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Keywords: Colebrook-White equation; Lambert W-function; Turbulent flow friction; Hydraulic resistance; Analytical approximations

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Dear editor/reviewers,

This is revised version of my manuscript:

W solutions of CW equation for flow friction

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Sincerely,

Dejan Brkić, PhD

*Detailed Response to Reviewers DB

Thank you for your comments. I appreciate them. They were useful for me and I have improved my text according to your suggestions.

1. I have noted in text that CW equation is empirical (first word in abstract also). I have noted this fact several times through text.
2. I have noted that coefficient 2.51 is replaced with 2.825 for gas flow as recommended by American Gas Agency AGA and I have associated the paper of Haaland from 1983 with this fact
3. I have corrected my error regarding electrical resistance (word current is replaced with resistance). Word resistance is more suitable, but with constant voltage, change of resistance implies also change of current. But you are note is absolutely good, changes of resistance is causing changes in current (or flow in case of fluid flow). It is not acceptable to replace cause and its consequence.
4. Paper of D.J. Zigrang, N.D. Sylvester, (1985) is also cited.
5. Definition of Lambert W-function is now added in text (second row in section 2).
6. According to solution for observed branch of W function by Barry et al I have added improved solution of CW function. These are equations (16) and (17). Also I have added solution for CW function based on solution of W function proposed by Winitzki (section 3.5). I have also added graph of error distribution of one solution of CW based on solution of W by Barry et al.
7. Name of author is now added after title.
8. I have rearranged term ‘in our paper’to ‘here’or similar….
9. Now, I use only numbers for equations, i.e. (10) and not (10a)
10. I have not found graphical abstract in any paper in Applied Mathematics Letters. But I have added graph of W and CW function in text (figure 1). This figure can be also used as graphical abstract if suitable.

Figure 1 DB

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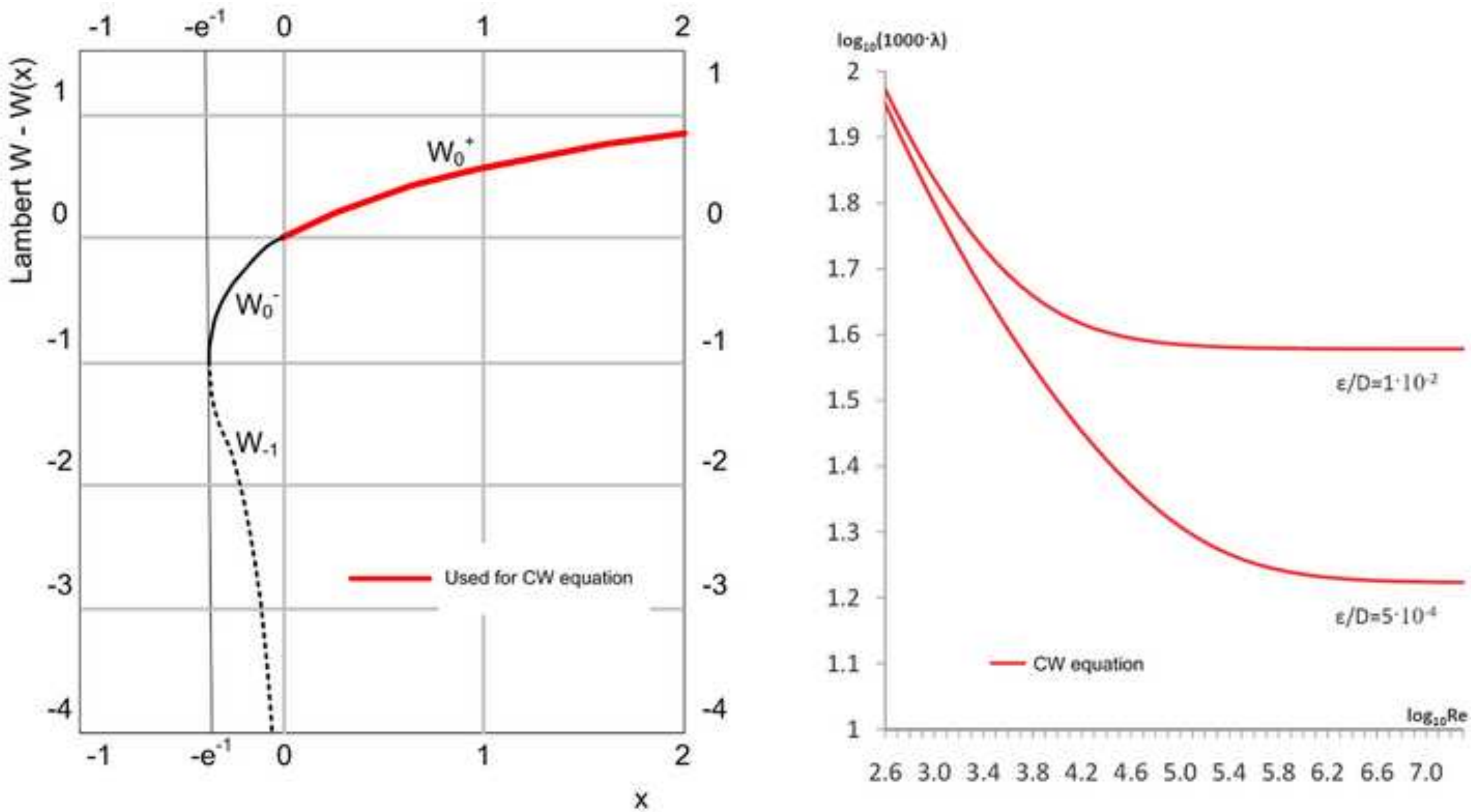

