Jnsmac, **46**(2006), 13-19.

[2] Hamza, A. S. A. and N. O. Al-Shehri, On left derivations of BCI-algebras, Soochow Journal of Mathematics, **33**(2007), No. 6, 435-444.

[3] Harmaitree, S. and U. Leerawat, On f-derivations in Lattices, To appear in the Far East Journal of Mathematical Sciences (FJMS).

[4] Y. B. Jun and X. L. Xin, On derivations of BCI-algebras, Information Sciences, **159**(2004), 167-176.

[5] R. Lidl and G. Pilz, Applied Abstract Algebra, Springer-Verlag New York Inc, U.S.A, 1984.

[6] N. O. Alshehri, Generalized Derivation of Lattices, Int. J. Contemp. Math. Science, **5**(2010), No. 13, 629-640.

[7] Prabprayak, C. and U. Leerawat, On derivations of BCC-algebras, Kasetsart Journal, **53**(2009), 398-401.

[8] L. X. Xin, T. Y. Li and J. H. Lu, On derivations of lattice, Information Sciences, **178**(2008), 307-316.

[9] J. Zhan, and Y. L. Liut, On f-derivations of BCI-algebras, International Journal of Mathematics and Mathematical Sciences, **11**(2005), 1675-1684.

SCIENTIA MAGNA

An international journal

ISSN 1556- 6706