Some New Intrinsically 3-Linked Graphs

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Abstract

In [2], it was shown that every spatial embedding of K_{10}, the complete graph on ten vertices, contains a non-split 3-component link (K_{10} is intrinsically 3-linked). We improve this result by exhibiting two different subgraphs of K_{10} that also have this property. In addition, we also exhibit several families of graphs that are intrinsically 3-linked.

1 Introduction

Conway, Gordon [1], and Sachs [7], [6] showed that K_6 has the property that in every (tame) spatial embedding there exist two cycles that form a non-split link. That is, K_6 is *intrinsically linked*. Sachs also showed that each of the seven graphs obtained from K_6 by $\Delta - Y$ and $Y - \Delta$ exchanges are intrinsically linked. It is customary to call these seven graphs the Petersen family of graphs, because one member of the family is the classic Petersen graph. Sachs conjectured that this family was the complete set of minor minimal intrinsically linked graphs. (Recall that a graph H is said to be a *minor* of a graph G if H can be obtained from G by deleting and/or contracting a finite number of edges. A graph G is said to be *minor minimal* with respect to a property if G has the property but no minor of G has the property.) Robertson, Seymour, and Thomas [5] later proved that Sachs had indeed found the entire minor minimal set of intrinsically linked graphs.

A natural generalization of an intrinsically linked graph is an intrinsically n-linked graph, for an integer $n \geq 2$. A graph is *intrinsically n-linked* if there exists a non-split n-component link in every spatial embedding. It follows from the result of Robertson and Seymour [4] that there are only finitely many minor minimal intrinsically n-linked graphs. Ultimately we would like to determine this set. Flapan, Foisy, Naimi, and Pommersheim

[3] constructed a minor minimal intrinsically n-linked graph for each n≥3. In another paper, Flapan, Naimi, and Pommersheim, [2], proved that K_9 was not intrinsically 3-linked, and that K_{10} was intrinsically 3-linked. The question was left open as to whether or not K_{10} was minor minimal with respect to this property.

In this paper, we will concentrate on intrinsically 3-linked graphs. In particular, we show that K_{10} is not minor minimal with respect to being intrinsically 3-linked. More specifically, we show that the graph obtained from K_{10} by removing four edges incident to the same vertex is intrinsically 3-linked, and we show that the graph obtained from K_{10} by removing two non-adjacent edges is also intrinsically 3-linked. We do not know if either of these graphs is minor minimal with respect to intrinsic 3-linking.

We also exhibit several families of intrinsically 3-linked graphs, partly inspired by techniques from [3]. We believe that the graphs in these families are probably not minor minimal with respect to being intrinsically 3-linked, but that minors of graphs in these families will be new minor-minimal intrinsically 3-linked graphs.

In this paper we will take "linked" to mean "non-split linked," and "3-component link" to mean "non-split 3-component link."

2 Key Lemmas

For any pair of disjoint simple closed curves (embedded cycles), B and C in $\mathbb{R}^3$, we let $lk(B,C)$ denote the mod 2 linking number of B and C. Let B and C be cycles in an embedded graph, whose intersection is an arc, in this case define $B+C$ to be the cycle with edges given by $[E(B)\cup E(C)] - [E(B)\cap E(C)]$ and vertices given by all the vertices incident to those edges. We will need the following lemmas.

Lemma 1 *[2] If C_1, C_2, and C_3 are cycles in an embedded graph, C_1 disjoint from C_2 and C_3, and $C_2 \cap C_3$ is an arc, then $lk(C_1,C_2) + lk(C_1,C_3) = lk(C_1, C_2 + C_3)$.*

Lemma 2 *[2] Suppose that G is a graph embedded in $\mathbb{R}^3$ and contains the simple closed curves C_1, C_2, C_3, and C_4. Suppose that C_1 and C_4 are disjoint from each other and both are disjoint from C_2 and C_3, and that $C_2 \cap C_3$ is an arc. If $lk(C_1,C_2) = 1$ and $lk(C_3,C_4) = 1$, then there is a non-split 3-component link in that embedding.*

Lemma 3 *In an embedded graph with mutually disjoint simple closed curves, C_1,C_2,C_3, and C_4, and two disjoint paths x_1 and x_2 such that x_1 and x_2 begin in C_2 and end in C_3, if $lk(C_1, C_2) = lk(C_3, C_4) = 1$ then the embedding contains a non-split 3-component link.*