Figure 2 DB
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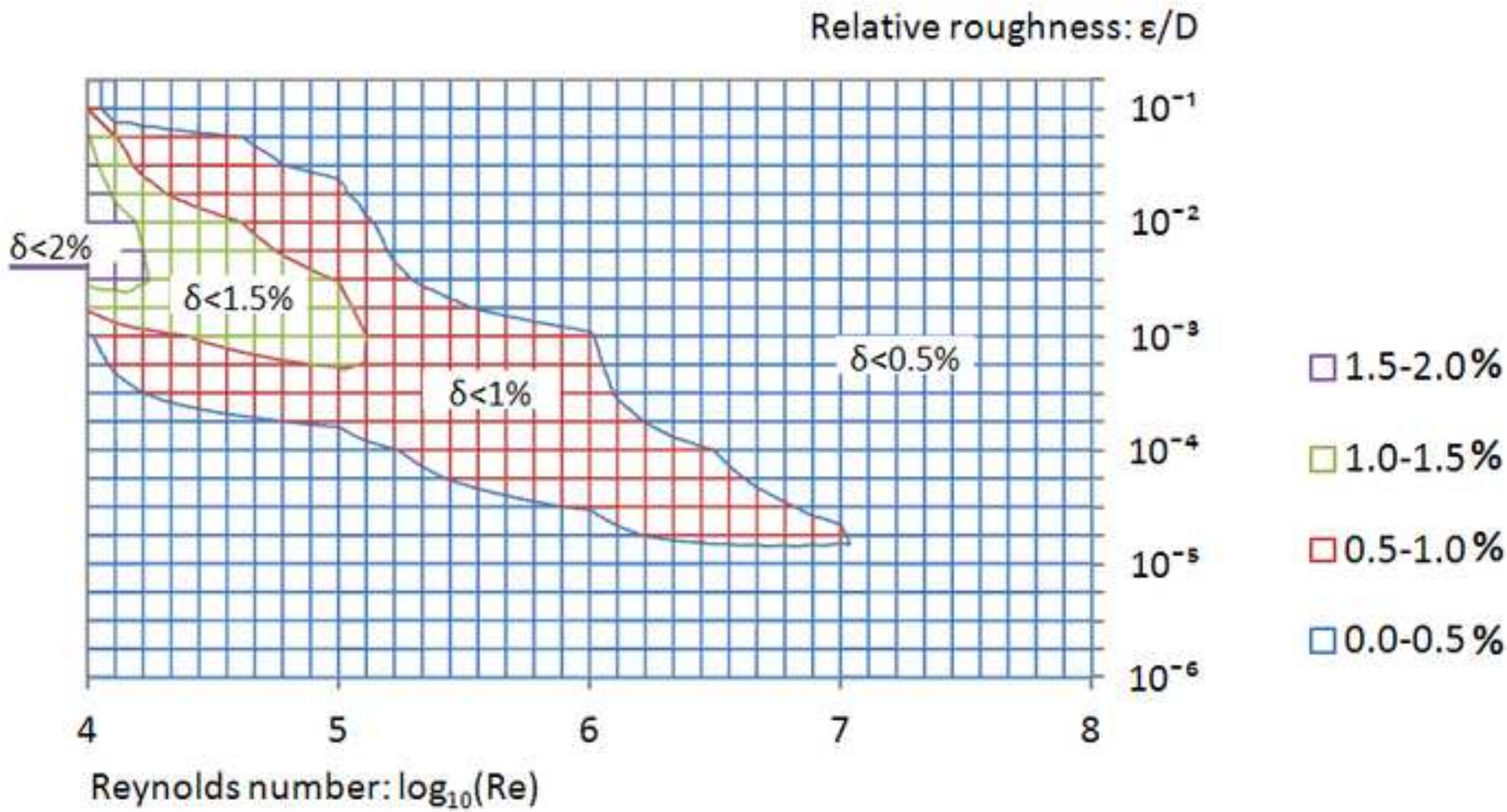

