Kinetic models and quantum effects : a modified Boltzmann equation forFermi-Dirac particles

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Abstract. We study a modified Boltzmann equation, which takes into account quantum effects for a gas of Fermi-Dirac particles and gives in the classical limit the Boltzmann equation.

Key-words

kinetic models – theory of gases – Fermi-Dirac particles – Boltzmann equation – collision operator – collision kernel – quantum effects – Cauchy problem – kinetic energy – entropy – H theorem – classical limit – equilibrium states – fixed-point methods – renormalization – renormalized Boltzmann equation

Summary

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1. Introduction

One way to study the statistical evolution of rarefied gases of particles consists in writing down kinetic equations. Let us denote $f = f(t, x, \xi)$ the density of probability of presence of gas particles at time t and position x with velocity ξ. In this paper, we are not interested in boundary or relativistic effects, so we shall assume that $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$ with $N \geq 1$.

When no interaction holds, the particles move with a constant velocity ξ and the density f is solution of the free transport equation :

$$\partial_t f + \xi \cdot \partial_x f = 0 \quad . \tag{1}$$

In a rarefied gas without external force, the interactions reduce to collisions in which only two particles interact and one can assume at the first level of approximation that these collisions are elastic. Denoting by (ξ, ξ_*) and (ξ', ξ'_*) the velocities of the particles respectively before and after collision, the conservation of momentum and energy gives

$$\begin{aligned} \xi' + \xi'_* &= \xi + \xi_* \\ |\xi'|^2 + |\xi'_*|^2 &= |\xi|^2 + |\xi_*|^2 \end{aligned}$$

which can be solved in

$$\begin{aligned} \xi' &= \xi - (\xi - \xi_*).\omega\ \omega \\ \xi'_* &= \xi_* + (\xi - \xi_*).\omega\ \omega \end{aligned}$$

where ω denotes a unit vector of $\mathbb{R}^d$: $\omega \in S^{N-1}$.

Assuming that there was no correlation between particles before and after collision, Boltzmann showed that equation (1) has to be modified as follows :

$$\partial_t f + \xi \cdot \partial_x f = Q(f, f) \tag{2}$$

with :

$$Q(f, f) = \int\int_{\mathbb{R}^d \times S^{N-1}} B(\xi - \xi_*, \omega)(f' f'_* - f f_*)\ d\xi_* d\omega \quad .$$

where B denotes the cross section and where we used the following notations :

$$f = f(t, x, \xi) \qquad f_* = f(t, x, \xi_*) \qquad f' = f(t, x, \xi') \qquad f'_* = f(t, x, \xi'_*)$$

Now, if we want to describe a gas of Fermi-Dirac particles satisfying Sommerfeld's degeneracy condition (see [C,C]), one has to modify the collision integral in order to take into account quantum effects. For example, it is necessary for some light atoms at very low tmperature like ^3H_e, or for a gas of electrons in a metall (but in this case, the gas is dense and of course, one must add electromagnetic forces to get a realistic description). Two particles which interact are not any more uncorrelated before and after collision (this is a consequence of Pauli's exclusion principle). One can show that equation (2) has to be replaced by

$$\partial_t f + \xi \cdot \partial_x f = C(f) \tag{3}$$

with

$$C(f) = \iint_{\mathbb{R}^d \times S^{N-1}} B(\xi - \xi_*, \omega)(f' f'_*(1-\varepsilon f)(1-\varepsilon f_*) - f f_*(1-\varepsilon f')(1-\varepsilon f'_*))\, d\xi_* d\omega\ ,$$

where ε is a positive constant, proportional to h^3 (h is Planck's constant). Let us notice that (3) reduces to (2) when $\varepsilon = 0$. In the general case (when Sommerfeld's degeneracy condition is not satisfied), equation (3) remains true for gases of Fermi-Dirac particles (εf is very small and therefore $C(f)$ is approximatively equal to $Q(f, f)$) : equation (3) provides a good approximation for the Boltzmann equation (2).

In this paper, we shall give existence and uniqueness results for the Cauchy problem associated to (3) in $\mathbb{R}^d$. We shall study the conservation of macroscopic quantities : mass, kinetic momentum and kinetic energy, and also the evolution of the entropy; letting the small parameter ε tend to 0, we shall prove that, up to the extraction of a subsequence, a sequence of solutions of (3) indexed by ε gives at the limit a solution of (2) in the sense of R. DiPerna and P-L. Lions.

Indeed, it is a natural assumption to ask that, when the quantic parameter tends to 0, we get at the limit a solution of the classical problem. The interest of this method of approximation is due to the fact that every solution of (3) has a natural $L^\infty-$bound, which of course depends on ε, while the solutions of (2) do not have any natural $L^\infty-$estimate.

Finally, in the last section, we shall give some indications on equilibrium states of equation (3).

Notations : In this paper, we denote the derivative with respect to the time t by ∂_t and the gradient with respect to the position x by ∂_x. We do not specify the target space for the functional spaces when it is $\mathbb{R}$:
$L^p(\mathbb{R}^d) = L^p(\mathbb{R}^d, \mathbb{R})$. $L^p(\mathbb{R}^+_{\text{loc}} \times \mathbb{R}^d \times \mathbb{R}^d)$ means $L^p([0, T] \times \mathbb{R}^d \times \mathbb{R}^d)$ for all $T > 0$. χ_A is the characteristic function of the set A.

2. Existence and uniqueness results

In this section, the parameter ε is assumed to be a strictly positive real constant. We want to solve the Cauchy problem :

$$\begin{array}{l} \partial_t f + \xi \cdot \partial_x f = C(f) \\ f|_{t=0} = f_0 \end{array} \tag{4}$$

Let us assume that the following assumptions are satisfied :

(i) the cross section satisfies (this is a very strong assumption)

$$B \in L^1(\mathbb{R}^d \times S^{N-1}) \quad \text{and} \quad B \geq 0 \text{ a.e.} \tag{5}$$

(ii) the initial data satisfies

$$f_0 \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d) \quad \text{and} \quad 0 \leq f_0 \leq \varepsilon^{-1} \text{ a.e.} \tag{6}$$

Theorem 1 : *Under assumptions (5) and (6), (4) has a unique solution f satisfying*

$$f \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d) \quad \text{and} \quad 0 \leq f \leq \varepsilon^{-1} \text{ a.e.} \tag{7}$$

Moreover, f is absolutely continuous with respect to t.

First, let us mention a classical result of linear transport theory (see [DP,L 1]).

Lemma 1 : *Let $f, h \in L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d)$. f is solution of*

$$\partial_t f + \xi \cdot \partial_x f = h \quad \text{in } D'(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d)$$

if and only if for almost all $(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$, $f(t,x,\xi)^\sharp$ is absolutely continuous with respect to t, $h(t,x,\xi)^\sharp \in L^1_{\text{loc}}(\mathbb{R})$, and

$$f(t_2,x,\xi) - f(t_1,x,\xi) = \int_{t_1}^{t_2} h(s, x-(s-t)\xi, \xi)\, ds \quad \forall\, (t_1,t_2) \in \mathbb{R}^2 \quad .$$

Here $g^\sharp$ denotes, for any function g measurable on $\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$, the following measurable function :

$$g^\sharp(t,x,\xi) = g(t, x-t\xi, \xi) \quad .$$

Proof : It is enough to prove the theorem with $\varepsilon = 1$. Indeed, if we replace f by (εf), f_0 by (εf_0) and B by $(\varepsilon^{-1}B)$, the general case is reduced to the case $\varepsilon = 1$. In the following, we shall assume that $\varepsilon = 1$.

There exists $\theta > 0$ such that

$$\begin{aligned} &\partial_t f + \xi \cdot \partial_x f = C(\overline{f}) \\ &f|_{t=0} = f_0 \end{aligned} \tag{8}$$

has a unique solution f satisfying

$$f \in L^\infty([0,\theta] \times \mathbb{R}^d \times \mathbb{R}^d) \quad \text{and} \quad 0 \le f \le 1 \text{ a.e.} \tag{9}$$

Here $\overline{f}$ is defined as follows :

$$\begin{aligned} \overline{f}(t,x,\xi) = f(t,x,\xi) \qquad &\text{if} \quad 0 \le f(t,x,\xi) \le 1 \quad , \\ \overline{f}(t,x,\xi) = 0 \qquad &\text{if} \quad f(t,x,\xi) \le 0 \quad , \\ \overline{f}(t,x,\xi) = 1 \qquad &\text{if} \quad f(t,x,\xi) \ge 1 \quad . \end{aligned}$$

Indeed, according to lemma 1, a function satisfying assumption (9) is solution of (8) if and only if f is a fixed-point of the nonlinear operator T defined on $L^\infty([0,\theta] \times \mathbb{R}^d \times \mathbb{R}^d)$ by setting

$$Tf(t,x,\xi) = f_0(x - t\xi, \xi) + \int_0^t C(\overline{f})(s, x - (s-t)\xi, \xi)\, ds \quad ,$$

for all $t \in [0,\theta]$ and for almost all $(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$.

The operator T is a Lipschitz operator. Indeed, let us consider $h_1, h_2 \in L^\infty([0,\theta] \times \mathbb{R}^d \times \mathbb{R}^d)$.

$$\begin{aligned} &\|Th_1 - Th_2\|_{L^\infty([0,\theta]\times\mathbb{R}^d\times\mathbb{R}^d)} \\ &= \| \quad \int_0^t ds \int\int_{\mathbb{R}^d \times S^{N-1}} B(\xi - \xi_*, \omega) \cdot \Big(F((\overline{h_1}), (\overline{h_1})_*, (\overline{h_1})', (\overline{h_1})'_*)^\sharp \\ &\qquad - F((\overline{h_2}), (\overline{h_2})_*, (\overline{h_2})', (\overline{h_2})'_*)^\sharp \Big)\, d\xi_* d\omega \quad \|_{L^\infty([0,\theta]\times\mathbb{R}^d\times\mathbb{R}^d)} \quad , \end{aligned}$$

with

$$F(x^1,x^2,x^3,x^4) = x^3 x^4 (1-x^1)(1-x^2) - x^1 x^2 (1-x^3)(1-x^4)$$

$$\forall\, x = (x^1,x^2,x^3,x^4) \in [0,1]^4 \quad .$$

But

$$\sup_{x\in[0,1]^4} \left|\frac{\partial F}{\partial x^i}\right| \;\le\; 2 \quad \forall\, i = 1,2,3,4 \quad ,$$

and therefore

$$\begin{aligned}\|Th_1 - Th_2 \quad &\|_{L^\infty([0,\theta]\times\mathbb{R}^d\times\mathbb{R}^d)} \\ \le \| \quad & \int_0^t ds \int\int_{\mathbb{R}^d\times S^{N-1}} B(\xi-\xi_*,\omega) \\ & \cdot 2\Big(|(h_1)-(h_2)|^\sharp + |(h_1)_* - (h_2)_*|^\sharp + \\ & |(h_1)' - (h_2)'|^\sharp + |(h_1)'_* - (h_2)'_*|^\sharp \Big)\, d\xi_* d\omega \quad \|_{L^\infty} \\ \le \; & 8\theta b \cdot \|h_1 - h_2\|_{L^\infty([0,\theta]\times\mathbb{R}^d\times\mathbb{R}^d)}\end{aligned}$$

with

$$b = \|B\|_{L^1(\mathbb{R}^d\times S^{N-1})} \quad .$$

Finally if

$$\theta < \frac{1}{8b} \quad ,$$

T is a contracting operator.