Proof: Choose c to be a path in C_3 that connects x_1 to x_2, and choose d to be a path in C_4 that connects x_1 to x_2. Consider the cycle, X that is formed by joining together the paths c, x_1, d, and x_2. Since X intersects along an arc with C_2, by Lemma 1, $lk(C_1, X) + lk(C_1, C_2) = lk(C_1, X + C_2$. If $lk(C_1, X + C_2) = 1$, then we can apply Lemma 2 to the cycles $\{C_1, C_2 + X, C_3, C_4\}$ to conclude that there is a 3-component link. Otherwise $lk(C_1, X) = 1$ and we can again use Lemma 2 (now on the cycles $\{C_1, X, C_3, C_4\}$) to conclude that the embedded graph has a 3-component link. □

Finally we state a result of [2] regarding 4-patterns and 6-patterns. A *4-pattern* within an embedded graph, G, consists of a 3-cycle, B, that is linked with four other 3-cycles that can be described as follows. For vertices q,r in G, each 3-cycle linked to B is of the form (q, r, x) where x is one of any four vertices of G other than B, q, and r. (see Figure 1.)

A *6-pattern* within an embedded graph, G, consists of a 3-cycle, B, that is linked with six other 3-cycles that can be described as follows. For vertices p,q,r in G, each 3-cycle linked to B is either of the form (p, q, x) or (p, r, x) where x is one of any three vertices of G other than B, p, q, and r. (see Figure 1.) We may now state the following Lemma:

Lemma 4 *[2] There exists an embedding of K_9 without any 3-component links. For any embedding of K_9 every linked 3-cycle is in a 4-pattern, a 6-pattern, or a 3-component link.*

3 Intrinsically 3-linked subgraphs of K_{10}

Lemma 5 *Let G be an embedded graph that contains a 4-pattern (respectively 6-pattern), as well as a vertex, A, that is disjoint from the 4-pattern (6-pattern). Let S denote the vertices of the 4-pattern (6-pattern) minus the vertices of B, where B is as defined above. If the vertices of $S \cup \{A\}$ induce a complete subgraph of G, then a non-split 3-component link is present in the embedding of G.*

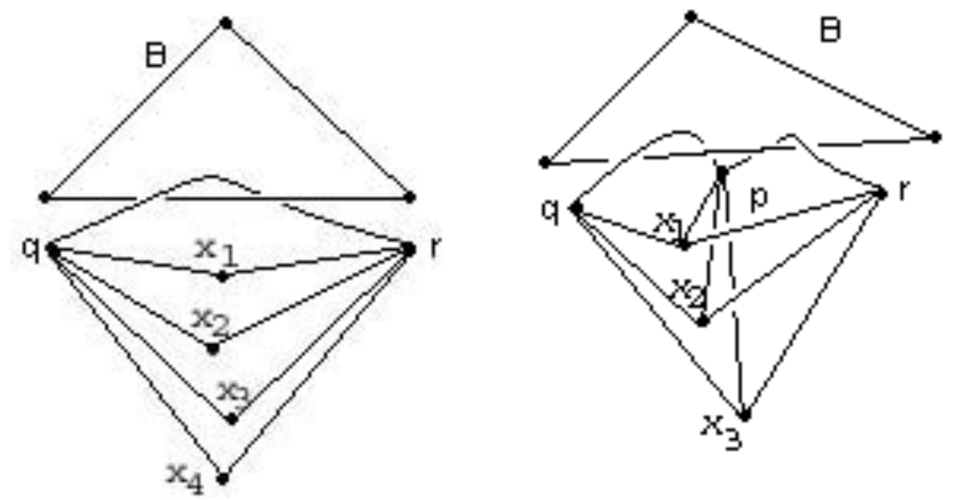

Figure 1: A possible 4-pattern on the left, and a possible 6-pattern on the right.

Proof: We consider two cases:
Case 1: A is connected to a 4-Pattern
In this case the subgraph induced by the vertices $\{A, x_1, x_2, x_3, x_4, q\}$ is a K_6, thus, we know there exist a pair of linked 3-cycles, C and D through those vertices in the embedding. If C or D is linked with B we are done, else we should consider the possible vertices of C and D. We consider two subcases. Without loss of generality, we assume the vertex A is in cycle C.

Subcase 1.1: Without loss of generality, $C = (A, x_3, x_4)$ and $D = (q, x_1, x_2)$. Since the three cycle (q, x_1, r) is linked to B, by Lemma 2 there is a 3-component link.

Subcase 1.2: Without loss of generality C=(q,x_1,A) and D=(x_2,x_3,x_4). Since the cycle (q, x_1, r) is linked to B, by Lemma 2 there is a 3-component link.

Case 2: A is connected to a 6-Pattern
In this case the subgraph induced by the vertices $\{A, p, q, x_1, x_2, x_3\}$ forms a K_6. This implies that two linked 3-cycles exist among these vertices. Let us call these 3-cycles C and D. If either C or D is linked with B, we are finished, otherwise we need to consider three subcases. Without loss of generality, we assume the vertex A is in the cycle C.

Subcase 2.1: Without loss of generality $C = (p, x_1, A)$, and $D = (q, x_2, x_3)$. Since the cycle (p, x_1, r) is linked to B, by Lemma 2 there is a 3-component link.

Subcase 2.2: Without loss of generality, $C = (p, q, A)$ and $D = (x_1, x_2, x_3)$. Since $(x_1, x_2, x_3) = [(x_1, x_2, r) + (x_2, x_3, r)] + (x_3, x_1, r)$, by applying Lemma 1 twice, one of $lk(C, (x_1, x_2, r))$, $lk(C, (x_2, x_3, r))$, or $lk(C, (r, x_3, x_1,))$ is one. Without loss of generality, we will assume that $lk(C, (x_1, x_2, r)) = 1$. Now, since $(p, q, A) = [(p, q, x_3) + (A, x_3, q)] + (A, p, x_3)$, by applying Lemma 1 twice, we have either $lk((A, p, x_3), (r, x_1, x_2))$ or $lk((p, q, x_3), (r, x_1, x_2))$, or $lk(A, x_3, q), (r, x_1, x_2))$ is one. If $lk((A, p, x_3)), (r, x_1, x_2))$ is one, then we may apply Lemma 2 with $C_1 = B$, $C_2 = (p, q, x_3)$, $C_3 = (A, p, x_3)$ and $C_4 = (r, x_1, x_2)$. If $lk((p, q, x_3), (r, x_1, x_2)) = 1$, then the cycles $B, (p, q, x_3)$ and (r, x_1, x_2) form a 3−component link. If $lk((A, x_3, q), (r, x_1, x_2)) = 1$, then we may apply Lemma 2 with $C_1 = B$, $C_2 = (p, q, x_3)$, $C_3 = (A, x_3, q)$ and $C_4 = (r, x_1, x_2)$.

Subcase 2.3: Without loss of generality, $C = (A, x_1, x_2)$ and $D = (p, q, x_3)$. In this case, the cycles B, C and D form a 3-component link. □

Theorem 1 *The graph obtained from K_{10} by removing four edges incident* *to a common vertex is intrinsically 3-linked.*

Proof: Let G denote the graph obtained from K_{10} obtained by removing four edges incident to a common vertex. Denote the vertices of G by $\{v_1, v_2, \ldots, v_{10}\}$, and we shall assume that there is no edge from v_1 to any of the vertices in the set $\{v_7, v_8, v_9, v_{10}\}$. Embed G, and considering the K_6 induced by the vertices $\{v_1, v_2, v_3, v_4, v_5, v_6\}$; the vertex v_1 is in a linked 3-cycle, let us call that linked 3-cycle B. Now temporarily put back in the missing edges from v_1 to the vertices v_7, v_8, and v_9. The embedded subgraph induced by $\{v_1, v_2, \ldots, v_9\}$ is K_9, so by Lemma 4, B is in either a 3-component link, a 4-pattern, or a 6-pattern. If we remove the temporary edges, any such 3-component link, 4-pattern or 6-pattern that contained B will remain. Suppose there is no 3-component link. Since v_{10} is connected to all vertices of the 4-pattern or 6-pattern (except to all vertices of B), and since the vertices of the 4-pattern or 6-pattern induce a complete subgraph, by Lemma 5, there is a 3-component link in that embedding. □

Theorem 2 *The graph obtained from K_{10} by removing two nonadjacent edges is intrinsically 3-linked.*

Proof: Let H denote the graph obtained by removing two non-adjacent edges from K_8. It suffices to show that in any embedding of H there exists a linked 3-cycle with two vertices of degree six. This is because adding back in the two edges and a ninth vertex that is connected to all other vertices, would create an embedding of K_9. By Lemma 4, this linked 3-cycle in H would be in either a 4-pattern, a 6-pattern, or a three component link in K_9. Removing the non-adjacent edges would preserve the 4-pattern, 6-pattern, or three-component link. Furthermore, the vertices of the 4-pattern or 6-pattern, minus the vertices of B (where B is defined as in Lemma 5), induce a complete subgraph.

Let G denote the graph obtained from K_{10} by removing two non-adjacent edges. By the reasoning given in the previous paragraph, any embedding of G will have an embedded 9-vertex subgraph with either a 3-component link, 4-pattern, or a 6-pattern in such a way that Lemma 5 applies to establish that G has a 3-component link in any embedding.

Now we show that in any embedding of H there exists a linked 3-cycle with two vertices of degree six. Embed H. If we partition the vertices of H into two sets: $\{a_1, a_2, a_3, a_4\}$ and $\{b_1, b_2, b_3, b_4\}$ so that the endpoints of the two missing edges, say (a_1, a_3) and (a_2, a_4) form one partition, then we only need to show that there is an edge within that partition that is part of a linked 3-cycle. By temporarily putting in the edges (a_1, a_3) and (a_2, a_4), we can consider the embedded $K_{4,4}$ induced by the same partitions. By Sachs [7] there are (without loss of generality) two linked 4-cycles of the form (a_1, b_1, a_3, b_2) and (a_2, b_3, a_4, b_4) or (a_1, b_1, a_2, b_2) and (a_3, b_3, a_4, b_4). In the case of the latter, since the edge (a_3, a_4) exists, by Lemma 1, either (a_3, b_3, a_4) or (a_3, b_4, a_4) is a linked 3-cycle and we are done, since both a_3 and a_4 have degree 6.

If the two linked cycles are $A = (a_1, b_1, a_3, b_2)$ and $B = (a_2, b_3, a_4, b_4)$, then by Lemma 1, $lk(A, B) = lk(B, (a_1, b_1, b_2)) + lk(B, (b_1, a_3, b_2))$, and without loss of generality, $lk(B, (a_1, b_1, b_2))$ is non-zero. Now, we have that $B = (a_2, b_3, a_4, a_3) + (a_2, a_3, a_4, b_4)$. Once again by Lemma 1, either $lk((a_1, b_1, b_2), (a_2, b_3, a_4, a_3))$ or $lk((a_1, b_1, b_2), (a_2, a_3, a_4, b_4))$ is non-zero. Without loss of generality, assume the latter. Since $(a_2, b_3, a_4, a_3) = (b_3, a_4, a_3) + (b_3, a_3, a_2)$, by Lemma 1 we know one of the 3-cycles, (b_3, a_4, a_3) or (b_3, a_3, a_2) is linked, proving our theorem, since a_2, a_3, and a_4 all have degree 6. □

4 Families of intrinsically 3-linked graphs

Here we establish several families of intrinsically 3-linked graphs. We do not know if they are minor minimal with respect to being intrinsically 3-linked.

The following result was shown for $K_{4,4}$ in [6] and [3]:

Lemma 6 *Let G be a spatial embedding of K_7 or $K_{4,4}$, then every edge of G is in a non-split linked cycle.*

Proof: First embed K_7, then consider an edge $e_1 = (v_1, v_2)$ in K_7. The vertices of $G - v_2$ induce a K_6. Then vertex v_1 is in a linked cycle in this embedded K_6, say (v_1, v_3, v_4) is linked to cycle C. By Lemma 1, $lk((v_1, v_3, v_4), C) = lk((v_1, v_3, v_2), C) + lk((v_1, v_2, v_3, v_4), C)$, and thus e_1 is in a linked cycle.

□

The following result was shown in [3] for A=B=$K_{4,4}$:

Theorem 3 *Let G be a graph formed by identifying an edge of a graph G_1 with an edge from another graph G_2, where G_1 and G_2 are either K_7 or $K_{4,4}$. Then every such G is intrinsically 3-linked.*

Proof: Embed G. Let C_1 and C_2 denote the linked cycles in G_1 such that C_2 contains the edges that is identified with an edge in G_2. We know such cycles exist by Lemma 6. Similarly, let C_3 and C_4 denote the linked cycles in G_2, where C_3 contains an edge that is identified with an edge in C_2. By Lemma 2, there is a non-split 3-component link in this embedding. □

Theorem 4 *Let G be a graph containing two disjoint graphs from the Petersen family, G_1 and G_2, as subgraphs. If there are edges between two subgraphs G_1 and G_2 such that the edges form a 6-cycle with vertices whose endpoints alternate between vertices of G_1 and vertices of G_2, then G is intrinsically 3-linked.*

Proof: Let $\{a_1, a_2, a_3\}$ be the set of three vertices in G_1 and $\{b_1, b_2, b_3\}$ be the set of three vertices in G_2, as described above. Embed G. By the pigeonhole principle, at least two vertices in the set $\{a_1, a_2, a_3\}$ are in a linked cycle within the embedded G_1 (without loss of generality, a_1 and a_2), and likewise we may assume that the vertices b_1 and b_2 are in a linked cycle in G_2. Because of the edges between $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, we know that there are two disjoint edges between the sets $\{a_1, a_2\}$ and $\{b_1, b_2\}$. By Lemma 3, a 3-component link is present in the embedding. □

Acknowledgments-We would like to thank Barbara Chervenka, Eman Kunz, Quincy Looney, and April Siegler for several helpful conversations. We would also like to thank the referee for offering many helpful suggestions. December 23, 2003.

References

[1] J. Conway and C. Gordon, *Knots and links in spatial graphs*, J. Graph Theory **7** (1983), 445-453.

[2] E. Flapan, R. Naimi, J. Pommersheim, *Intrinsically triple linked complete graphs*, Topology Appl. **115**, no. 2 (2001), 239-246.

[3] E. Flapan, J. Foisy, R. Naimi, J. Pommersheim, *Intrinsically n-linked graphs*, J. Knot Theory Ramifications **10** (2001), no. 8, 1143-1154.

[4] N. Robertson, P. Seymour, *Graph minors XVI. Wagner's conjecture*, preprint.

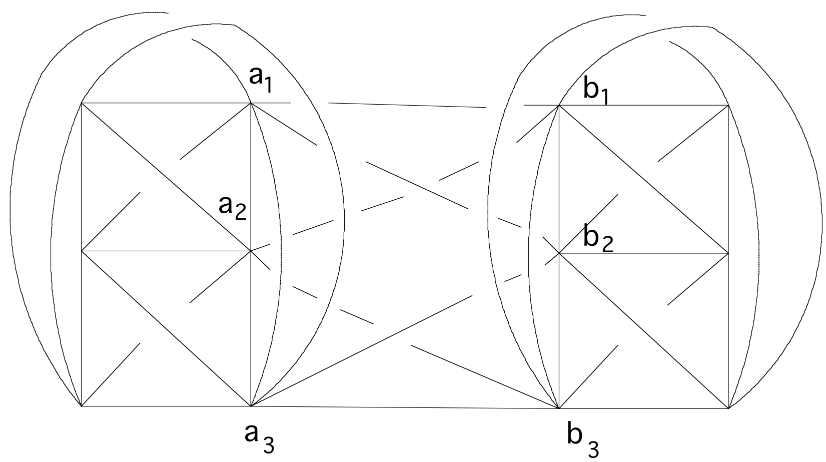

Figure 2: An example of a graph G, as in Theorem 4, with $G_1 = G_2 = K_6$.

[5] N. Robertson, P. Seymour, R. Thomas, *Sachs' linkless embedding conjecture*, J. Combin. Theory Ser. B **64** (1995), pp. 185-227.

[6] H. Sachs, *On a spatial analogue of Kuratowski's Theorem on planar graphs - an open problem*, in *Graph Theory, Lagów*, 1981, Lecture Notes in Math, 1018, Springer-Verlag, Berlin-Heidelberg, (1983), 230-241.

[7] H. Sachs, *On spatial representations of finite graphs, finite and infinite sets*, (A. Hajnal, L. Lovasz, and V. T. Sós, eds), colloq. Math. Soc. János Bolyai, vol. 37, North-Holland, Budapest, (1984), 649-662.

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