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Abstract: Empirical Colebrook-White (CW) equation belongs to the group of transcendental functions. CW function is used for determination of hydraulic resistances associated with fluid flow through pipes, flow of rivers, etc. Since CW equation is implicit in fluid flow friction factor, it has to be approximately solved using iterative procedure or using some of the approximate explicit formulas developed by many authors. Alternate mathematically equivalents to the original expression of the CW equation, but now in explicit form developed using Lambert W-function, are shown (with related solutions). W function is also transcendental, but it is used more general compared with CW function. Hence, solution to W function developed by mathematicians can be used effectively for CW function which is of interest only for hydraulics.

MSC2010 codes: 00A06, 33B99, 33F05, 65D20, 65H05, 76F06

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1. Introduction

Problem of flow in pipes and open conduits was one which had been of considerable interest to engineers for nearly 250 years. Even today, this problem is not solved definitively [1]. The difficulty to solve the turbulent flow problems lies in the fact that the friction factor is a complex function of relative surface roughness (ε/D) and the Reynolds number (Re). Precisely, hydraulic resistance depends on flow rate. Similar situation is with electrical resistance when diode is in circuit. To be more complex, widely used, empirical Colebrook-White (CW) equation valuable for determination of hydraulic resistances for turbulent regime in smooth and rough pipes is iterative (implicit in fluid flow friction factor). The unknown friction factor appears on both sides of the equation i.e., both the right and left-hand terms contain friction factor [2,3]. Colebrook-White equation is also known as Colebrook equation or simple CW equation (1):

$$\frac{1}{\sqrt{\lambda}}=-2\cdot\log_{10}\left(\frac{2.51}{\mathrm{Re}\cdot\sqrt{\lambda}}+\frac{\varepsilon}{3.71\cdot D}\right) \qquad (1)$$

Empirical implicit CW equation itself can produce an error of more than 5% but today even apropos this fact it is accepted standard for calculation of flow friction factor in hydraulically smooth and rough pipes [4]. Many researchers adopt a modification of the CW equation, using the 2.825 constant instead of 2.51 especially for gas flow calculation [5,6]. This adoption for gas flow produced deviation of maximal 3.2% compared with classical CW equation.

As an alternative to the implicit CW equation, many approximate explicit formulas were given. Gregory and Fogarasi [7], Yıldırım [8] and Brkić [9] made comparisons of in that time available approximations of CW equation.

2. Explicit reformulation of the CW equation based on Lambert W-function

Lambert W and CW are transcendental functions. The (real-valued) Lambert W-function is solution of the nonlinear equation $W\cdot e^{W}=x$. The range of the lower branch of inverse Lambert function is $-1\leq W_{-1}$, while the upper branch W_0 is divided into $-1\leq W_0^- \leq 0$ and $0\leq W_0^+$. W_0 is referred to as principal branch of the Lambert W-function. Only W_0^+ part of the principal branch of the Lambert W function will be used for solution of here presented problem. The Lambert function is implemented in many mathematical systems like Mathematica by Wolfram Research under the name ProductLog or Matlab by MathWorks under the name Lambert. Note that name "W" for Lambert function is not as old as the related function [10]. The modern history of Lambert W began in the 1980s, when a version of the function was built into the Maple computer-algebra system and given the name W. Corless et al [11] proposed name Lambert W for this function and this name is also used here. But in formulas is used only letter W for related function because this notation is shorter. Lambert W-function is somewhere known as Omega [11]. For CW equation, only the positive part of the principal branch of the Lambert W-function is considered (Figure 1) because the other branches correspond to nonphysical solutions of CW equations, so the simplified notation W is not ambiguous. Fortuitously, the letter W has additional significance because pioneering work on many aspects of W by Wright [12]. Although White was not actually a co-author of the paper in which CW equation was presented [2], Colebrook made a special point of acknowledging important contribution of White to the development of the equation [3]. So letter W has additional symbolic value in here reformulated CW equation (2):

$$\frac{1}{\sqrt{\lambda}} = -2 \cdot \log_{10}\left(10^{\frac{-W(x)}{\ln(10)}} + \frac{\varepsilon}{3.71 \cdot D}\right) = -2 \cdot \log_{10}\left(\frac{5.02 \cdot W(x)}{\mathrm{Re} \cdot \ln(10)} + \frac{\varepsilon}{3.71 \cdot D}\right) \quad (2)$$