Moreover, for all f in $L^\infty([0,\theta]\times\mathbb{R}^d\times\mathbb{R}^d)$, we have

$$\begin{array}{rcl} -b\cdot\max(f,0) & = -b\; \cdot\overline{f} \;\le & C(\overline{f}) \quad , \\ C(\overline{f}) & \le \; b\; \cdot(1-\overline{f}) \; = & b\cdot(1-\min(1,f)) \quad . \end{array}$$

This ensures that

$$-b\cdot\max(f,0) \quad \le \quad \partial_t(Tf) + \xi\cdot\partial_x(Tf) \quad \le \quad b\cdot(1-\min(1,f))$$

and

$$\begin{aligned}\partial_t\Big(\max((Tf)^\sharp,0)\cdot e^{bt}\Big) &\ge 0 \\ \partial_t\Big((1-\min(1,(Tf)^\sharp))\cdot e^{bt}\Big) &\ge 0\end{aligned}$$

Finally, for all $t\in[0,\theta]$ and for almost all $(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d$, we have :

$$0 \le f_0(x-t\xi,\xi)\cdot e^{-bt} \le (Tf)(t,x,\xi) \le 1-(1-f_0(x-t\xi,\xi))\cdot e^{-bt} \; . \quad (10)$$

The set $\{\; f\in L^\infty([0,\theta]\times\mathbb{R}^d\times\mathbb{R}^d) \mid 0\le f\le 1 \text{ a.e. } \}$ is stable under the action of T. This proves that (8) has a unique solution satisfying assumption (9), and we have :

$$f(t,x,\xi) = f_0(x-t\xi,\xi) + \int_0^t C(f)(s,x-(s-t)\xi,\xi)\,ds$$

$$\forall\, t\in[0,\theta] \quad (x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d \text{ a.e.} \quad .$$

According to lemma 1, the solution of (8) is absolutely continuous with respect to t : $f(\theta, ., .)$ is making sense and verifies the same conditions as f_0 because of inequalities (10). Iterating the previous method, we get a solution of equation (8) in $L^\infty(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d)$ which is also a solution of (4) satisfying (7) because (10) ensures that $\overline{f} = f$. Theorem 1 is therefore proved and we have

$$f(t,x,\xi) = f_0(x - t\xi, \xi) + \int_0^t C(f)(s, x - (s-t)\xi, \xi)\, ds \tag{11}$$

$$\forall\, t \in \mathbb{R}^+ \quad (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d \text{ a.e.} \quad .$$

Remark 1 : One can notice that f and $(\varepsilon^{-1} - f)$ play the same role : if f solves (4) with initial data f_0, $(\varepsilon^{-1} - f)$ solves (4) with initial data $(\varepsilon^{-1} - f_0)$. This explains why 0 and ε^{-1} are natural bounds.

3. Conservation of mass, kinetic momentum and kinetic energy

We are now interested in conserved integral quantities associated to (4). Physical cross sections are generally supposed to verify

$$B(\xi,\omega) = q(|\xi|, |\xi.\omega|) \quad \forall\, (\xi,\omega) \in \mathbb{R}^d \times S^{N-1} \text{ a.e.} \tag{12}$$

where q is a function defined on $\mathbb{R}^+ \times \mathbb{R}^+$. As in section 2, the conservation of mass, kinetic momentum and kinetic energy is essentially a fixed-point result.

3.1 Conservation of mass

Proposition 1 : *Let us assume that the initial data f_0 satisfies (6), belongs to $L^1(\mathbb{R}^d \times \mathbb{R}^d)$, and that the cross-section satisfies assumptions (5) and (12). Then the solution of (4) given in theorem 1 belongs to $C^0(\mathbb{R}^+, L^1(\mathbb{R}^d \times \mathbb{R}^d))$ and satisfies*

$$\int\!\!\int_{\mathbb{R}^d \times \mathbb{R}^d} f(t,x,\xi)\, dx d\xi \quad = \quad \int\!\!\int_{\mathbb{R}^d \times \mathbb{R}^d} f_0(x,\xi)\, dx d\xi \quad \forall\, t \in \mathbb{R}^+ \quad . \tag{13}$$

Proof : As for the proof of theorem 1, let us assume that $\varepsilon = 1$ and consider the operator T. For all h_1, h_2 in

$$\{ \ f \in L^\infty([0,\theta] \times \mathbb{R}^d \times \mathbb{R}^d) \cap C^0([0,\theta], L^1(\mathbb{R}^d \times \mathbb{R}^d)) \mid 0 \le f \le 1 \text{ a.e. } \} \quad ,$$

equation (11) ensures that the following estimate holds

$$\begin{aligned} &\sup_{t\in[0,\theta]} \|Th_1(t,.,.) - Th_2(t,.,.)\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)} \\ &\le \ 8\theta b \cdot \sup_{t\in[0,\theta]} \|h_1(t,.,.) - h_2(t,.,.)\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)} \quad , \end{aligned} \tag{14}$$

which proves that the solution of equation (4) given in theorem 1 belongs to $C^0(\mathbb{R}^+, L^1(\mathbb{R}^d \times \mathbb{R}^d))$. Let us denote by f this solution. If f belongs to $C^0(\mathbb{R}^+, L^1(\mathbb{R}^d \times \mathbb{R}^d))$, $C(f)$ belongs also to $C^0(\mathbb{R}^+, L^1(\mathbb{R}^d \times \mathbb{R}^d))$, and by Fubini's theorem, we get for all $t \in [0,\theta]$

$$\begin{aligned} \int\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,\xi)\; dxd\xi \ &= \int\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,\xi)\; dxd\xi \\ &\quad + \int_0^t ds \int\int\int\int_{(\mathbb{R}^d)^3\times S^{N-1}} B(\xi-\xi_*,\omega) \\ &\qquad \cdot (f' f'_*(1-f)(1-f_*) - f f_*(1-f')(1-f'_*)) dxd\xi d\xi_* d\omega. \end{aligned}$$

Using the change of variables $(\xi, \xi_*) \to (\xi', \xi'_*)$ and according to (12), we get for all $t \in [0,\theta]$

$$\int_0^t ds \int\int\int\int_{(\mathbb{R}^d)^3\times S^{N-1}} B(\xi-\xi_*,\omega)(f' f'_*(1-f)(1-f_*) - f f_*(1-f')(1-f'_*)) dxd\xi d\xi_* d\omega = 0,$$

which proves (13).

3.2 Conservation of kinetic momentum

Proposition 2 : *Let us assume that the initial data f_0 satisfies (6) and*

$$\int\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,\xi).(|x|^2 + |\xi|^2)\; dxd\xi \quad < \quad +\infty \quad . \tag{15}$$

Let us assume also that the cross-section satisfies (5) and (12).

Then the solution of equation (4) given in theorem 1 is such that the function

$$(t,x,\xi) \mapsto f(t,x,\xi).|x|^2$$

belongs to $C^0(\mathbb{R}^+, L^1(\mathbb{R}^d \times \mathbb{R}^d))$ *and satisfies*

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,\xi).|x|^2 \, dxd\xi = \iint_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,\xi).|x+t\xi|^2 \, dxd\xi \quad \forall\, t \in \mathbb{R}^+ .$$

Proof : The proof follows that of proposition 1 (we assume that $\varepsilon = 1$). It is based on inequality (16), which replaces inequality (14) :

$$\begin{array}{l} \sup_{t\in[0,\theta]} \| \quad (Th_1(t,.,.) - Th_2(t,.,.)).|x|^2 \quad \|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)} \\ \qquad \leq \; 8\theta b \cdot \sup_{t\in[0,\theta]} \| \quad (h_1(t,.,.) - h_2(t,.,.)).|x|^2 \quad \|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)} \quad , \end{array} \tag{16}$$

for convenient h_1, h_2. Once again we use the change of variables $(\xi, \xi_*) \mapsto (\xi', \xi'_*)$ and property (12) to conclude.

3.3 Conservation of kinetic energy

Proposition 3 : *Let us assume that the initial data* f_0 *satisfies (6) and*

$$\int\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,\xi).|\xi|^2 \, dxd\xi \quad < \quad +\infty \quad .$$

Let us assume also that the cross-section satisfies assumptions (5) and (12).

Then the solution of equation (4) given in theorem 1 is such that the function

$$(t,x,\xi) \mapsto f(t,x,\xi).|\xi|^2$$

belongs to $C^0(\mathbb{R}^+, L^1(\mathbb{R}^d \times \mathbb{R}^d))$ *and satisfies*

$$\int\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,\xi).|\xi|^2 \, dxd\xi = \int\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,\xi).|\xi|^2 \, dxd\xi \quad \forall\, t \in \mathbb{R}^+ . \tag{17}$$

Once again, the proof follows that of proposition 1 (in th following, we assume that $\varepsilon = 1$). Only the fixed-point inequality is a little more difficult to establish (see lemma 2). The proof is a straightforward consequence of the following lemmata :

Lemma 2 : *Let us consider an initial data* f_0 *satisfying (6) and*

$$\int\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,\xi).(1+|\xi|^2) \, dxd\xi \quad < \quad +\infty \quad ,$$

and assume that the cross-section satisfies (5) and (12), and

$$a \;=\; \int\int_{\mathbb{R}^d\times S^{N-1}} B(\xi,\omega).|\xi|^2\, d\xi d\omega \quad < \quad \infty \quad .$$

Then the solution of equation (4) given in theorem 1 is such that the function

$$(t,x,\xi)\mapsto f(t,x,\xi).|\xi|^2$$

belongs to $C^0(\mathbb{R}^+, L^1(\mathbb{R}^d\times\mathbb{R}^d))$ *and satisfies*

$$\int\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,\xi).|\xi|^2\, dxd\xi = \int\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,\xi).|\xi|^2\, dxd\xi \quad \forall\, t\in\mathbb{R}^+ \;.$$

Lemma 3 : *Let us consider* $(f_0^n)_{n\in\mathbb{N}}$ *and* $(B^n)_{n\in\mathbb{N}}$ *such that for all* $n\in\mathbb{N}$, f_0^n *and* B^n *satisfy assumptions (5) and (6), and denote* f^n *the solution of*

$$\begin{aligned} &\partial_t f^n + \xi\cdot\partial_x f^n = C^n(f^n)\\ &f^n|_{t=0} = f_0^n \end{aligned}$$

where C^n *is the collision kernel associated to* B^n.

If $(f_0^n)_{n\in\mathbb{N}}$ *converges to* f_0 *in* $L^1(\mathbb{R}^d\times\mathbb{R}^d)$ *and* $(B^n)_{n\in\mathbb{N}}$ *converges to* B *in* $L^1(\mathbb{R}^d\times S^{N-1})$, *then* f^n *converges in* $C^0(\mathbb{R}^+_{\text{loc}}, L^1(\mathbb{R}^d\times\mathbb{R}^d))$ *to the solution* f *of equation (4) with initial data* f_0, *where* C *is the collision kernel associated to* B.

Proof of proposition 3 : Let us define $(B^n)_{n\in\mathbb{N}}$ by setting

$$B^n = B\cdot\chi_{|\xi|<n} \quad ,$$

and assume

$$f_0^n = f_0 \quad .$$

According to lemma 2,we have

$$\int\int_{\mathbb{R}^d\times\mathbb{R}^d} f^n(t,x,\xi).|\xi|^2\, dxd\xi \quad = \quad \int\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,\xi).|\xi|^2\, dxd\xi \quad ,$$

According to lemma 3, we can assume that, after extraction of a subsequence if necessary,

$$f^n(t,x,\xi).|\xi|^2 \to f(t,x,\xi).|\xi|^2 \quad \text{a.e.}$$

and by Fatou's lemma

$$\begin{array}{l}\int\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,\xi).|\xi|^2\,dxd\xi \\ \leq \liminf_{n\to+\infty}\int\int_{\mathbb{R}^d\times\mathbb{R}^d} f^n(t,x,\xi).|\xi|^2\,dxd\xi \\ \qquad = \quad \int\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,\xi).|\xi|^2\,dxd\xi \quad .\end{array}$$

As a consequence, the function

$$(t,x,\xi)\mapsto|\xi|^2\cdot C(f)(t,x,\xi)$$

belongs to $L^\infty(\mathbb{R}^+,L^1(\mathbb{R}^d\times\mathbb{R}^d))$. Indeed

$$\begin{array}{rcl}|\xi|^2.C_-(f)(t,x,\xi) & \leq & |\xi|^2.\int\int_{\mathbb{R}^d\times S^{N-1}} B(\xi-\xi_*,\omega)\,f(t,x,\xi)\,d\xi_*d\omega\cdot\varepsilon^{-1} \\ & \leq & |\xi|^2.f(t,x,\xi)\cdot b\varepsilon^{-1} \quad , \\ |\xi|^2.C_+(f)(t,x,\xi) & \leq & |\xi|^2.\int\int_{\mathbb{R}^d\times S^{N-1}} B(\xi-\xi_*,\omega)\,f'f'_*\,d\xi_*d\omega \\ & \leq & \int\int_{\mathbb{R}^d\times S^{N-1}} B(\xi-\xi_*,\omega)\,(f'|\xi'|^2+f'_*|\xi'_*|^2)\,d\xi_*d\omega\cdot\varepsilon^{-1},\end{array}$$

because

$$|\xi|^2\leq|\xi|^2+|\xi_*|^2=|\xi'|^2+|\xi'_*|^2 \quad .$$

Using the change of variables $(\xi,\xi_*)\to(\xi',\xi'_*)$, we get

$$\int_0^t ds\int\int_{\mathbb{R}^d\times\mathbb{R}^d} C(f)(s,x-s\xi,\xi).|\xi|^2\,dxd\xi \quad = \quad 0 \quad ,$$

and according to (11), we get (17).

Proof of lemma 2 : Let us consider $h_1,h_2\in L^\infty([0,\theta]\times\mathbb{R}^d\times\mathbb{R}^d)\cap C^0([0,\theta],L^1(\mathbb{R}^d\times\mathbb{R}^d))$ such that

$$\int\int_{\mathbb{R}^d\times\mathbb{R}^d} h_i(t,x,\xi).|\xi|^2\,dxd\xi \quad < \quad +\infty \quad \forall\,t\in[0,\theta] \quad (i=1,2) \quad .$$

We have

$$\begin{array}{l}|\,(Th_1 \;\; -Th_2).|\xi|^2\,| \\ \qquad \leq \quad 4\theta\int\int_{\mathbb{R}^d\times S^{N-1}} B(\xi-\xi_*,\omega)\;\cdot(|h'_2-h'_1|+|(h_2)'_*-(h_1)'_*| \\ \qquad\qquad +|h_2-h_1|+|(h_2)_*-(h_1)_*|).|\xi|^2\,d\xi_*d\omega \quad \forall\,t\in[0,\theta] \quad .\end{array}$$

Using the changes of variables

$$(\xi,\xi_*)\mapsto(\xi_*,\xi) \quad (\xi,\xi_*)\mapsto(\xi',\xi'_*) \quad (\xi,\xi_*)\mapsto(\xi'_*,\xi') \quad , \qquad (18)$$

we get successively

$$\begin{aligned}
&\int\int\int\int_{(\mathbb{R}^d)^3\times S^{N-1}} B(\xi-\xi_*,\omega)\cdot(|h_2'-h_1'|+|(h_2)_*'-(h_1)_*'| \\
&\qquad +|h_2-h_1|+|(h_2)_*-(h_1)_*|)\cdot|\xi|^2\, dxd\xi d\xi_* d\omega \\
&= \tfrac{1}{2}\int\int\int\int_{(\mathbb{R}^d)^3\times S^{N-1}} B(\xi-\xi_*,\omega)\ \cdot(|h_2'-h_1'|+|(h_2)_*'-(h_1)_*'| \\
&\qquad +|h_2-h_1|+|(h_2)_*-(h_1)_*|)\cdot(|\xi|^2+|\xi_*|^2)\, dxd\xi d\xi_* d\omega \\
&= \int\int\int\int_{(\mathbb{R}^d)^3\times S^{N-1}} B(\xi-\xi_*,\omega) \\
&\qquad \cdot(|h_2-h_1|+|(h_2)_*-(h_1)_*|)\cdot(|\xi|^2+|\xi_*|^2)\, dxd\xi d\xi_* d\omega \\
&= 2\int\int\int\int_{(\mathbb{R}^d)^3\times S^{N-1}} B(\xi-\xi_*,\omega)\cdot|h_2-h_1|\cdot(|\xi|^2+|\xi_*|^2)\, dxd\xi d\xi_* d\omega\ .
\end{aligned}$$

But

$$|\xi_*|^2 \le 2|\xi-\xi_*|^2+2|\xi|^2\ ,$$

therefore

$$\begin{aligned}
&\int\int\int\int_{(\mathbb{R}^d)^3\times S^{N-1}} B(\xi-\xi_*,\omega)\cdot|h_2-h_1|\cdot|\xi_*|^2\, dxd\xi d\xi_* d\omega \\
&\le 2a\int\int_{\mathbb{R}^d\times\mathbb{R}^d}|h_2-h_1|\, dxd\xi + 2b\int\int_{\mathbb{R}^d\times\mathbb{R}^d}|h_2-h_1|\cdot|\xi|^2\, dxd\xi\ ,
\end{aligned}$$

and finally

$$\begin{aligned}
&\sup_{t\in[0,\theta]}\int\int_{\mathbb{R}^d\times\mathbb{R}^d} |\ (Th_1-Th_2).(1+|\xi|^2)\ |\, dxd\xi \\
\le\quad & 16\theta a\cdot\int\int_{\mathbb{R}^d\times\mathbb{R}^d}|h_2-h_1|\, dxd\xi + 12\theta b\cdot\int\int_{\mathbb{R}^d\times\mathbb{R}^d}|h_2-h_1|.|\xi|^2\, dxd\xi \\
\le\quad & 8\theta\max(2a,3b)\cdot\int\int_{\mathbb{R}^d\times\mathbb{R}^d} |\ h_2\cdot(1+|\xi|^2)-h_1\cdot(1+|\xi|^2)\ |\, dxd\xi.
\end{aligned}$$

Once again we use the change of variables $(\xi,\xi_*)\mapsto(\xi',\xi_*')$ and property (12) to conclude.

Proof of lemma 3 : According to (11), let us compute directly $\|f^n - f\|_{L^\infty\left([0,\theta],L^1(\mathbb{R}^d\times\mathbb{R}^d)\right)}$

$$\begin{aligned}
&\sup_{t\in[0,\theta]}\int\int_{\mathbb{R}^d\times\mathbb{R}^d}|f^n-f|\, dxd\xi \\
\le\quad & \int\int_{\mathbb{R}^d\times\mathbb{R}^d}|f_0^n-f_0|\, dxd\xi \\
& +2\theta\cdot\|B^n-B\|_{L^1(\mathbb{R}^d\times S^{N-1})}\cdot\int\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,\xi)\, dxd\xi \\
& +8\theta\Big(b+\|B^n-B\|_{L^1(\mathbb{R}^d\times S^{N-1})}\Big)\cdot\sup_{t\in[0,\theta]}\int\int_{\mathbb{R}^d\times\mathbb{R}^d}|f^n-f|\, dxd\xi,
\end{aligned}$$

$$\begin{aligned}
\|f^n &-f\|_{L^\infty\left([0,\theta],L^1(\mathbb{R}^d\times\mathbb{R}^d)\right)} \\
&\le \frac{1}{1-8\theta\left(b+\|B^n-B\|_{L^1(\mathbb{R}^d\times S^{N-1})}\right)}\ \cdot\Big(\|f_0^n-f_0\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)} \\
&\qquad +2\theta\|B^n-B\|_{L^1(\mathbb{R}^d\times S^{N-1})}\cdot\int\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,\xi)\, dxd\xi\Big) \\
&\qquad\qquad \to 0\quad ,
\end{aligned}$$

when n goes to infinity (for θ small enough). It is not difficult to iterate the method and prove for all $M \in \mathbb{N}$ that

$$\begin{aligned} \|f^n & - f\|_{L^\infty([0,M\theta],L^1(\mathbb{R}^d\times\mathbb{R}^d))} \\ & \leq \left(\frac{1}{1-8\theta\left(b+\|B^n-B\|_{L^1(\mathbb{R}^d\times S^{N-1})}\right)}\right)^M \cdot \Big(\|f_0^n - f_0\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)} \\ & \quad +2\theta\|B^n - B\|_{L^1(\mathbb{R}^d\times S^{N-1})} \cdot \int\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,\xi)\, dxd\xi\Big) \\ & \qquad \to 0 \quad , \end{aligned}$$

when n goes to infinity.

Remark 2 : Using the same methods as in the proofs of propositions 2 and 3, one can prove (under assumptions of proposition 2) that

$$\int\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,\xi).(x.\xi)\, dxd\xi = \int\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,\xi).(x+t\xi.\xi)\, dxd\xi \quad \forall\, t \in \mathbb{R}^+,$$

and then

$$\int\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,\xi).|x-t\xi|^2\, dxd\xi = \int\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,\xi).|x|^2\, dxd\xi \quad \forall\, t \in \mathbb{R}^+. \tag{19}$$

4. Decrease of the entropy : the H theorem

In this section, we give some results about the entropy $s(f)$ where s is a real function defined by setting

$$s(\tau) = \tau\log\tau + \frac{1}{\varepsilon}(1-\varepsilon\tau)\log(1-\varepsilon\tau) \quad .$$

Let us define $e(f)$ by setting

$$\begin{aligned} e(f) = \tfrac{1}{4}\int\int_{\mathbb{R}^d\times S^{N-1}} B(\xi-\xi_*,\omega) \quad & \left(f'f'_*(1-\varepsilon f)(1-\varepsilon f_*) - ff_*(1-\varepsilon f')(1-\varepsilon f'_*)\right) \\ & \cdot\log\left(\frac{f'f'_*(1-\varepsilon f)(1-\varepsilon f_*)}{ff_*(1-\varepsilon f')(1-\varepsilon f'_*)}\right) d\xi_* d\omega \quad . \end{aligned}$$

Proposition 4 : *Let us assume that assumptions (5), (6), (12) and (15) are satisfied, as in proposition 2. Then the solution given in theorem 1 satisfies*

$$\begin{aligned} & \int\int_{\mathbb{R}^d\times\mathbb{R}^d} |f(t,x,\xi)\log f(t,x,\xi)|\, dxd\xi \\ \leq & \quad \frac{C(N)}{\varepsilon} + \int\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,\xi).(|\log\varepsilon| + |x|^2 + |\xi|^2)\, dxd\xi \quad \forall\, t \in \mathbb{R}^+, \end{aligned} \tag{20}$$

where $C(N)$ is a nonnegative constant which depends only on N. As a consequence,

$$s(f) \in L^\infty(\mathbb{R}^+, L^1(\mathbb{R}^d \times \mathbb{R}^d)) \quad . \tag{21}$$

Moreover

$$s(f) \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^d \times \mathbb{R}^d)) \quad . \tag{22}$$

$$e(f) \in L^1(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d) \quad , \tag{23}$$

and we have the following assertion :

$$\begin{array}{l} \int\int_{\mathbb{R}^d\times\mathbb{R}^d} s(f)(t,x,\xi)\ dxd\xi + \int_0^t \int\int_{\mathbb{R}^d\times\mathbb{R}^d} e(f)(s,x,\xi)\ dsdxd\xi \\ = \int\int_{\mathbb{R}^d\times\mathbb{R}^d} s(f_0)(x,\xi)\ dxd\xi \quad \forall\, t \in \mathbb{R}^+ \quad . \end{array} \tag{24}$$

Proof : First, let us prove (20) and (21). The basic tool is the following classical lemma :

Lemma 4 : *Let us consider $h \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that*

$$\int\int_{\mathbb{R}^d\times\mathbb{R}^d} h(x,\xi).(|x|^2 + |\xi|^2)\ dxd\xi \quad < \quad +\infty \quad ,$$

and

$$0 \le h(x,\xi) \le 1 \qquad (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d \quad \text{a.e.} \quad .$$

Then $h \log h \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$ and for all $t \in \mathbb{R}$, we have

$$\begin{array}{ll} \int\int_{\mathbb{R}^d\times\mathbb{R}^d} & |h(x,\xi)\log h(x,\xi)|\ dxd\xi \\ & \le \quad C(N) + \int\int_{\mathbb{R}^d\times\mathbb{R}^d} h(x,\xi).(|x-t\xi|^2 + |\xi|^2)\ dxd\xi \quad . \end{array}$$

Let us notice that in this lemma, t is only a real parameter.

Let us choose $t \in \mathbb{R}^+$ and apply lemma 4 with

$$h(x,\xi) = \varepsilon f(t,x,\xi) \quad .$$

$$\begin{array}{ll} \int\int_{\mathbb{R}^d\times\mathbb{R}^d} & |(\varepsilon f)(x,\xi)\log(\varepsilon f)(x,\xi)|\ dxd\xi \\ & \le \quad C(N) + \int\int_{\mathbb{R}^d\times\mathbb{R}^d} (\varepsilon f)(x,\xi).(|x-t\xi|^2 + |\xi|^2)\ dxd\xi \ , \end{array}$$

$$\begin{array}{l} \int\int_{\mathbb{R}^d\times\mathbb{R}^d} |f(t,x,\xi)\log f(t,x,\xi)|\ dxd\xi \\ \le \quad \frac{C(N)}{\varepsilon} + \int\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,\xi).(|\log\varepsilon| + |x-t\xi|^2 + |\xi|^2)\ dxd\xi \ . \end{array}$$

(20) is a straightforward consequence of (13), (17) and (19).

Now, for all $\tau \in]0, \varepsilon^{-1}[$, we have

$$\tau\log\tau - \tau \le s(\tau) \le \tau\log\tau \quad ,$$

and therefore

$$|s(\tau)| \le |\tau \log \tau| + \tau \quad ,$$

which, according to (20), proves (21).

Let us prove (23) and (24). First, let us assume that

$$\eta e^{-(|x|^2+|\xi|^2)} \le \varepsilon f_0(x,\xi) \le 1-\eta \qquad \forall\,(x,\xi)\in \mathbb{R}^d\times\mathbb{R}^d \quad , \tag{25}$$

for some $\eta > 0$. (10) ensures that for all $t \in \mathbb{R}^+$, for almost all $(x,\xi) \in \mathbb{R}^d\times\mathbb{R}^d$, we have

$$\eta e^{-(|x-t\xi|^2+|\xi|^2+\frac{bt}{\varepsilon})} \le \varepsilon f(t,x,\xi) \le 1-\eta e^{-\frac{bt}{\varepsilon}} \quad ,$$

$$|\log(\frac{f}{1-\varepsilon f})| \le |\log\eta| + (|x-t\xi|^2+|\xi|^2+\frac{bt}{\varepsilon}) + |\log\varepsilon| \quad . \tag{26}$$

It is then easy to prove that $s(f)$ is solution of

$$\begin{aligned}&\partial_t s(f) + \xi\cdot\partial_x s(f) = C(f)\cdot\log(\tfrac{f}{1-\varepsilon f})\\ &s(f)|_{t=0} = s(f_0)\end{aligned}$$

and we have for all $t\in\mathbb{R}^+$, for almost all $(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d$

$$s(f)(t,x,\xi) = s(f_0)(x-t\xi,\xi) + \int_0^t C(f)\cdot\log(\frac{f}{1-\varepsilon f})(s,x-(s-t)\xi,\xi)\,ds \quad . \tag{27}$$

According to (26), the function $(t,x,\xi)\mapsto C(f).\log(\frac{f}{1-\varepsilon f})(t,x,\xi)$ belongs to $L^1(\mathbb{R}^+_{\text{loc}}\times\mathbb{R}^d\times\mathbb{R}^d)$. Using the changes of variables (18)

$$(\xi,\xi_*)\mapsto(\xi_*,\xi),\quad (\xi,\xi_*)\mapsto(\xi',\xi'_*),\quad (\xi,\xi_*)\mapsto(\xi',\xi'_*),$$

we get

$$\begin{aligned}&\int\int_{\mathbb{R}^d\times\mathbb{R}^d} dxd\xi \int_0^t C(f)\cdot\log(\tfrac{f}{1-\varepsilon f})(s,x-(s-t)\xi,\xi)\,ds\\ &= -\tfrac14 \int\int\int\int\int_{[0,t]\times(\mathbb{R}^d)^3\times S^{N-1}} B(\xi-\xi_*,\omega)\big(f' f'_*(1-\varepsilon f)(1-\varepsilon f_*) - f f_*(1-\varepsilon f')(1-\varepsilon f'_*)\big)\\ &\qquad\cdot\log\left(\tfrac{f' f'_*(1-\varepsilon f)(1-\varepsilon f_*)}{f f_*(1-\varepsilon f')(1-\varepsilon f'_*)}\right) dsdxd\xi d\xi_* d\omega\\ &= -\int_0^t ds \int\int_{\mathbb{R}^d\times\mathbb{R}^d} e(f)(s,x,\xi)\,dxd\xi \quad ,\end{aligned}$$

$$\begin{aligned}&\int\int_{\mathbb{R}^d\times\mathbb{R}^d} s(f)(t,x,\xi)\,dxd\xi + \int_0^t ds \int\int_{\mathbb{R}^d\times\mathbb{R}^d} e(f)(s,x,\xi)\,dxd\xi\\ &= \int\int_{\mathbb{R}^d\times\mathbb{R}^d} s(f_0)(x,\xi)\,dxd\xi \quad \forall\, t\in\mathbb{R}^+ \quad .\end{aligned} \tag{28}$$

Now, we go back to the general case and consider the sequence $(f^n)_{n>2}$, where f^n is solution of

$$\begin{array}{l} \partial_t f^n + \xi \cdot \partial_x f^n = C(f^n) \\ f^n|_{t=0} = f_0^n \end{array}$$

and where f_0^n is defined by setting

$$f_0^n(x,\xi) = \frac{\varepsilon^{-1}}{n} e^{-(|x|^2+|\xi|^2)} + (1-\frac{2}{n}) f_0(x,\xi) \quad \forall\, (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d \quad .$$

According to lemma 3, the sequence $(f^n)_{n\in\mathbb{N}}$ converges to the solution f of (4) with initial data f_0. For all $n \in \mathbb{N}$, we have

$$\begin{array}{rl} |s(f_0^n)| \le f_0^n + |f_0^n \log f_0^n| \le & (1+|x|^2+|\xi|^2)e^{-(|x|^2+|\xi|^2)} \\ & +(1+\log(1+\varepsilon^{-1}) f_0 \in L^1(\mathbb{R}^d \times \mathbb{R}^d) \ . \end{array}$$

Lebesgue's theorem ensures that

$$\lim_{n\to+\infty} \int\int_{\mathbb{R}^d\times\mathbb{R}^d} s(f_0^n)(x,\xi)\; dxd\xi = \int\int_{\mathbb{R}^d\times\mathbb{R}^d} s(f_0)(x,\xi)\; dxd\xi \quad .$$

s is a convex function and $(x,y) \mapsto (x-y)\log(\frac{x}{y})$ is convex on $\mathbb{R}^+ \times \mathbb{R}^+$: for all $t \in \mathbb{R}^+$, we have

$$\int\int_{\mathbb{R}^d\times\mathbb{R}^d} s(f)(t,x,\xi)\; dxd\xi \le \liminf_{n\to+\infty} \int\int_{\mathbb{R}^d\times\mathbb{R}^d} s(f^n)(t,x,\xi)\; dxd\xi \quad ,$$

$$\begin{array}{l} 0 \le \int_0^t ds\ \int\int_{\mathbb{R}^d\times\mathbb{R}^d} e(f)(s,x-(t-s)\xi,\xi)\; dxd\xi \\ \le \liminf_{n\to+\infty} \int_0^t ds\ \int\int_{\mathbb{R}^d\times\mathbb{R}^d} e(f^n)(s,x-(t-s)\xi,\xi)\; dxd\xi \quad . \end{array}$$

Finally

$$\begin{array}{rl} \int\int_{\mathbb{R}^d\times\mathbb{R}^d} s(f)(t,x,\xi)\; dxd\xi + & \int_0^t ds\ \int\int_{\mathbb{R}^d\times\mathbb{R}^d} e(f)(s,x,\xi)\; dsdxd\xi \\ & \le \int\int_{\mathbb{R}^d\times\mathbb{R}^d} s(f_0)(x,\xi)\; dxd\xi \quad \forall\, t \in \mathbb{R}^+ \quad . \end{array}$$

Let us notice that $e(f)$ is nonnegative, and that for all $t \in \mathbb{R}^+$, we have, according to (20),

$$\begin{array}{l} \int_0^t ds\ \int\int_{\mathbb{R}^d\times\mathbb{R}^d} e(f)(s,x,\xi)\; dxd\xi \\ \le \int\int_{\mathbb{R}^d\times\mathbb{R}^d} (|s(f)| + |s(f_0)|)\; dxd\xi \\ \le \frac{2C(N)}{\varepsilon} + 2\int\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,\xi).(|\log\varepsilon| + |x|^2 + |\xi|^2)\; dxd\xi \quad , \end{array}$$

which proves assertion (23). The function $(t,x,\xi) \mapsto C(f).\log(\frac{f}{1-\varepsilon f})(t,x,\xi)$ belongs therefore to $L^1(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d)$. Assumption (25) is not any more necessary to get (27) and (28) : this proves assertions (22) and (24).

Proof of lemma 4 : We have

$$\tau \log \tau < 2\sqrt{\tau} \quad \forall\, \tau \in]0,1[\quad .$$

$$\begin{aligned}
&\int\int_{\mathbb{R}^d\times\mathbb{R}^d} |h(x,\xi)\log h(x,\xi)|\ dxd\xi \\
&= \int\int_{\mathbb{R}^d\times\mathbb{R}^d} h(x,\xi)\log(\tfrac{1}{h(x,\xi)})\ dxd\xi \\
&\leq \int\int_{0\leq h\leq e^{-(|x-t\xi|^2+|\xi|^2)}} 2\sqrt{h}\ dxd\xi \\
&\qquad\qquad + \int\int_{e^{-(|x-t\xi|^2+|\xi|^2)}\leq h\leq 1} h(x,\xi)(|x-t\xi|^2+|\xi|^2) \\
&\leq C(N) + \int\int_{\mathbb{R}^d\times\mathbb{R}^d} h(x,\xi)(|x-t\xi|^2+|\xi|^2)\ dxd\xi \quad ,
\end{aligned}$$

with

$$C(N) = 2\int\int_{\mathbb{R}^d\times\mathbb{R}^d} e^{-\frac{1}{2}(|x-t\xi|^2+|\xi|^2)}\ dxd\xi \quad ,$$

$$C(N) = 2^{N+1}\int\int_{\mathbb{R}^d\times\mathbb{R}^d} e^{-(|x|^2+|\xi|^2)}\ dxd\xi \quad .$$

Finally, let us mention a result which is useful to get a limit for $(f^{\varepsilon})_{\varepsilon<1}$ when ε tends to zero (see section 5).

Proposition 5 : *Under assumptions of proposition 4, we have, for all* $t \in \mathbb{R}^+$

$$\begin{aligned}
&\int\int_{\mathbb{R}^d\times\mathbb{R}^d} |f(t,x,\xi)\log f(t,x,\xi)|\ dxd\xi \\
\leq\quad & 2C(N) + \int\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,\xi).(1+2|x|^2+2|\xi|^2+|\log f_0(x,\xi)|)\ dxd\xi \quad .
\end{aligned}$$

Proof :

$$\begin{aligned}
&\int\int_{\mathbb{R}^d\times\mathbb{R}^d} |f\log f(t,x,\xi)|\ dxd\xi \\
&= \int\int_{\mathbb{R}^d\times\mathbb{R}^d} f\log f(t,x,\xi)\ dxd\xi - 2\int\int_{f\leq 1} f\log f(t,x,\xi)\ dxd\xi \\
&= \int\int_{\mathbb{R}^d\times\mathbb{R}^d} f\log f(t,x,\xi)\ dxd\xi + 2\int\int_{f\leq 1} |f\log f(t,x,\xi)|\ dxd\xi \quad .
\end{aligned}$$

Using lemma 4, we get

$$\begin{aligned}
\int\int_{f\leq 1}\quad & |f(t,x,\xi)\log f(t,x,\xi)|\ dxd\xi \\
\leq\quad & C(N) + \int\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,\xi).(|x-t\xi|^2+|\xi|^2)\ dxd\xi \quad ,
\end{aligned}$$

$$\begin{aligned}
&\int\int_{\mathbb{R}^d\times\mathbb{R}^d} |f(t,x,\xi)\log f(t,x,\xi)|\ dxd\xi \\
&\leq \int\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,\xi)\log f(t,x,\xi)\ dxd\xi \\
&\qquad +2C(N) + 2\int\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,\xi).(|x|^2+|\xi|^2)\ dxd\xi \quad .
\end{aligned}$$

But identity (24) ensures that

$$\begin{aligned}\iint_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,\xi)\log f(t,x,\xi)\;dxd\xi &\le \iint_{\mathbb{R}^d\times\mathbb{R}^d}(s(f)+f)(t,x,\xi)\;dxd\xi\\ &\le \iint_{\mathbb{R}^d\times\mathbb{R}^d}(s(f_0)+f_0)(x,\xi)\;dxd\xi\\ &\le \iint_{\mathbb{R}^d\times\mathbb{R}^d}(f_0\log f_0+f_0)(x,\xi)\;dxd\xi,\end{aligned}$$

and proposition 5 is proved.

5. The classical limit

In this section, we deal with the classsical limit. We prove that we can find a solution of the renormalized Boltzmann equation through the use of a sequence $(f^\varepsilon)_{\varepsilon<1}$ of solutions of equation (3). Let us assume that the following assumptions are satisfied (these are the assumptions of R. DiPerna and P-L. Lions in [DP-L])

$$f_0 \in L^1(\mathbb{R}^d\times\mathbb{R}^d)\;,\quad f_0\ge 0\quad \text{a.e.}\quad , \tag{29}$$

$$\int\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,\xi).(1+|x|^2+|\xi|^2)\;dxd\xi\quad <+\infty\quad , \tag{30}$$

$$\int\int_{\mathbb{R}^d\times\mathbb{R}^d} |f_0(x,\xi)\log f_0(x,\xi)|\;dxd\xi\quad <+\infty\quad , \tag{31}$$

$$B\in L^1_{\text{loc}}(\mathbb{R}^d\times\mathbb{R}^d)\quad \text{and}\quad B\ge 0\quad \text{a.e.}\quad , \tag{32}$$

$$B(\xi,\omega)=q(|\xi|,|\xi.\omega|)\quad (\xi,\omega)\in\mathbb{R}^d\times S^{N-1}\ \text{a.e.}\quad , \tag{33}$$

$$\int_{|\xi_*|<1} A(\xi-\xi_*)\;d\xi_*\quad = o(1+|\xi|^2)\quad \text{when}\quad |\xi|\to+\infty\quad , \tag{34}$$

where q is a function defined on $\mathbb{R}^+\times\mathbb{R}^+$ and $A\in L^1_{\text{loc}}(\mathbb{R}^d)$ verifies

$$\forall\,\xi\in\mathbb{R}^d\quad A(\xi)=\int_{S^{N-1}} B(\xi,\omega)\;d\omega\quad . \tag{35}$$

Now, let us define $(f_0^\varepsilon)_{\varepsilon<1}$ and $(B^\varepsilon)_{\varepsilon<1}$ by setting

$$f_0^\varepsilon=\min(f_0,\varepsilon^{-1})\quad , \tag{36}$$

$$B^\varepsilon = B\cdot\chi_{|\xi|<\varepsilon^{-1}}\quad . \tag{37}$$

f^ε is the unique solution (see theorem 1) in $L^\infty(\mathbb{R}^+_{\text{loc}}, L^1(\mathbb{R}^d\times\mathbb{R}^d))$ of

$$\begin{aligned}&\partial_t f+\xi\cdot\partial_x f = C^\varepsilon(f)\\ &f|_{t=0}=f_0\end{aligned} \tag{38}$$

with

$$C^\varepsilon(f) = \int\int_{\mathbb{R}^d\times S^{N-1}} B^\varepsilon(\xi-\xi_*,\omega)\cdot\big(f'f'_*(1-\varepsilon f)(1-\varepsilon f_*)-ff_*(1-\varepsilon f')(1-\varepsilon f'_*)\big)d\xi_* d\omega \ . \tag{39}$$

According to propositions 1-4, we have

$$f^\varepsilon \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^d\times\mathbb{R}^d)) \quad ,$$

$$0 \le f^\varepsilon \le \varepsilon^{-1} \text{ a.e.} \quad ,$$

$$\begin{aligned} \int\int_{\mathbb{R}^d\times\mathbb{R}^d} & f^\varepsilon(t,x,\xi).(1+|x|^2+|\xi|^2)\,dxd\xi \\ \le & \quad \int\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,\xi).(1+2|x|^2+(2t^2+1)|\xi|^2)\,dxd\xi \ , \end{aligned} \tag{40}$$

$$\begin{aligned} &\int\int_{\mathbb{R}^d\times\mathbb{R}^d} |f^\varepsilon(t,x,\xi)\log f^\varepsilon(t,x,\xi)|\,dxd\xi \\ \le & \quad 2C(N) + \int\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,\xi).(1+2|x|^2+2|\xi|^2+|\log f_0(x,\xi)|)\,dxd\xi \quad . \end{aligned} \tag{41}$$

$$\begin{aligned} &\int\int_{\mathbb{R}^d\times\mathbb{R}^d} s^\varepsilon(f^\varepsilon)(t,x,\xi)\,dxd\xi + \int_0^t ds\ \int\int_{\mathbb{R}^d\times\mathbb{R}^d} e^\varepsilon(f^\varepsilon)(s,x,\xi)\,dxd\xi \\ &= \int\int_{\mathbb{R}^d\times\mathbb{R}^d} s^\varepsilon(f_0)(x,\xi)\,dxd\xi \quad , \end{aligned}$$

where

$$s^\varepsilon(f^\varepsilon) = f^\varepsilon\log f^\varepsilon + \varepsilon^{-1}(1-\varepsilon f^\varepsilon)\log(1-\varepsilon f^\varepsilon) \quad ,$$

$$\begin{aligned} e^\varepsilon(f^\varepsilon) = \frac{1}{4}\int\int_{\mathbb{R}^d\times S^{N-1}} \quad & B^\varepsilon(\xi-\xi_*,\omega) \\ & \cdot\big(f^{\varepsilon\prime}f_*^{\varepsilon\prime}(1-\varepsilon f^\varepsilon)(1-\varepsilon f_*^\varepsilon)-f^\varepsilon f_*^\varepsilon(1-\varepsilon f^{\varepsilon\prime})(1-\varepsilon f_*^{\varepsilon\prime})\big) \\ & \cdot\log\Big(\frac{f^{\varepsilon\prime}f_*^{\varepsilon\prime}(1-\varepsilon f^\varepsilon)(1-\varepsilon f_*^\varepsilon)}{f^\varepsilon f_*^\varepsilon(1-\varepsilon f^{\varepsilon\prime})(1-\varepsilon f_*^{\varepsilon\prime})}\Big)\ d\xi_* d\omega \quad . \end{aligned}$$

Theorem 2 : *Under assumptions (29)-(39), there exists a sequence* $(\varepsilon_n)_{n\in\mathbb{N}}$, *with* $\lim_{n\to+\infty}\varepsilon_n = 0$ *such that*

$$f^{\varepsilon_n} \to f \quad \text{in} \quad L^1(\mathbb{R}^+_{\text{loc}}\times\mathbb{R}^d\times\mathbb{R}^d)\text{−weak} \quad ,$$

where f *is a function of* $C^0(\mathbb{R}^+, L^1(\mathbb{R}^d\times\mathbb{R}^d))$, *solution in the renormalized sense of the Boltzmann equation*

$$\begin{aligned} &\partial_t f + \xi\cdot\partial_x f = Q(f,f) \\ &f|_{t=0} = f_0 \end{aligned} \tag{42}$$

such that, for all $t\in\mathbb{R}^+$

$$\begin{aligned} &\int\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,\xi).(1+|x|^2+|\xi|^2)\,dxd\xi \\ &\le \int\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,\xi).(1+2|x|^2+(2t^2+1)|\xi|^2)\,dxd\xi \ , \end{aligned}$$

$$\int\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,\xi)\log f(t,x,\xi)\ dxd\xi + \int_0^t ds\ \int\int_{\mathbb{R}^d\times\mathbb{R}^d} E(f)(s,x,\xi)\ dxd\xi$$
$$\leq \int\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,\xi)\log f_0(x,\xi)\ dxd\xi \quad , \tag{43}$$

where

$$E(f) = \frac{1}{4}\int\int_{\mathbb{R}^d\times S^{N-1}} B(\xi-\xi_*,\omega)(f'f'_* - ff_*)\cdot\log(\frac{f'f'_*}{ff_*})\ d\xi_* d\omega \quad .$$

According to the definition of R. DiPerna and P-L. Lions, we say that f solves (42) in the renormalized sense if and only if

$$\frac{Q_\pm(f,f)}{1+f} \in L^1_{\text{loc}}(\mathbb{R}^+\times\mathbb{R}^d\times\mathbb{R}^d) \quad ,$$

and $\beta(f)$ verifies, in the sense of the distributions, the equation

$$\partial_t\beta(f) + \xi\cdot\partial_x\beta(f) = \frac{Q(f,f)}{1+f} \quad ,$$

with

$$\beta(t) = \log(1+t) \quad .$$

Here

$$Q_+(f,f) = \int_{\omega\in S^{N-1}}\int_{\xi_*\in\mathbb{R}^d} B(\xi-\xi_*,\omega)f'f'_*\ d\xi_* d\omega \quad ,$$
$$Q_-(f,f) = \int_{\omega\in S^{N-1}}\int_{\xi_*\in\mathbb{R}^d} B(\xi-\xi_*,\omega)ff_*\ d\xi_* d\omega \quad ,$$

and of course

$$Q(f,f) = Q_+(f,f) - Q_-(f,f) \quad .$$

According to (40) and (41), it is clear that $(f^\varepsilon)_{\varepsilon<1}$ is weakly relatively compact in $L^1(\mathbb{R}^+_{\text{loc}}\times\mathbb{R}^d\times\mathbb{R}^d)$: there exists a function $f\in C^0(\mathbb{R}^+, L^1(\mathbb{R}^d\times\mathbb{R}^d))$ and a sequence $(\varepsilon_n)_{n\in N}$ tending to zero, such that

$$f^{\varepsilon_n} \to f \quad \text{in} \quad L^1(\mathbb{R}^+_{\text{loc}}\times\mathbb{R}^d\times\mathbb{R}^d)-\text{weak} \quad .$$

We will not reproduce the arguments given in [DP,L 1]. We shall only give the most important modifications that are necessary to adapt their proof.

1^{st} **modification** : One has to prove that $\partial_t f^\varepsilon + \xi \cdot \partial_x f^\varepsilon$ is bounded in $L^1(\mathbb{R}^+_{\text{loc}} \times \mathbb{R}^d \times \mathbb{R}^d)$. In the proof of DiPerna and Lions, a crucial ingredient is the following identity :

$$Q_\pm(f,f) \leq K \cdot Q_\mp(f,f) + \frac{4}{\log K} \cdot E(f) \quad \forall\, K > 1 \quad .$$

Let us define $C^\varepsilon_\pm(f^\varepsilon)$ by setting

$$C^\varepsilon_+(f^\varepsilon) = \int\int_{S^{N-1}\times\mathbb{R}^d} B^\varepsilon(\xi-\xi_*,\omega)(f^{\varepsilon\prime} f^{\varepsilon\prime}_*(1-\varepsilon f^\varepsilon)(1-\varepsilon f^\varepsilon_*))\, d\xi_* d\omega \quad ,$$

$$C^\varepsilon_-(f^\varepsilon) = \int\int_{S^{N-1}\times\mathbb{R}^d} B^\varepsilon(\xi-\xi_*,\omega)(f^\varepsilon f^\varepsilon_*(1-\varepsilon f^{\varepsilon\prime})(1-\varepsilon f^{\varepsilon\prime}_*))\, d\xi_* d\omega \quad .$$

Then we have the inequality

$$C^\varepsilon_\pm(f^\varepsilon) \leq K \cdot C^\varepsilon_\pm(f^\varepsilon) + \frac{4}{\log K} \cdot e^\varepsilon(f^\varepsilon) \quad \forall\, K > 1 \quad . \tag{44}$$

Indeed, for a given t in $[0,+\infty[$

$$\begin{aligned} C^\varepsilon_+(f^\varepsilon) \leq \quad & K \cdot \int\int_{\Omega^\varepsilon} B^\varepsilon(\xi-\xi_*,\omega)(f^\varepsilon f^\varepsilon_*(1-\varepsilon(f^\varepsilon)')(1-\varepsilon(f^\varepsilon)'_*))\, d\xi_* d\omega \\ & + \int\int_{(\Omega^\varepsilon)^c} B^\varepsilon(\xi-\xi_*,\omega)((f^\varepsilon)'(f^\varepsilon)'_*(1-\varepsilon f^\varepsilon)(1-\varepsilon f^\varepsilon_*))\, d\xi_* d\omega \;, \end{aligned}$$

where

$$\begin{aligned} \Omega^\varepsilon = \{\ (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d \mid \quad & (f^\varepsilon)'(f^\varepsilon)'_*(1-\varepsilon f^\varepsilon)(1-\varepsilon f^\varepsilon_*)(t,x,\xi) \\ & \geq K \cdot f^\varepsilon f^\varepsilon_*(1-\varepsilon(f^\varepsilon)')(1-\varepsilon(f^\varepsilon)'_*)(t,x,\xi)\ \}\ . \end{aligned}$$

On $(\Omega^\varepsilon)^c$, we have

$$((f^\varepsilon)'(f^\varepsilon)'_*(1-\varepsilon f^\varepsilon)(1-\varepsilon f^\varepsilon_*))(t,x,\xi) \leq K(f^\varepsilon f^\varepsilon_*(1-\varepsilon(f^\varepsilon)')(1-\varepsilon(f^\varepsilon)'_*))(t,x,\xi),$$

$$\frac{1}{\log K} \cdot \log\left(\frac{(f^\varepsilon)'(f^\varepsilon)'_*(1-\varepsilon f^\varepsilon)(1-\varepsilon f^\varepsilon_*)}{f^\varepsilon f^\varepsilon_*(1-\varepsilon(f^\varepsilon)')(1-\varepsilon(f^\varepsilon)'_*)}\right) \geq 1 \quad ,$$

$$\begin{aligned} & \int\int_{(\Omega^\varepsilon)^c} B^\varepsilon(\xi-\xi_*,\omega)\big((f^\varepsilon)'(f^\varepsilon)'_*(1-\varepsilon f^\varepsilon)(1-\varepsilon f^\varepsilon_*) - f^\varepsilon f^\varepsilon_*(1-\varepsilon(f^\varepsilon)')(1-\varepsilon(f^\varepsilon)'_*)\big)\, d\xi_* d\omega \\ & \leq \tfrac{1}{\log K} \int\int_{(\Omega^\varepsilon)^c} B^\varepsilon(\xi-\xi_*,\omega)\big((f^\varepsilon)'(f^\varepsilon)'_*(1-\varepsilon f^\varepsilon)(1-\varepsilon f^\varepsilon_*) - f^\varepsilon f^\varepsilon_*(1-\varepsilon(f^\varepsilon)')(1-\varepsilon(f^\varepsilon)'_*)\big) \\ & \qquad\qquad \cdot \log\left(\tfrac{(f^\varepsilon)'(f^\varepsilon)'_*(1-\varepsilon f^\varepsilon)(1-\varepsilon f^\varepsilon_*)}{f^\varepsilon f^\varepsilon_*(1-\varepsilon(f^\varepsilon)')(1-\varepsilon(f^\varepsilon)'_*)}\right) d\xi_* d\omega \\ & \leq \tfrac{4}{\log K} \cdot e^\varepsilon(f) \quad . \end{aligned}$$

because $e^\varepsilon(f)$ is nonnegative almost everywhere. We can exchange $C^\varepsilon_+(f^\varepsilon)$ and $C^\varepsilon_-(f^\varepsilon)$, and the same arguments lead to the other case : equation (44) is proved.

2^{nd} **modification** : We must prove that

$$\frac{1}{1+f^\varepsilon}(Q(f^\varepsilon, f^\varepsilon) - C^\varepsilon(f^\varepsilon)) \to 0 \quad \text{weakly in } L^1(\mathbb{R}^+_{\text{loc}} \times \mathbb{R}^d \times \mathbb{R}^d) \quad ,$$

or, equivalently, that

$$\frac{1}{1+f^\varepsilon}(Q_-(f^\varepsilon, f^\varepsilon) - C^\varepsilon_-(f^\varepsilon)) \to 0 \quad \text{weakly in } L^1(\mathbb{R}^+_{\text{loc}} \times \mathbb{R}^d \times \mathbb{R}^d) \quad .$$

Proving that

$$\varepsilon \cdot \int\int_{S^{N-1}\times\mathbb{R}^d} B^\varepsilon(\xi-\xi_*, \omega)(f^\varepsilon)_*(f^\varepsilon)' \; d\xi_* d\omega \to 0 \text{ weakly in } L^1(\mathbb{R}^+_{\text{loc}} \times \mathbb{R}^d \times \mathbb{R}^d)$$

$$\varepsilon \cdot \int\int_{S^{N-1}\times\mathbb{R}^d} B^\varepsilon(\xi-\xi_*, \omega)(f^\varepsilon)_*(f^\varepsilon)'_* \; d\xi_* d\omega \to 0 \text{ weakly in } L^1(\mathbb{R}^+_{\text{loc}} \times \mathbb{R}^d \times \mathbb{R}^d)$$

is enough, but this is clear since

$$\int\int_{S^{N-1}\times\mathbb{R}^d} B^\varepsilon(\xi - \xi_*, \omega)(f^\varepsilon)_* \; d\xi_* d\omega$$

is bounded in $L^\infty(\mathbb{R}^+_{\text{loc}} \times \mathbb{R}^d \times \mathbb{R}^d)$ and since f^ε is contained in a weakly relatively compact set of $L^1(\mathbb{R}^+_{\text{loc}} \times \mathbb{R}^d \times \mathbb{R}^d)$:

$$\varepsilon \cdot (f^\varepsilon)'_* \to 0 \quad \text{in } L^1(\mathbb{R}^+_{\text{loc}} \times \mathbb{R}^d \times \mathbb{R}^d) \quad .$$

3^{rd} **modification** : Let us prove inequality (43). According to (24)

$$\begin{aligned}
&\textstyle\int\int_{\mathbb{R}^d\times\mathbb{R}^d}(f^\varepsilon(t,x,\xi)\log f^\varepsilon(t,x,\xi) - f^\varepsilon(t,x,\xi))\; dxd\xi \\
&\qquad \textstyle + \int_0^t ds \; \int\int_{\mathbb{R}^d\times\mathbb{R}^d} e^\varepsilon(f^\varepsilon)(s,x,\xi)\; dxd\xi \\
&\textstyle \leq \int\int_{\mathbb{R}^d\times\mathbb{R}^d} s^\varepsilon(f^\varepsilon)(t,x,\xi)\; dxd\xi + \int_0^t ds \; \int\int_{\mathbb{R}^d\times\mathbb{R}^d} e^\varepsilon(f^\varepsilon)(s,x,\xi)\; dxd\xi \\
&\qquad \textstyle = \int\int_{\mathbb{R}^d\times\mathbb{R}^d} s^\varepsilon(f_0)(x,\xi)\; dxd\xi \leq \int\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0^\varepsilon(x,\xi)\log f_0^\varepsilon(x,\xi)\; dxd\xi.
\end{aligned}$$

The function $\tau \mapsto \tau\log\tau$ is convex :

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \quad & \textstyle\int\int_{\mathbb{R}^d\times\mathbb{R}^d}(f^\varepsilon(t,x,\xi)\log f^\varepsilon(t,x,\xi) - f^\varepsilon(t,x,\xi))\; dxd\xi \\
& \textstyle\geq \int\int_{\mathbb{R}^d\times\mathbb{R}^d}(f(t,x,\xi)\log f(t,x,\xi) - f(t,x,\xi))\; dxd\xi \;,
\end{aligned}$$

and Lebesgue' s theorem of dominated convergence ensures that

$$\begin{aligned}
\lim_{\varepsilon\to 0} \quad & \textstyle\int\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0^\varepsilon(x,\xi)\log f_0^\varepsilon(x,\xi)\; dxd\xi \\
& \textstyle= \int\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,\xi)\log f_0(x,\xi)\; dxd\xi \;.
\end{aligned}$$

Let us prove that

$$\begin{array}{ll} \liminf_{\varepsilon\to 0} & \int_0^t ds \int\int_{\mathbb{R}^d\times\mathbb{R}^d} e^\varepsilon(f^\varepsilon)(s,x,\xi)\, dxd\xi \\ & \geq \int_0^t ds \int\int_{\mathbb{R}^d\times\mathbb{R}^d} E(f^\varepsilon)(s,x,\xi)\, dxd\xi \quad \forall\, t\in[0,+\infty[\,. \end{array} \tag{45}$$

$$\begin{array}{l} B^\varepsilon(\xi-\xi_*,\omega)(f^{\varepsilon\prime}f_*^{\varepsilon\prime}(1-\varepsilon f^\varepsilon)(1-\varepsilon f_*^\varepsilon) - f^\varepsilon f_*^\varepsilon(1-\varepsilon f^{\varepsilon\prime})(1-\varepsilon f_*^{\varepsilon\prime})) \\ \qquad \cdot\log(\frac{f^{\varepsilon\prime}f_*^{\varepsilon\prime}(1-\varepsilon f^\varepsilon)(1-\varepsilon f_*^\varepsilon)}{f^\varepsilon f_*^\varepsilon(1-\varepsilon f^{\varepsilon\prime})(1-\varepsilon f_*^{\varepsilon\prime})}) \\ = \beta^\varepsilon(\xi-\xi_*,\omega)(g^{\varepsilon\prime}g_*^{\varepsilon\prime}\frac{1-\varepsilon f^\varepsilon}{1-\varepsilon h^\varepsilon}\frac{1-\varepsilon f_*^\varepsilon}{1-\varepsilon h_*^\varepsilon} - g^\varepsilon g_*^\varepsilon\frac{1-\varepsilon f^{\varepsilon\prime}}{1-\varepsilon h^{\varepsilon\prime}}\frac{1-\varepsilon f_*^{\varepsilon\prime}}{1-\varepsilon h_*^{\varepsilon\prime}}) \\ \qquad \cdot\log(\frac{g^{\varepsilon\prime}g_*^{\varepsilon\prime}\frac{1-\varepsilon f^\varepsilon}{1-\varepsilon h^\varepsilon}\frac{1-\varepsilon f_*^\varepsilon}{1-\varepsilon h_*^\varepsilon}}{g^\varepsilon g_*^\varepsilon\frac{1-\varepsilon f^{\varepsilon\prime}}{1-\varepsilon h^{\varepsilon\prime}}\frac{1-\varepsilon f_*^{\varepsilon\prime}}{1-\varepsilon h_*^{\varepsilon\prime}}}) \end{array}$$

where

$$\begin{array}{l} \beta^\varepsilon = B^\varepsilon.(1-\varepsilon h^\varepsilon)(1-\varepsilon h_*^\varepsilon)(1-\varepsilon h^{\varepsilon\prime})(1-\varepsilon h_*^{\varepsilon\prime}) \\ h^\varepsilon = \min(f^\varepsilon,1) \\ g^\varepsilon = \frac{1}{1-\varepsilon h^\varepsilon} \end{array}$$

It is obvious that

$$g^{\varepsilon_n}\to f \quad \text{in} \quad L^1(\mathbb{R}^+_{\text{loc}}\times\mathbb{R}^d\times\mathbb{R}^d)-\text{weak} \quad ,$$

$$\|\frac{1-\varepsilon_n f^{\varepsilon_n}}{1-\varepsilon_n h^{\varepsilon_n}}\|_{L^\infty(\mathbb{R}^+_{\text{loc}}\times\mathbb{R}^d\times\mathbb{R}^d)} \quad , \qquad \frac{1-\varepsilon_n f^{\varepsilon_n}}{1-\varepsilon_n h^{\varepsilon_n}}\to 1 \quad \text{a.e.}$$

Therefore, for all $\delta>0$, we have, as in [DP,L 1] :

$$\frac{\beta^{\varepsilon_n}g^{\varepsilon_n}g_*^{\varepsilon_n}}{1+\delta\int_{\mathbb{R}^d}g^{\varepsilon_n}\,d\xi}\frac{1-\varepsilon_n f^{\varepsilon_n\prime}}{1-\varepsilon_n h{\varepsilon_n}'}\frac{1-\varepsilon_n f_*^{\varepsilon_n\prime}}{1-\varepsilon_n h_*^{\varepsilon_n\prime}} \to \frac{Bff_*}{1+\delta\int_{\mathbb{R}^d}f\,d\xi}$$

$$\text{in} \quad L^1(\mathbb{R}^+_{\text{loc}}\times(\mathbb{R}^d)^3\times S^{N-1})-\text{weak} \quad ,$$

$$\frac{\beta^{\varepsilon_n}g{\varepsilon_n}'g{\varepsilon_n}_*'}{1+\delta\int_{\mathbb{R}^d}g^{\varepsilon_n}\,d\xi}\frac{1-\varepsilon_n f^{\varepsilon_n}}{1-\varepsilon_n h^{\varepsilon_n}}\frac{1-\varepsilon_n f_*^{\varepsilon_n}}{1-\varepsilon_n h_*^{\varepsilon_n}} \to \frac{Bf'f_*'}{1+\delta\int_{\mathbb{R}^d}f\,d\xi}$$

$$\text{in} \quad L^1(\mathbb{R}^+_{\text{loc}}\times(\mathbb{R}^d)^3\times S^{N-1})-\text{weak} \quad .$$

The function $(x,y)\mapsto(x-y)\log(\frac{x}{y})$ is convex on $\mathbb{R}^+\times\mathbb{R}^+$:

$$\liminf_{n\to+\infty}\int_0^t ds \int\int_{\mathbb{R}^d\times\mathbb{R}^d} e^{\varepsilon_n}(f^{\varepsilon_n})(s,x,\xi)\,dxd\xi$$

$$\geq \liminf_{n\to+\infty}\int_0^t ds \int\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{e^{\varepsilon_n}(f^{\varepsilon_n})(s,x,\xi)}{1+\delta\int_{\mathbb{R}^d}g^{\varepsilon_n}\,d\xi}\,dxd\xi$$

$$\geq \int_0^t ds \int\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{E(f^{\varepsilon_n})(s,x,\xi)}{1+\delta\int_{\mathbb{R}^d}f\,d\xi}\,dxd\xi \quad ,$$

which proves assertion (45).

6. Equilibrium states

In this section, we give some indications about stationary states and equilibrium states of the modified Boltzmann equation :

$$\partial_t f + \xi \cdot \partial_x f = C(f) \quad .$$

We look only for solutions satisfying the following L^∞-bounds :

$$0 \le f \le \varepsilon^{-1} \text{ a.e.} \quad ,$$

which are natural in view of assertion (10). We shall say that a solution is stationary if it does not depend on t. More generally, we define equilibrium solutions as solutions such that their entropy is constant :

$$\frac{d}{dt} \int \int s(f)(t,x,\xi)\, dx d\xi \quad = 0 \quad ,$$

or equivalently, such that the corresponding term of decrease of entropy is equal to zero :

$$\int \int e(f)(s,x,\xi)\, dx d\xi \quad = 0 \quad \forall\, s \in \mathbb{R}^+ \quad .$$

Equilibrium solutions appear naturally in the long time asymptotic problem associated to the modified Boltzmann equation (see [De,Do], and also remark 3).

Of course, if we do not assume that the position x remains in a bounded set, the only long time asymptotic solution is zero. Indeed, let us consider a solution f. For a given $R_0 \in]0,+\infty[$, we have, according to proposition 2

$$\begin{aligned} \int \int_{|x|<R_0} f(t,x,\xi)\, dx d\xi \quad &\le \quad \int \int_{|x|<R_0, |x-t\xi|<R} f(t,x,\xi)\, dx d\xi \\ &\quad + \quad \int \int_{|x-t\xi|\ge R} f(t,x,\xi)\, dx d\xi \end{aligned}$$

$$\begin{aligned} \int \int_{|x|<R_0} f(t,x,\xi)\, dx d\xi \quad &\le \quad \varepsilon^{-1} |S^{N-1}|^2 \cdot R_0^N (R+R_0)^N \cdot t^{-N} \\ &\quad + \quad \frac{1}{R^2} \cdot \int \int_{\mathbb{R}^d \times \mathbb{R}^d} f_0(x,\xi).|\xi|^2\, dx d\xi \quad , \end{aligned}$$

for all $R > 0$, and therefore

$$f(t,x,\xi) \to 0 \quad (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d \text{ a.e.} \quad \text{when } t \to +\infty \quad .$$

To avoid some technicalities to deal with the boundary conditions, we shall assume in the following that f is periodic in x (see [A]) : f is defined on $\mathbb{R}^+ \times (\mathbb{R}^d/Z^N) \times \mathbb{R}^d$. This is a natural assumption if we look at the Cauchy problem for the modified Boltzmann equation with an initial data f_0 such that

$$\begin{aligned} &f_0(x,\xi) = f_0(x+\tau,\xi) \quad \forall\, \tau \in Z^N \ , \quad (x,\xi) \in [0,1]^N \times \mathbb{R}^d \text{ a.e.} \\ &0 \leq f_0(x,\xi) \leq \varepsilon^{-1} \quad (x,\xi) \in [0,1]^N \times \mathbb{R}^d \text{ a.e.} \end{aligned}$$

Indeed, looking at the proof of theorem 1, we can see that there exists a unique solution f, which is also periodic in x, such that

$$0 \leq f(t,x,\xi) \leq \varepsilon^{-1} \quad \forall\, \tau \in Z^N \ , \quad (x,\xi) \in [0,1]^N \times \mathbb{R}^d \text{ a.e.}$$

and it is not difficult to prove that the results of sections 3 and 4 can be adapted to this case.

Let us assume that

$$B > 0 \quad \text{a.e. on } \mathbb{R}^d \times S^{N-1}$$

and try to exhibit the equilibrium solutions. It is not difficult to see that

$$\int_0^t ds \int\int_{[0,1]^N \times \mathbb{R}^d} e(f)(s,x,\xi)\, dx d\xi = 0$$

implies for almost all $(t,x,\xi,\xi_*,\omega) \in \mathbb{R}^+ \times [0,1]^N \times \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1}$

$$f' f'_*(1-\varepsilon f)(1-\varepsilon f_*) = f f_*(1-\varepsilon f')(1-\varepsilon f'_*)$$

with standard notations, because for all stricly positive real numbers x, y

$$(x-y)\log(\frac{x}{y}) = 0$$

if and only if $x = y$. Difficulties arise when $f = \varepsilon^{-1}$, and we are not able to deal with the general case. We will give the expressions of the solutions for two cases

first case : f is an equilibrium solution such that $1 - \varepsilon f > 0$ a.e.

second case : f is stationary and continuous.

Let us notice that the limit - when the time goes to infinity - of a uniformly continuous solution is a continuous equilibrium solution (see remark 3).

1^{st} **case** : Let us assume that $1 - \varepsilon f > 0$ a.e. and define m by setting

$$m = \frac{f}{1 - \varepsilon f} \quad .$$

We have

$$m' m'_* = m m_* \quad ,$$

which ensures (see [T,M]) that m is a maxwellian. Therefore (see [De]) the solution f (and also m) solves the free transport equation

$$\partial_t f + \xi \cdot \partial_x f = 0 \quad ,$$

and

$$f(t,x,\xi) = \frac{m(t,x,\xi)}{1 + \varepsilon m(t,x,\xi)} \quad \forall\, t \in \mathbb{R}^+ \quad (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d \text{ a.e.}$$

The case of x belonging to an open set with specified boundary conditions can be treated in a similar way (see [De] for a detailed study of the maxwellian solutions of the free transport equation).

2^{nd} **case** : Let us assume that f does not depend on t and belongs to $C^0([0,1]^N \times \mathbb{R}^d)$. Then one of the following assertions is satisfied

$$\begin{array}{ll} \text{(i)} & 1 - \varepsilon f(x,\xi) > 0 \quad \forall\, (x,\xi) \in [0,1]^N \times \mathbb{R}^d \\ \text{(ii)} & f(x,\xi) = \varepsilon^{-1} \quad \forall\, (x,\xi) \in [0,1]^N \times \mathbb{R}^d \end{array}$$

If (i) is satisfied, the results of the first case apply : $m = \frac{f}{1-\varepsilon f}$ is a maxwellian solution of the free transport equation. Conversly, let us asssume that

$$\exists (x_0, \overline{\xi}) \in [0,1]^N \times \mathbb{R}^d \quad \text{such that} \quad f(x_0, \overline{\xi}) = \varepsilon^{-1} \quad .$$

If there exists $(x,\xi) \in [0,1]^N \times \mathbb{R}^d$ such that

$$f(x,\xi) < \varepsilon^{-1} \tag{H}$$

then

$$\exists (x_1, \xi_1) \in [0,1]^N \times \mathbb{R}^d \quad \text{such that} \quad f(x_1,\xi_1) \in]0, \varepsilon^{-1}[\quad .$$

because f is continuous. But $f^{-1}(]0,\varepsilon^{-1}[)$ is an open set :

$$\exists R > 0 \quad \forall\,(x,\xi) \in B(x_1,R) \times B(\xi_1,R) \quad f(x,\xi) \in]0,\varepsilon^{-1}[\quad .$$

The equation for f is now $\xi \cdot \partial_x f = 0$:

$$f(x+t\xi,\xi) = f(x,\xi) \quad \forall\, t \in \mathbb{R} \quad \forall\,(x,\xi) \in [0,1]^N \times \mathbb{R}^d \quad .$$

We can therefore find $\overline{x} \in [0,1]^N$ and $\overline{\xi_*} \in B(\xi_1,R)$ such that

$$f(\overline{x},\overline{\xi}) = \varepsilon^{-1} \quad \text{and} \quad f(\overline{x},\overline{\xi_*}) \in]0,\varepsilon^{-1}[\quad .$$

Using the continuity of $\xi \mapsto f(\overline{x},\xi)$, it is clear that there exists $\rho > 0$ such that

$$\forall\,\xi_* \in B(\overline{\xi_*},\rho) \quad f(\overline{x},\xi_*) \in]0,\varepsilon^{-1}[\quad .$$

Let $\omega \in S^{N-1}$. With the notations

$$\begin{aligned} \xi &= \overline{\xi} \\ \xi' &= \xi - (\xi-\xi_*).\omega\,\omega \\ \xi'_* &= \xi_* + (\xi-\xi_*).\omega\,\omega \end{aligned}$$

we have, for $x = \overline{x}$:

$$ff_*(1-\varepsilon f')(1-\varepsilon f'_*) = 0$$

and if $\xi_* \in B(\overline{\xi_*},\rho)$ and $\xi'_* \in B(\overline{\xi_*},\rho)$, then

$$f(\overline{x},\xi'_*) = \varepsilon^{-1} \quad ,$$

which proves that for all $\xi' \in B(\overline{\xi},\rho)$

$$f(\overline{x},\xi') = \varepsilon^{-1} \quad .$$

Finally, $f(\overline{x},.)^{-1}(\{\varepsilon^{-1}\})$ is a non empty open set :

$$f(\overline{x},.)^{-1}\{\varepsilon^{-1}\} = \mathbb{R}^d \quad .$$

(H) is not possible, which proves assertion (ii).

Remark 3 : First, let us notice that the solution of the modified Boltzmann equation is continuous if the initial data is continuous. Indeed, looking at the proof of theorem 1, one can see that the sequence $(f_n)_{n\in N}$ defined by setting

$$f^{n+1} = Tf^n \quad (\forall\, n \in \mathbb{N})$$

converges uniformly for all $M \in \mathbb{N}$ on every compact set of $[M\theta, (M+1)\theta] \times [0,1]^N \times \mathbb{R}^d$ to f : f is continuous.

Now, if f is a uniformly continuous solution, let us consider the family $(f^\tau)_{\tau>0}$ defined for all $\tau > 0$ by setting

$$f^\tau(t,x,\xi) = f(t+\tau,x,\xi) \quad \forall\, t \in \mathbb{R}^+ \quad \forall\,(x,\xi) \in [0,1]^N \times \mathbb{R}^d \quad .$$

$(f^\tau)_{\tau>0}$is obviously equicontinuous and bounded : according to Ascoli' s theorem, there exists a sequence $(\tau_n)_{n\in\mathbb{N}}$ going to infinity and a continuous function g such that f^{τ_n} converges uniformly to g on every compact set of $\mathbb{R}^+ \times [0,1]^N \times \mathbb{R}^d$, and g is an equilibrium solution.
Indeed, for all $(t,x,\xi) \in \mathbb{R}^+ \times [0,1]^N \times \mathbb{R}^d$

$$\begin{aligned}
&\lim_{n\to+\infty} f^{\tau_n\prime} f_*^{\tau_n\prime}(1-\varepsilon f^{\tau_n})(1-\varepsilon f_*^{\tau_n}) = g' g_*'(1-\varepsilon g)(1-\varepsilon g_*)\\
&\lim_{n\to+\infty} f^{\tau_n} f_*^{\tau_n}(1-\varepsilon f^{\tau_n\prime})(1-\varepsilon f_*^{\tau_n\prime}) = g g_*(1-\varepsilon g')(1-\varepsilon g_*')
\end{aligned}$$

which ensures that

$$\begin{aligned}
0 &= \lim_{n\to+\infty} \int_{\tau_n}^{+\infty} ds \int\int_{[0,1]^N\times\mathbb{R}^d} e(f)(s,x,\xi)\, dxd\xi\\
&= \int_0^{+\infty} ds \int\int_{[0,1]^N\times\mathbb{R}^d} e(f^{\tau_n})(s,x,\xi)\, dxd\xi\\
&= \int_0^{+\infty} ds \int\int_{[0,1]^N\times\mathbb{R}^d} e(g)(s,x,\xi)\, dxd\xi \quad .
\end{aligned}$$

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