Here, argument of Lambert W-function can be noted as (3):

$$x = \frac{\mathrm{Re} \cdot \ln(10)}{5.02} \quad (3)$$

In the papers of Moore [13], Nandakumar [14] and Goudar and Sonnad [15-17], other possible transformations of CW equation using Lambert-W function are shown. But relations shown in these papers have limitation in applicability for high values of Reynolds number and relative roughness because today available computers cannot operate with extremely large numbers [18]. In the paper of Keady [19], CW equation is expressed in Maple notation using Lambert W-function. Clamond [20] also provides Matlab and FORTRAN codes for CW relation expressed in term of the Lambert W-function.

3. Possible solutions of CW equation based on Lambert W-function

Besides the relative simplicity of an explicit from of CW equation transformed using Lambert W-function, it allows highly accurate estimation of friction factor as the Lambert W-function can be evaluated accurately [21].

3.1 Formal solution

Since the Lambert W is transcendental function, formal solution of the Lambert W-function can be expressed only in endless form (4):

$$W(x) = \ln \frac{x}{\ln\left(\frac{x}{\ln\left(\frac{x}{\ln(\cdot\cdot)}\right)}\right)} \tag{4}$$

3.2 Solution using series expansion

Functions such as exponential, logarithmic or square root are useful tools in solving broad classes of mathematical problems. With just the four basic operations of arithmetic any linear equation can be solved. Adding square roots quadratic equations can be solved as well. For some class of problems are useful trigonometric functions. These functions can be classified as elementary. But for solution of implicit equations such as CW, best function is Lambert W. Logical question which can arise is why Lambert W-function is not elementary function, while trigonometric, logarithmic, exponential, etc are. Note that Taylor series is used in principle in pocket calculators. This means that W button can be added very easy. Using series expansion, principal branch of the Lambert W-function can be noted as (5):

$$W_0(x) \approx x - x^2 + \frac{3}{2}\cdot x^3 - \frac{8}{3}\cdot x^4 + \frac{125}{24}\cdot x^5 - \frac{54}{5}\cdot x^6 + O(x^7) = \sum_{1}^{+\infty}\frac{(-n)^{n-1}}{n!}\cdot x^n \qquad (5)$$

Whether W ultimately attains such canonical status will depend on whether the mathematical community at large finds it sufficiently useful [10].

3.3 Solution using Boyd shifted function

It is convenient to define a new function and new parameter [22] such that both the domain and range are the nonnegative real axis as (6):

$$\omega(y) = W(x) + 1 \Leftrightarrow W(x) = \omega(y) - 1 \qquad (6)$$

With also 'shifted' argument of the function [22] (7):

$$y = 1 + x\cdot e^1 \approx 1 + 2.71\cdot x \Leftrightarrow x = \frac{y-1}{e^1} \approx \frac{y-1}{2.71} = 0.367\cdot(y-1) \qquad (7)$$

Then, the Lambert W function can be transformed in ω function [22] (8):

$$(\omega - 1)\cdot e^{\omega} = y - 1 \qquad (8)$$

Approximate solution[1] of ω function can be expressed [22] (9):

$$\omega_0 \approx \{\ln(y+10) - \ln(\ln(y+10))\}\cdot \tanh\left(\frac{\sqrt{2\cdot y}}{\ln(10) - \ln(\ln(10))}\right) \qquad (9)$$

In previous equation (9) hyperbolic tangent function can be defined $\tanh(\xi)=(e^{\xi}-e^{-\xi})/(e^{\xi}+e^{-\xi})$. Improved solution after Boyd [22] is also available (10):

[1] Note that log is actually ln, i.e. $\log_e$ in the paper of Boyd [20]

$$\varpi_0 = \omega_0 \cdot (1+\Omega) \tag{10}$$

Where Ω is (11):

$$\Omega = \frac{\left(\ln(y) - \frac{7}{5}\right) \cdot e^{-\frac{3}{40}\cdot\left(\ln(y)-\frac{7}{5}\right)^2}}{10} \tag{11}$$

When the improved approximation is used as the first guess for Newton's iteration scheme (12) only four iterations reduce the relative error significantly over the entire domain of 'shifted' function [22]:

$$\omega_{i+1} = \omega_i - \frac{(\omega_i - 1) - e^{(-\omega)\cdot(y-1)}}{\omega_i} \tag{12}$$

This error does not have uniform distribution for entire domain of 'shifted' function [22]. Values of errors presented for 'shifted' function i.e. ω(y) have limited meaning, because our interest is in evaluation of error for W(x).

3.4 Solution proposed by Barry et al.

Barry et al. [23] gives their approximation for the upper branch of the Lambert W-function valid for CW equation (13):

$$W_0^+(x) \approx \ln\left(\frac{6}{5}\frac{x}{\ln\left(\frac{12}{5}\right)\left(\frac{x}{\ln\left(1+\frac{12\cdot x}{5}\right)}\right)}\right) \tag{13}$$

For CW equation, previous equation can be arranged (14):

$$\frac{1}{\sqrt{\lambda}} \approx -2 \cdot \log_{10}\left(10^{-0.4343 \cdot S} + \frac{\varepsilon}{3.71 \cdot D}\right) \approx -2 \cdot \log_{10}\left(\frac{2.18 \cdot S}{\mathrm{Re}} + \frac{\varepsilon}{3.71 \cdot D}\right) \quad (14)$$

Where S is (15):

$$\mathrm{S} \approx \ln \frac{\mathrm{Re}}{1.816 \cdot \ln\left(\frac{1.1 \cdot \mathrm{Re}}{\ln(1 + 1.1 \cdot \mathrm{Re})}\right)} \quad (15)$$

Presented relation produce maximal relative error no more than 3% compared to original implicit CW equation. More accurate procedure, but also more complex is available in the paper of Barry et al. [23]. To reduce error below 2%, S should be replaced with S^* (16) or S^{**} (17):

$$\mathrm{S}^* \approx 1.4586887 \cdot \mathrm{S} - 0.4586887 \cdot \ln\left(\frac{\mathrm{Re} \cdot 0.917365}{\ln(1 + \mathrm{Re} \cdot 0.917365)}\right) \quad (16)$$

Distribution of relative error for approximate equation (17) over the entire practical domain of relative roughness (ε/D) and Reynolds number (Re) is shown in figure 2.

$$\mathrm{S}^{**} \approx \ln\left(0.488 \cdot \mathrm{Re} \cdot \left[\ln\left(\frac{\mathrm{Re}}{\mathrm{S}}\right)\right]^{-1}\right) \quad (17)$$

Since the error derived using presented procedures and real errors produced by using implicit CW equation are of both signs the errors will add in some cases and cancel in other cases.

3.5 Solution proposed by Winitzki

According to Winitzki [24] approximation for Lambert W-function can be obtained by using (18):

$$\mathrm{W}(\mathrm{x}) \approx \ln(1 + \mathrm{x}) \cdot \left(1 - \frac{\ln(1 + \ln(1 + \mathrm{x}))}{2 + \ln(1 + \mathrm{x})}\right) \quad (18)$$

Using (14) and previous equation one new approximate formula for CW equation can be obtained (19):

$$S \approx \ln(1+0.458 \cdot \mathrm{Re}) \cdot \left(1 - \frac{\ln(1+\ln(1+0.458 \cdot \mathrm{Re}))}{2+\ln(1+0.458 \cdot \mathrm{Re})}\right) \qquad (19)$$

This approximation is accurate as those developed by using procedure proposed by Barry et al. [23].

4. Conclusion

CW equation is widely used in the petroleum industry for calculations of oil and gas pipelines, in civil engineering for calculation of water distribution systems, in chemical engineering, and in all fields of engineering where fluid flow can be occurred. The problem is that, since CW equation is implicit, containing the friction factor, the Reynolds number and the pipe roughness, it has to be solved iteratively. Even today in the era of advance computer technology, explicit approximations of the implicit CW relation are very often used for calculation of friction factor in pipes. Presented methods for solution of CW equation based on Lambert W-function provide equally satisfied accuracy of solution for observed problem.

Details about work and life of Johann Heinrich Lambert can be read in the paper of Gray and Tilling [25].

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Notations:

λ-Darcy friction factor (dimensionless)

Re-Reynolds number (dimensionless)

D-inner pipe diameter (m)

ε-roughness of the inner surface of pipe (m)

ε/D-relative roughness of the inner surface of pipe (dimensionless)

W-Lambert function

ω-shifted, auxiliary function proposed by Boyd

x-argument of the Lambert W-function

y- argument of the shifted, auxiliary function proposed by Boyd

n-positive integer number

ϖ , Ω, ζ, S-auxiliary terms defined in text

List of Figures: