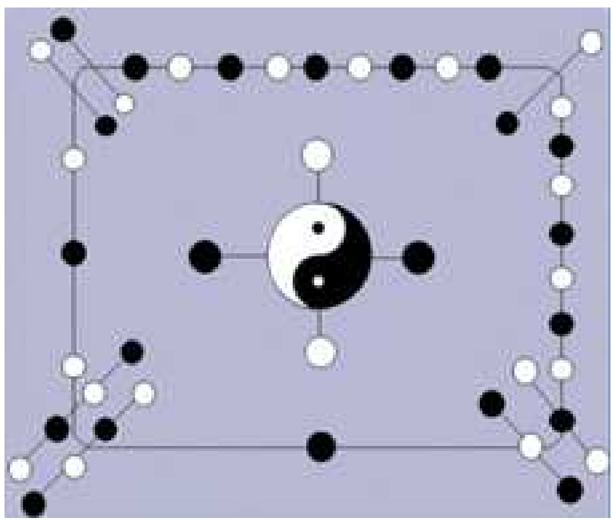
ISSN 1937 - 1055

VOLUME 1, 2009

INTERNATIONAL JOURNAL OF

MATHEMATICAL COMBINATORICS



EDITED BY

THE MADIS OF CHINESE ACADEMY OF SCIENCES

April, 2009

Vol.1, 2009 ISSN 1937-1055

International Journal of

Mathematical Combinatorics

Edited By

The Madis of Chinese Academy of Sciences

April, 2009

Aims and Scope: The **International J.Mathematical Combinatorics** (*ISSN 1937-1055*) is a fully refereed international journal, and published quarterly comprising 100-150 pages approx. per volume, which publishes original research papers and survey articles in all aspects of Smarandache multi-spaces, Smarandache geometries, mathematical combinatorics, non-euclidean geometry and topology and their applications to other sciences. Topics in detail to be covered are:

Smarandache multi-spaces with applications to other sciences, such as those of algebraic multi-systems, multi-metric spaces,$\cdots$, etc.. Smarandache geometries;

Differential Geometry; Geometry on manifolds;

Topological graphs; Algebraic graphs; Random graphs; Combinatorial maps; Graph and map enumeration; Combinatorial designs; Combinatorial enumeration;

Low Dimensional Topology; Differential Topology; Topology of Manifolds;

Geometrical aspects of Mathematical Physics and Relations with Manifold Topology;

Applications of Smarandache multi-spaces to theoretical physics; Applications of Combinatorics to mathematics and theoretical physics;

Mathematical theory on gravitational fields; Mathematical theory on parallel universes;

Other applications of Smarandache multi-space and combinatorics.

Generally, papers on mathematics with its applications not including in above topics are also welcome.

It is also available from the below international databases:

Serials Group/Editorial Department of EBSCO Publishing
10 Estes St. Ipswich, MA 01938-2106, USA
Tel.: (978) 356-6500, Ext. 2262 Fax: (978) 356-9371
http://www.ebsco.com/home/printsubs/priceproj.asp

and

Gale Directory of Publications and Broadcast Media, Gale, a part of Cengage Learning
27500 Drake Rd. Farmington Hills, MI 48331-3535, USA
Tel.: (248) 699-4253, ext. 1326; 1-800-347-GALE Fax: (248) 699-8075
http://www.gale.com

Indexing and Reviews: Mathematical Reviews(USA), Zentralblatt fur Mathematik(Germany), Referativnyi Zhurnal (Russia), Mathematika (Russia), Computing Review (USA), Institute for Scientific Information (PA, USA), Library of Congress Subject Headings (USA).

Subscription A subscription can be ordered by a mail or an email directly to

Linfan Mao
The Editor-in-Chief of *International Journal of Mathematical Combinatorics*
Chinese Academy of Mathematics and System Science
Beijing, 100080, P.R.China
Email: maolinfan@163.com

Price: US$48.00

No object is mysterious. The mystery is your eye.

By Elizabeth, a British female writer.

International J.Math. Combin. Vol.1 (2009), 01-08

Study of the Problems of Persons with Disability (PWD) Using FRMs

W.B. Vasantha Kandasamy

(Department of Mathematics,Indian Institute of Technology, Chennai - 600 036)

E-mail: vasanthakandasamy@gmail.com

A. Praveen Prakash

(Department of Mathematics, Guru Nanak College, Chennai - 600 042)

K. Thirusangu

(Department of Mathematics, Anna University, Chennai - 600 025)

Abstract: In this paper we find the interrelations and the hidden pattern of the problems faced by the PWDs and their caretakers using Fuzzy Relational Maps (FRMs). Here we have taken the problems faced by the rural persons with disabilities in Melmalayanur and Kurinjipadi Blocks, Tamil Nadu, India. This paper is organized with the following four sections. Section one is introductory in nature giving the overall contents from the survey made about PWDs in the above said Blocks. Section two gives description of FRM models and the attributes taken for the study related with the PWDs and the caretakers, the FRM model formed using these attributes and their analysis. The third section gives the suggestions and conclusions derived from the survey as well as the FRM model.

Key Words: FRM model, fixed point, hidden pattern, relational matrix, limit cycle.

AMS(2000): 04A72.

§1. Introduction

A study was conducted taking 93 village panchayats from the Kurinjipadi and Melmalayanur Blocks. The data reveals only 1.64 percent of the population are PWDs. The male population is comparatively higher. (60% males and 40% females). 51% are orthopedic followed by 16% with speech and hearing impaired. Also it is observed from the data that 60% are not married in the reproductive age group; however 73% are found married in the non reproductive age group. It is still unfortunate to see among the 3508 PWDs in the age group 4 yrs and above 59% of them have not even entered school. Further in the age group 4 to 14, 37% are yet to be enrolled in the school. Thus the education among the PWDs is questionably poor. Their living conditions are poor with no proper toilet facilities who are under nourished.

We use FRMs to study the problem taking the attributes of the domain space as the problems faced by the PWD and the range attributes are taken as the problems felt by the caretakers of the PWD. We just describe the FRM model and proceed on to justify why FRM model is used in this study.

[1]Received December 8, 2008. Accepted January 6, 2009.

§2. Description of FRM Model and its Application to the Problem

Fuzzy Relational Maps (FRMs) are constructed analogous to FCMs. FRMs are divided into two disjoint units. We denote by R the set of nodes $R_1, \cdots, R_m$ of the range space where $R_j = \{(x_1, \cdots, x_m) | x_j = 0 \text{ or } 1\}$ for $j = 1, 2, \cdots, m$. $D_1, \cdots, D_n$ denote the nodes of the domain space where $D_i = \{(y_1, \cdots, y_n) | y_i = 0 \text{ or } 1\}$ for $i = 1, 2, \cdots, n$. Here, $y_i = 0$ denotes the *off state* and $y_i = 1$ the *on state* of any state vector. Similarly $x_i = 1$ denotes the *on state* and $x_i = 0$ the *off state* of any state vector.

Thus a FRM is a directed graph or a map from D to R with concepts like policies or events etc as nodes and causalities as edges. It represents causal relations between the spaces D and R.

Let D_i and R_j denote the nodes of an FRM. The directed edge from D_i to R_j denotes the causality of D_i on R_j called relations. Every edge in the FRM is weighted with a number of the set $\{0, +1\}$. Let e_{ij} be the weight of the edge D_iR_j; $e_{ij} \in \{0, +1\}$. The weight of the edge D_iR_j is positive if increase in D_i implies increase in R_j or decrease in D_i implies decrease in R_j i.e., causality of D_i on R_j is 1. If $e_{ij} = 0$ then D_i does not have any effect on R_j. When increase in D_i implies decrease in R_j or decrease in D_i implies increase in R_j then the causality of D_i on R_j is -1.

A FRM is a directed graph or a map from D to R with concepts like policies or events etc, as nodes and causalities as edges. It represents causal relations between spaces D and R.

For the FRM with $D_1, \cdots, D_n$ as nodes of the domain space D and $R_1, \cdots, R_n$ as the nodes of the range space R, E defined as $E = (e_{ij})$, where e_{ij} is the weight of the directed edge D_iR_j (or R_jD_i); E is called the relational matrix of the FRM. $A = (a_1, \cdots, a_n)$, $a_i \in \{0, 1\}$; A is called the instantaneous state vector of the domain space and it denotes the on-off position of the nodes at any instant. Similarly for the range space $a_i = 0$ if a_i is off and $a_i = 1$ if a_i is on. Let the edges form a directed cycle. A FRM with directed cycle is said to be a FRM with feed back. A FRM with feed back is said to be the dynamical system and the equilibrium of the dynamical system is called the hidden pattern; it can be a fixed point or a limit cycle.

For example let us start the dynamical system by switching on R_1 (or D_1). Let us assume that the FRM settles down with R_1 and R_m or (D_1 and D_n) on i.e., $(10000 \cdots 1)$ or $(100 \cdots 01)$. Then this state vector is a fixed point. If the FRM settles down with a state vector repeating in the form, i.e., $A_1 \to A_2 \to \cdots A_i \to A_1$ or $B_1 \to B_2 \to \cdots \to B_i \to B_1$, then this equilibrium is called a limit cycle.

Now we would be using FRM models to study the problem.

2.1 Justification for Using FRM

(1) We see the problems of Persons With Disability (PWD) is distinctly different from the problems of the caretakers of the PWD. Thus at the outset we are justified in using FRM i.e., a set of domain attributes and a set of range attributes.

(2) All the attributes under study cannot be quantified as numbers. So the data is one involving a large quantity of feelings. Hence fuzzy models is the best suited, as the data is an unsupervised one.

(3) Also this model alone can give the effect of problems faced by the caretakers on the PWDs and vice versa. So this model is best suited for our problem.
(4) Finally this model gives hidden pattern i.e., it gives a pair of resultant state vectors i.e., hidden pattern related with the PWDs as well as hidden pattern related with the caretakers. Thus we use this model to analyze the problem.

Now the attributes related with the PWDs are taken as the domain space of the FRM and the attributes related with the caretakers of the PWDs are taken as the range space of the FRM. We shall describe each of the attributes related with the PWDs and that of the caretakers in a line or two.

2.2 Attributes Related with the PWDs

The following attributes are given by an expert. The problems of PWDs are taken as the nodes of the domain space and the attributes associated with the close caretakers are taken as the nodes of the range space. The attributes associated with the PWDs are given below. They are in certain cases described in line or two.

D_1 – Depressed. From the survey majority of the PWDs looked and said they were depressed because of their disability and general treatment.

D_2 – Suffer from inferiority complex.

D_3 – Mental stress/agony - They often were isolated and sometimes kept in a small hut outside the house which made them feel sad as well as gave time to think about their disability with no other work. So they were often under stress and mental tension.

D_4 – Self Image - Majority did not possess any self image. It was revealed from the discussions and survey.

D_5 – Happy and contended.

D_6 – Uninterested in life.

D_7 – Dependent on others for every thing.

D_8 – Lack of mobility.

D_9 – Illtreated by close relatives.

Now the attributes $D_1, D_2, \cdots, D_9$ are taken as the nodes of the domain space of the FRM. We give the attributes associated with the range space.

R_1 – Poor. So cannot find money to spend on basic requirements. The PWDs go to work for their livelihood.

R_2 – Ashamed - relatives were ashamed of the PWDs.

R_3 – Indifferent - They were treated indifferently by their caretakers.

R_4 – PWDs are a burden to them. So they neglected them totally.

R_5 – Fatalism - They said it was fate that they have a PWD as their child / relative.

R_6 – Sympathetic.

R_7 – Caring.

R_8 – Show hatred towards the PWDs.

R_9 – The caretakers were not interested in marrying them off.

R_{10} – The PWDs are an economic burden to them.

R_{11} – They were isolated from others for reasons best known to the caretakers.

Thus $R_1, R_2, \cdots, R_{11}$ are taken as the nodes of the range space of the FRM.

The directed graph related with the FRM is shown in Fig.2.1, in which we have omitted the direction $D_i \to R_j$ on each edge D_iR_j for simplicity.

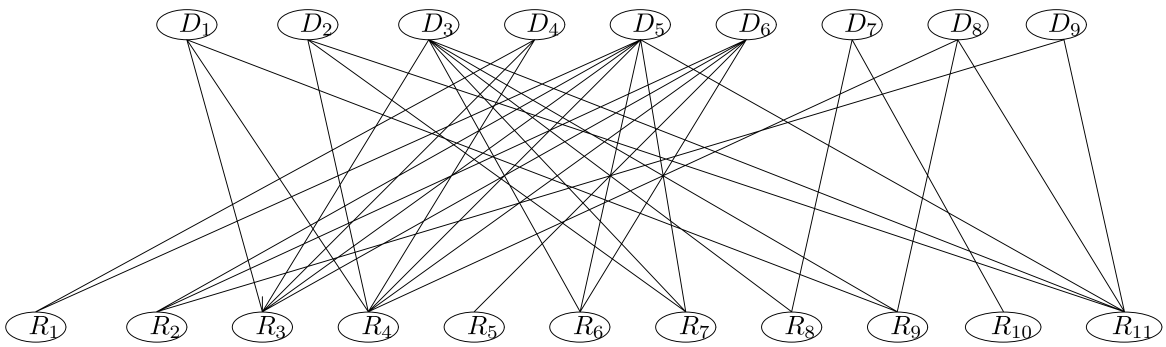


Fig.2.1

Let the relation matrix associated with the directed graph be given by T, where T is a 9×11 matrix with entries from the set $\{0, -1, 1\}$ following.

$$T = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & 1 \\ -1 & 0 & -1 & -1 & 0 & 1 & 1 & -1 & 0 & -1 & 1 \\ -1 & -1 & -1 & -1 & 0 & 1 & 1 & -1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Now we study the effect of the state vectors on the dynamical system T.

Suppose the expert wishes to study the *on state* of the node D_1 and all other nodes are in the *off state*. Let the state vector be $X = (100000000)$. The effect of X on the dynamical system T is given by

$$XT = (00110000100) = Y(\text{say}),$$

$$YT^t = (312\ -2\ -2\ 2020) \to (111001010) = X_1(\text{say}),$$

where $\to$ denotes that the resultant state vector YT^t is updated and thresholded, i.e., all negative values and 0 are replaced by 0 and all positive values greater than or equal to one are replaced by 1. By updating we mean the co ordinate which we started in the *on state* should remain in the *on state* till the end.

Now we find that

$$X_1T \to (01111001101) = Y_1(\text{say}),$$

$$Y_1T^t \to (111001111) = X_2(\text{say}),$$

$$X_2T \to (01111001111) = Y_2(\text{say}),$$

$$Y_2T^t \to (111001111) = X_3(\text{say}) = X2.$$

Thus the hidden pattern gives a fixed pair given by $\{(111001111), (01111001111)\}$.

Thus when the node depressed alone in the domain space is in the *on state* we see this makes the nodes $D_2, D_3, D_6, D_7, D_8, D_9$ to come to *on state* in the domain space and R_2, R_3, R_4, R_5, R_8, R_9, R_{10} and R_{11} in the *on state* in the range space.

Thus we see except the nodes the PWD has self image and she/he is happy and contended all other nodes come to *on state*. Thus this reveals if a PWD is depressed certainly he has no self image and he is not happy and contended. Further it also reveals from the state vector in the domain space poverty is not a cause of depression for R_1 is in the *off state*. Also R_6 and R_7 alone do not come to *on state* which clearly shows that the caretakers are not sympathetic and caring which is one of the reasons for the PWDs to be depressed. Thus we see all negative attributes come to *on state* in both the spaces when the PWD is depressed.

Next the expert is interested in studying the effect of the *on state* of the node in the range space viz. R_6 i.e., the caretakers are sympathetic towards the PWDs. Let $Y = (00000100000)$ be the state vector of the range space. To study the effect of Y on the dynamical system T^t.

$$YT^t \to (000110000) = X_1(\text{say}),$$

$$X_1T \to (00000110000) = Y_1(\text{say}),$$

$$Y_1T^t \to (000110000) = X_2(\text{say}).$$

But $X_2 = X_1$. Thus we see the hidden pattern of the state vector is a fixed pair of points given by $\{(00000110000), (000110000)\}$. It is clear when the PWD is treated with sympathy it makes him feel their caretakers are caring. So R_1 come to *on state*. On the other hand, we see she/he is happy and contended with a self image. Next the expert wishes to find the hidden pattern of the *on state* of the domain node D_4 i.e., self image of the PWD alone is in the *on state*.

Let $P = (000100000)$ be the given state vector. The effect of P on T is given by

$$PT \to (00000110000) = S_1(say),$$

$$S_1T^t \to (000110000) = P_1(\text{say}),$$

$$P_1T \to (00000110000) = S_2(\text{say}).$$

But $S_2 = S_1$ resulting in a fixed pair. Thus the hidden pattern of P is a fixed pair. We see self image of the PWD makes him happy and contended. He/she also feel that the caretakers are caring and sympathetic towards them. Now the expert studies the effect of the state vector in the range space when the PWD is isolated from the other, i.e., when R_{11} is in the *on state*.

Let $X = (00000000001)$ be the given state vector. Its effect on the dynamical system T is given by

$$XT^t \to (011000010) = Y(\text{say}),$$

$$YT \to (00110001101) = X_1(\text{say}).$$

The effect of X_1 on T is given by

$$X_1T^t \to (111001111) = Y_1(\text{say}),$$

$$Y_1T \to (01111001111) = X_2(\text{say}),$$

$$X_2T^t \to (111001111) = Y_2(say).$$

We see $Y_2 = Y_1$. Thus the hidden pattern of the state vector is a fixed pair given by $\{(01111001111), (111001111)\}$. Thus when the PWD is isolated from others he/she suffers all negative attributes and it is not economic condition that matters. Isolation directly means they are taken care of and the caretakers are not sympathetic towards them. When they are isolated they are not happy and contended and they do not have self image. All this is evident from the hidden patterns in which R_1, R_6 and R_7 are 0 and D_4 and D_5 are 0, i.e., in the *off state*. We have worked with the several *on state*s and the conclusions are based on that as well as from the survey we have taken. This is given in the following sections of this paper.

§3. Suggestions and Conclusions

3.1 Conclusions based on the model

1. From the hidden pattern given by the FRM model we see when the PWDs suffer from depression all negative attributes from both the range space and the domain space come to *on state* and their by showing its importance and its impact on the PWDs. It is clear that the nodes *self image* and *happy and contended* is in the *off state*s where as all other nodes in the domain of attributes are in the *on state*. Further the nodes economic condition, caring and sympathetic are in the *off state* in the range of attributes. Thus it is suggested the caretakers must be caring and sympathetic towards the PWDs to save them from depression.

2. When the node the caretakers are sympathetic towards the PWDs alone was in the *on state* the FRM model gave the hidden pattern which was a fixed pair in which only the nodes self image and happy and contended was alone in the *on state* from the domain vectors. In fact it was surprising to see all other negative nodes in the domain space was in the *off state*. Further in the range space of vectors we saw only the node caring came to *on state* and all other nodes were in the *off state*. Thus we see a small positive quality like sympathetic towards the PWDs can make a world of changes in their lives.

3. When the node PWDs are isolated from others was in the *on state* in the state vectors of the range space it is surprising to see that in the hidden pattern only the nodes happy and contended and self image are in the *off state* and all other nodes come to *on state* in the domain attributes and in the range attributes only the nodes poor cannot find time to spend with PWDs, caring and sympathetic remain in the *off state* and all other nodes in the range off attributes come to *on state*. Thus when the PWD is isolated from others he is depressed, not interested in life under goes mental stress, suffers from inferiority complex has no self image, is not happy or contended and is illtreated by the relatives. Also when the caretakers isolates a PWD it clearly implies they are not sympathetic or caring for the PWD and infact they are ashamed of the PWD and are indifferent to him/her. They also feel he/she is a burden and it is a fate that he/she is present in their house and show hatred towards him/her and are least bothered marrying off the PWD and infact feel the PWD is an economic burden on them.
4. It is verified the 'on state' of any one of the negative attributes gives the hidden pattern of the model in which all the negative attributes in both the domain and range space come to *on state* and the positive attributes remain in the *off state*.

5. Further the hidden pattern in almost all the cases resulted only in the fixed point which clearly proves that the changes in the behavioral pattern of the PWDs or the caretakers do not fluctuate infact remains the same.

3.2 Observations and suggestions based on the survey and the data

1. The survey proved the family in which PWDs were present were looked down by others in the rural areas. Thus it was difficult to perform the marriages of PWDs as well as their close relatives. This is one of the reasons the PWDs are not given in marriage at the productive age however data proved they got married after the non productive age. This is clearly evident from the data that out of 1191 PWDs in the marriageable age group a majority of 715 PWDs are not married i.e., 60% of them are not married. Above the reproductive age we find out of 1589 PWDs the majority 1163 constituting 73 percent are found to be married. One has to make analysis in this direction alone.

2. From the data it is surprising to see that out of a total of 3316 PWDs 56% of them are not educated. Out of 580 children in the age group $7 - 18$ years 105 children dropped out. Out of 483 children in the age group 4 to 14, 37% are yet to be enrolled in the school. Thus we see from the data that they deny education to PWDs. The study of education and related problems faced by PWDs will have to be taken up separately.

3. 44% of caretakers have not planned about the future of the PWDs. This is also a sensitive issue for the PWDs may be feeling insecure about their future.

4. Providing money to these PWDs as stipend or to their caretakers will not solve the problems of PWDs. It is thus suggested these PWDs are taught some trade and paid for their work. When they are earning naturally the caretakers have to take proper care of the PWD for otherwise the PWD can opt to stay away from them. Also when they (PWD) earn their bread they will have self image also can be contended to some extent.

5. Further the survey showed the PWDs were happy and interactive in the group of PWDs so it would be nice if some opt to work for them so that the PWDs live in communities taken care of by some one. This will at large solve several of the problems addressed. Also this is possible if they earn on their own.

6. It is also suggested that a marriage bureau should operate solely for the PWDs so that their marriage is not unnecessarily delayed.

7. The caretakers must be given counseling to deal the PWDs with care and sympathy. We have considered PWD who are not employed in this study. We thank Lamp Net for giving information.

References

[1] *Lamp Net Network for Human Rights*, India, 2007.

[2] Vasantha Kandasamy and Yasmin Sultana, Knowledge Processing Using Fuzzy relational Maps, *Ultra Sci.* 12 (2000), 242 - 245.

[3] Vasantha Kandasamy and Florentin Smarandache, *Fuzzy Cognitive Maps and Neutrosophic Cognitive Maps*, Xiquan, AZ 2003.

International J.Math. Combin. Vol.1 (2009), 08-17

Topological Multi-groups and Multi-fields

Linfan MAO

(Chinese Academy of Mathematics and System Science, Beijing 100080, P.R.China)

E-mail: maolinfan@163.com

Abstract: Topological groups, particularly, Lie groups are very important in differential geometry, analytic mechanics and theoretical physics. Applying Smarandache multi-spaces, topological spaces, particularly, manifolds and groups were generalized to combinatorial manifolds and multi-groups underlying a combinatorial structure in references. Then *whether can we generalize their combination, i.e., topological group or Lie group to a multiple one*? The answer is YES. In this paper, we show how to generalize topological groups and the homomorphism theorem for topological groups to multiple ones. By applying the classification theorem of topological fields, the topological multi-fields are classified in this paper.

Key Words: Smarandache multi-space, combinatorial system, topological group, topological multi-group, topological multi-field.

AMS(2000): 05E15, 08A02, 15A03, 20E07, 51M15.

§1. Introduction

In the reference [9], we formally introduced the conceptions of Smarandachely systems and combinatorial systems as follows:

Definition 1.1 *A rule in a mathematical system $(\Sigma;\mathcal{R})$ is said to be Smarandachely denied if it behaves in at least two different ways within the same set Σ, i.e., validated and invalided, or only invalided but in multiple distinct ways.*

A Smarandache system $(\Sigma;\mathcal{R})$ is a mathematical system which has at least one Smarandachely denied rule in $\mathcal{R}$.

Definition 1.2 *For an integer $m \geq 2$, let $(\Sigma_1;\mathcal{R}_1)$, $(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$ be m mathematical systems different two by two. A Smarandache multi-space is a pair $(\widetilde{\Sigma};\widetilde{\mathcal{R}})$ with*

$$\widetilde{\Sigma} = \bigcup_{i=1}^{m} \Sigma_i, \quad and \quad \widetilde{\mathcal{R}} = \bigcup_{i=1}^{m} \mathcal{R}_i.$$

Definition 1.3 *A combinatorial system $\mathscr{C}_G$ is a union of mathematical systems $(\Sigma_1;\mathcal{R}_1)$,$(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$ for an integer m, i.e.,*

[1]Received December 12, 2008. Accepted January 8, 2009.

$$\mathscr{C}_G = (\bigcup_{i=1}^{m} \Sigma_i;\ \bigcup_{i=1}^{m} \mathcal{R}_i)$$

with an underlying connected graph structure G, where

$$V(G) = \{\Sigma_1, \Sigma_2, \cdots, \Sigma_m\},$$
$$E(G) = \{\ (\Sigma_i, \Sigma_j) \mid \Sigma_i \bigcap \Sigma_j \neq \emptyset, 1 \leq i, j \leq m\}.$$

These notions enable us to establish combinatorial theory on geometry, particularly, *combinatorial differential geometry* in [8], also those of combinatorial theory for other sciences [7], for example, algebra systems, etc..

By definition, a *topological group* is nothing but the combination of a group associated with a topological space structure, i.e., an algebraic system $(\mathscr{H}; \circ)$ with conditions following hold ([16]):

(i) $(\mathscr{H}; \circ)$ is a group;

(ii) $\mathscr{H}$ is a topological space;

(iii) the mapping $(a, b) \to a \circ b^{-1}$ is continuous for $\forall a, b \in \mathscr{H}$,

Application of topological group, particularly, Lie groups shows its importance to differential geometry, analytic mechanics, theoretical physics and other sciences. Whence, it is valuable to generalize topological groups to a multiple one by algebraic multi-systems.

Definition 1.4 *A topological multi-group $(\mathscr{S}_G; \mathscr{O})$ is an algebraic multi-system $(\widetilde{\mathscr{A}}; \mathscr{O})$ with $\widetilde{\mathscr{A}} = \bigcup\limits_{i=1}^{m} \mathscr{H}_i$ and $\mathscr{O} = \bigcup\limits_{i=1}^{m} \{\circ_i\}$ with conditions following hold:*

(i) *$(\mathscr{H}_i; \circ_i)$ is a group for each integer $i,\ 1 \leq i \leq m$, namely, $(\mathscr{H}, \mathscr{O})$ is a multi-group;*

(ii) *$\widetilde{\mathscr{A}}$ is a combinatorially topological space $\mathscr{S}_G$, i.e., a combinatorial topological space underlying a structure G;*

(iii) *the mapping $(a, b) \to a \circ b^{-1}$ is continuous for $\forall a, b \in \mathscr{H}_i$, $\forall \circ \in \mathcal{O}_i$, $1 \leq i \leq m$.*

A *combinatorial Euclidean space* is a combinatorial system $\mathscr{C}_G$ of Euclidean spaces $\mathbf{R}^{n_1}$, $\mathbf{R}^{n_2}, \cdots, \mathbf{R}^{n_m}$ with an underlying structure G, denoted by $\mathscr{E}_G(n_1, \cdots, n_m)$ and abbreviated to $\mathscr{E}_G(r)$ if $n_1 = \cdots = n_m = r$. It is obvious that a topological multi-group is a topological group if $m = 1$ in Definition 1.4. Examples following show the existence of topological multi-groups.

Example 1.1 Let $\mathbf{R}^{n_i}, 1 \leq i \leq m$ be Euclidean spaces with an additive operation $+_i$ and scalar multiplication $\cdot$ determined by

$$(\lambda_1 \cdot x_1, \lambda_2 \cdot x_2, \cdots, \lambda_{n_i} \cdot x_{n_i}) +_i (\zeta_1 \cdot y_1, \zeta_2 \cdot y_2, \cdots, \zeta_{n_i} \cdot y_{n_i})$$
$$= (\lambda_1 \cdot x_1 + \zeta_1 \cdot y_1, \lambda_2 \cdot x_2 + \zeta_2 \cdot y_2, \cdots, \lambda_{n_i} \cdot x_{n_i} + \zeta_{n_i} \cdot y_{n_i})$$

for $\forall \lambda_l, \zeta_l \in \mathbf{R}$, where $1 \leq \lambda_l, \zeta_l \leq n_i$. Then each $\mathbf{R}^{n_i}$ is a continuous group under $+_i$. Whence, the algebraic multi-system $(\mathscr{E}_G(n_1, \cdots, n_m); \mathscr{O})$ is a topological multi-group with a underlying structure G by definition, where $\mathscr{O} = \bigcup_{i=1}^{m}\{+_i\}$. Particularly, if $m = 1$, i.e., an n-dimensional Euclidean space $\mathbf{R}^n$ with the vector additive $+$ and multiplication $\cdot$ is a topological group.

Example 1.2 Notice that there is function $\kappa : M_{n\times n} \to \mathbf{R}^{n^2}$ from real $n \times n$-matrices $M_{n\times n}$ to $\mathbf{R}$ determined by

$$\kappa : \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{n\times n} \end{pmatrix} \to \begin{pmatrix} a_{11} & \cdots & a_{1n}, \cdots, a_{n1} & \cdots & a_{n\times n} \end{pmatrix}$$

Denoted all $n \times n$-matrices by $\mathbf{M}(n, \mathbf{R})$. Then the general linear group of degree n is defined by

$$GL(n, \mathbf{R}) = \{\ M \in \mathbf{M}(n, \mathbf{R}) \mid \mathrm{det}M \neq 0\ \},$$

where $\mathrm{det}M$ is the determinant of M. It can be shown that $GL(n, \mathbf{R})$ is a topological group. In fact, since the function $\det : M_{n\times n} \to \mathbf{R}$ is continuous, $\det^{-1}\mathbf{R} \setminus \{0\}$ is open in $\mathbf{R}^{n^2}$, and hence an open subset of $\mathbf{R}^{n^2}$.

We show the mappings $\phi : GL(n, \mathbf{R} \times GL(n, \mathbf{R})) \to GL(n, \mathbf{R})$ and $\psi : GL(n, \mathbf{R}) \to GL(n, \mathbf{R})$ determined by $\phi(a, b) = ab$ and $\psi(a) = a^{-1}$ are both continuous for $a, b \in GL(n, \mathbf{R})$. Let $a = (a_{ij})_{n\times n}$ and $b = (b_{ij})_{n\times n} \in \mathbf{M}(n, \mathbf{R})$. By definition, we know that

$$ab = ((ab)_{ij}) = (\sum_{k=1}^{n} a_{ik}b_{kj}).$$

Whence, $\phi(a, b) = ab$ is continuous. Similarly, let $\psi(a) = (\psi_{ij})_{n\times n}$. Then we know that

$$\psi_{ij} = \frac{a^*_{ij}}{\mathrm{det}a}$$

is continuous, where a^*_{ij} is the cofactor of a_{ij} in the determinant $\mathrm{det}a$. Therefore, $GL(n, \mathbf{R})$ is a topological group.

Now for integers $n_1, n_2, \cdots, n_m \geq 1$, let $\mathscr{E}_G(GL_{n_1}, \cdots, GL_{n_m})$ be a multi-group consisting of $GL(n_1, \mathbf{R})$, $GL(n_2, \mathbf{R})$, $\cdots$, $GL(n_m, \mathbf{R})$ underlying a combinatorial structure G. Then it is itself a combinatorial space. Whence, $\mathscr{E}_G(GL_{n_1}, \cdots, GL_{n_m})$ is a topological multi-group.

Conversely, a combinatorial space of topological groups is indeed a topological multi-group by definition. This means that there are innumerable such multi-groups.

§2. Topological multi-subgroups

A topological space S is *homogenous* if for $\forall a, b \in S$, there exists a continuous mapping $f : S \to S$ such that $f(b) = a$. We have a simple characteristic following.

Theorem 2.1 *If a topological multi-group $(\mathscr{S}_G; \mathscr{O})$ is arcwise connected and associative, then it is homogenous.*

Proof Notice that $\mathscr{S}_G$ is arcwise connected if and only if its underlying graph G is connected. For $\forall a, b \in \mathscr{S}_G$, without loss of generality, assume $a \in \mathscr{H}_0$ and $b \in \mathscr{H}_s$ and

$$P(a,b) = \mathscr{H}_0\mathscr{H}_1\cdots\mathscr{H}_s, \quad s \geq 0,$$

a path from $\mathscr{H}_0$ to $\mathscr{H}_s$ in the graph G. Choose $c_1 \in \mathscr{H}_0 \cap \mathscr{H}_1$, $c_2 \in \mathscr{H}_1 \cap \mathscr{H}_2, \cdots$, $c_s \in \mathscr{H}_{s-1} \cap \mathscr{H}_s$. Then

$$a \circ_0 c_1 \circ_1 c_1^{-1} \circ_2 c_2 \circ_3 c_3 \circ_4 \cdots \circ_{s-1} c_s^{-1} \circ_s b^{-1}$$

is well-defined and

$$a \circ_0 c_1 \circ_1 c_1^{-1} \circ_2 c_2 \circ_3 c_3 \circ_4 \cdots \circ_{s-1} c_s^{-1} \circ_s b^{-1} \circ_s b = a.$$

Let $L = a \circ_0 c_1 \circ_1 c_1^{-1} \circ_2 c_2 \circ_3 c_3 \circ_4 \cdots \circ_{s-1} c_s^{-1} \circ_s b^{-1} \circ_s$. Then L is continuous by the definition of topological multi-group. We finally get a continuous mapping $L : \mathscr{S}_G \to \mathscr{S}_G$ such that $L(b) = Lb = a$. Whence, $(\mathscr{S}_G; \mathscr{O})$ is homogenous. □

Corollary 6.4.1 *A topological group is homogenous if it is arcwise connected.*

A multi-subsystem $(\mathscr{L}_H; \mathcal{O})$ of $(\mathscr{S}_G; \mathscr{O})$ is called a *topological multi-subgroup* if it itself is a topological multi-group. Denoted by $\mathscr{L}_H \leq \mathscr{S}_G$. A criterion on topological multi-subgroups is shown in the following.

Theorem 2.2 *A multi-subsystem $(\mathscr{L}_H; \mathcal{O}_1)$ is a topological multi-subgroup of $(\mathscr{S}_G; \mathscr{O})$, where $\mathcal{O}_1 \subset \mathcal{O}$ if and only if it is a multi-subgroup of $(\mathscr{S}_G; \mathscr{O})$ in algebra.*

Proof The necessity is obvious. For the sufficiency, we only need to prove that for any operation $\circ \in \mathcal{O}_1$, $a \circ b^{-1}$ is continuous in $\mathscr{L}_H$. Notice that the condition (iii) in the definition of topological multi-group can be replaced by:

for any neighborhood $N_{\mathscr{S}_G}(a \circ b^{-1})$ of $a \circ b^{-1}$ in $\mathscr{S}_G$, there always exist neighborhoods $N_{\mathscr{S}_G}(a)$ and $N_{\mathscr{S}_G}(b^{-1})$ of a and b^{-1} such that $N_{\mathscr{S}_G}(a) \circ N_{\mathscr{S}_G}(b^{-1}) \subset N_{\mathscr{S}_G}(a \circ b^{-1})$, where $N_{\mathscr{S}_G}(a) \circ N_{\mathscr{S}_G}(b^{-1}) = \{x \circ y | \forall x \in N_{\mathscr{S}_G}(a), y \in N_{\mathscr{S}_G}(b^{-1})\}$

by the definition of mapping continuity. Whence, we only need to show that for any neighborhood $N_{\mathscr{L}_H}(x \circ y^{-1})$ in $\mathscr{L}_H$, where $x, y \in \mathscr{L}_H$ and $\circ \in \mathcal{O}_1$, there exist neighborhoods $N_{\mathscr{L}_H}(x)$ and $N_{\mathscr{L}_H}(y^{-1})$ such that $N_{\mathscr{L}_H}(x) \circ N_{\mathscr{L}_H}(y^{-1}) \subset N_{\mathscr{L}_H}(x \circ y^{-1})$ in $\mathscr{L}_H$. In fact, each neighborhood $N_{\mathscr{L}_H}(x \circ y^{-1})$ of $x \circ y^{-1}$ can be represented by a form $N_{\mathscr{S}_G}(x \circ y^{-1}) \cap \mathscr{L}_H$. By assumption, $(\mathscr{S}_G; \mathscr{O})$ is a topological multi-group, we know that there are neighborhoods $N_{\mathscr{S}_G}(x)$, $N_{\mathscr{S}_G}(y^{-1})$ of x and y^{-1} in $\mathscr{S}_G$ such that $N_{\mathscr{S}_G}(x) \circ N_{\mathscr{S}_G}(y^{-1}) \subset N_{\mathscr{S}_G}(x \circ y^{-1})$. Notice that $N_{\mathscr{S}_G}(x) \cap \mathscr{L}_H$, $N_{\mathscr{S}_G}(y^{-1}) \cap \mathscr{L}_H$ are neighborhoods of x and y^{-1} in $\mathscr{L}_H$. Now let $N_{\mathscr{L}_H}(x) = N_{\mathscr{S}_G}(x) \cap \mathscr{L}_H$ and $N_{\mathscr{L}_H}(y^{-1}) = N_{\mathscr{S}_G}(y^{-1}) \cap \mathscr{L}_H$. Then we get that $N_{\mathscr{L}_H}(x) \circ N_{\mathscr{L}_H}(y^{-1}) \subset N_{\mathscr{L}_H}(x \circ y^{-1})$ in $\mathscr{L}_H$, i.e., the mapping $(x, y) \to x \circ y^{-1}$ is continuous. Whence, $(\mathscr{L}_H; \mathcal{O}_1)$ is a topological multi-subgroup. □

Particularly, for the topological groups, we know the following consequence.

Corollary 2.2 *A subset of a topological group $(\Gamma; \circ)$ is a topological subgroup if and only if it is a subgroup of $(\Gamma; \circ)$ in algebra.*

§3. Homomorphism theorem on topological multi-subgroups

For two topological multi-groups $(\mathscr{S}_{G_1};\mathscr{O}_1)$ and $(\mathscr{S}_{G_2};\mathscr{O}_2)$, a mapping $\omega:(\mathscr{S}_{G_1};\mathscr{O}_1)\rightarrow(\mathscr{S}_{G_2};\mathscr{O}_2)$ is a *homomorphism* if it satisfies the following conditions:

(1) ω is a homomorphism from multi-groups $(\mathscr{S}_{G_1};\mathscr{O}_1)$ to $(\mathscr{S}_{G_2};\mathscr{O}_2)$, namely, for $\forall a,b\in\mathscr{S}_{G_1}$ and $\circ\in\mathcal{O}_1$, $\omega(a\circ b)=\omega(a)\omega(\circ)\omega(b)$;

(2) ω is a continuous mapping from topological spaces $\mathscr{S}_{G_1}$ to $\mathscr{S}_{G_1}$, i.e., for $\forall x\in\mathscr{S}_{G_1}$ and a neighborhood U of $\omega(x)$, $\omega^{-1}(U)$ is a neighborhood of x.

Furthermore, if $\omega:(\mathscr{S}_{G_1};\mathscr{O}_1)\rightarrow(\mathscr{S}_{G_2};\mathscr{O}_2)$ is an isomorphism in algebra and a homeomorphism in topology, then it is called an *isomorphism*, particularly, an *automorphism* if $(\mathscr{S}_{G_1};\mathscr{O}_1)=(\mathscr{S}_{G_2};\mathscr{O}_2)$ between topological multi-groups $(\mathscr{S}_{G_1};\mathscr{O}_1)$ and $(\mathscr{S}_{G_2};\mathscr{O}_2)$.

Let $(\mathscr{S}_G;\mathscr{O})$ be an associatively topological multi-subgroup and $(\mathscr{L}_H;\mathcal{O})$ one of its topological multi-subgroups with $\mathscr{S}_G=\bigcup\limits_{i=1}^{m}\mathscr{H}_i$, $\mathscr{L}_H=\bigcup_{i=1}^{m}\mathscr{G}_i$ and $\mathscr{O}=\bigcup\limits_{i=1}^{m}\{\circ_i\}$. In [8], we have know the following results on homomorphisms of multi-systems following.

Lemma 3.1([8]) *Let $(\mathscr{H},\widetilde{O})$ be an associative multi-operation system with a unit $1_\circ$ for $\forall\circ\in\widetilde{O}$ and $\mathscr{G}\subset\mathscr{H}$.*

(i) If $\mathscr{G}$ is closed for operations in $\widetilde{O}$ and for $\forall a\in\mathscr{G},\circ\in\widetilde{O}$, there exists an inverse element $a_\circ^{-1}$ in $(\mathscr{G};\circ)$, then there is a representation pair $(R,\widetilde{P})$ such that the quotient set $\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})}$ is a partition of $\mathscr{H}$, i.e., for $a,b\in\mathscr{H},\forall\circ_1,\circ_2\in\widetilde{O}$, $(a\circ_1\mathscr{G})\cap(b\circ_2\mathscr{G})=\emptyset$ or $a\circ_1\mathscr{G}=b\circ_2\mathscr{G}$.

(ii) For $\forall\circ\in\widetilde{O}$, define an operation $\circ$ on $\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})}$ by

$$(a\circ_1\mathscr{G})\circ(b\circ_2\mathscr{G})=(a\circ b)\circ_1\mathscr{G}.$$

Then $(\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})};\widetilde{O})$ is an associative multi-operation system. Particularly, if there is a representation pair $(R,\widetilde{P})$ such that for $\circ'\in\widetilde{P}$, any element in R has an inverse in $(\mathscr{H};\circ')$, then $(\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})},\circ')$ is a group.

Lemma 3.2([8]) *Let ω be an onto homomorphism from associative systems $(\mathscr{H}_1;\widetilde{O}_1)$ to $(\mathscr{H}_2;\widetilde{O}_2)$ with $(\mathcal{I}(\widetilde{O}_2);\widetilde{O}_2)$ an algebraic system with unit $1_{\circ^-}$ for $\forall\circ^-\in\widetilde{O}_2$ and inverse x^{-1} for $\forall x\in(\mathcal{I}(\widetilde{O}_2)$ in $((\mathcal{I}(\widetilde{O}_2);\circ^-)$. Then there are representation pairs $(R_1,\widetilde{P}_1)$ and $(R_2,\widetilde{P}_2)$, where $\widetilde{P}_1\subset\widetilde{O},\widetilde{P}_2\subset\widetilde{O}_2$ such that*

$$\frac{(\mathscr{H}_1;\widetilde{O}_1)}{(\widetilde{\mathrm{Ker}\omega};\widetilde{O}_1)}|_{(R_1,\widetilde{P}_1)}\cong\frac{(\mathscr{H}_2;\widetilde{O}_2)}{(\mathcal{I}(\widetilde{O}_2);\widetilde{O}_2)}|_{(R_2,\widetilde{P}_2)}$$

if each element of $\widetilde{\mathrm{Ker}\omega}$ has an inverse in $(\mathscr{H}_1;\circ)$ for $\circ\in\widetilde{O}_1$.

Whence, by Lemma 3.1, for any integer i, $1\le i\le m$, we get a quotient group $\mathscr{H}_i/\mathscr{G}_i$, i.e., a multi-subgroup $(\mathscr{S}_G/\mathscr{L}_H;\mathcal{O})=\bigcup\limits_{i=1}^{m}(\mathscr{H}_i/\mathscr{G}_i;\circ_i)$ on algebraic multi-groups.

Notice that for a topological space S with an equivalent relation $\sim$ and a projection $\pi: S\rightarrow S/\sim=\{[x]|\forall y\in[x],y\sim x\}$, we can introduce a topology on $S/\sim$ by defining its opened sets to be subsets V in $S/\sim$ such that $\pi^{-1}(V)$ is opened in S. Such topological space $S/\sim$

is called a *quotient space*. Now define a relation in $(\mathscr{S}_G;\mathscr{O})$ by $a \sim b$ for $a, b \in \mathscr{S}_G$ providing $b = h \circ a$ for an element $h \in \mathscr{L}_H$ and an operation $\circ \in \mathcal{O}$. It is easily to know that such relation is an equivalence. Whence, we also get an induced quotient space $\mathscr{S}_G/\mathscr{L}_H$.

Theorem 3.1 *Let $\omega : (\mathscr{S}_{G_1};\mathscr{O}_1) \to (\mathscr{S}_{G_2};\mathscr{O}_2)$ be an opened onto homomorphism from associatively topological multi-groups $(\mathscr{S}_{G_1};\mathscr{O}_1)$ to $(\mathscr{S}_{G_2};\mathscr{O}_2)$, i.e., it maps an opened set to an opened set. Then there are representation pairs $(R_1, \mathcal{P}_1)$ and $(R_2, \mathcal{P}_2)$ such that*

$$\frac{(\mathscr{S}_{G_1};\mathscr{O}_1)}{(\widetilde{\mathrm{Ker}\omega};\mathscr{O}_1)}|_{(R_1,\widetilde{P}_1)} \cong \frac{(\mathscr{S}_{G_2};\mathscr{O}_2)}{(\mathcal{I}(\widetilde{O}_2);\widetilde{O}_2)}|_{(R_2,\widetilde{P}_2)},$$

where $\mathcal{P}_1 \subset \mathscr{O}_1, \mathcal{P}_2 \subset \mathscr{O}_2$, $\mathcal{I}(\mathscr{O}_2) = \{1_\circ, \circ \in \mathscr{O}_2\}$ and

$$\widetilde{\mathrm{Ker}\omega} = \{\ a \in \mathscr{S}_{G_1} \mid \omega(a) = 1_\circ \in \mathcal{I}(\mathscr{O}_2)\ \}.$$

Proof According to Lemma 3.2, we know that there are representation pairs $(R_1, \mathcal{P}_1)$ and $(R_2, \mathcal{P}_2)$ such that

$$\frac{(\mathscr{S}_{G_1};\mathscr{O}_1)}{(\widetilde{\mathrm{Ker}\omega};\mathscr{O}_1)}|_{(R_1,\widetilde{P}_1)} \stackrel{\sigma}{\cong} \frac{(\mathscr{S}_{G_2};\mathscr{O}_2)}{(\mathcal{I}(\widetilde{O}_2);\widetilde{O}_2)}|_{(R_2,\widetilde{P}_2)}$$

in algebra, where $\sigma(a \circ \mathrm{Ker}\omega) = \sigma(a) \circ^{-1} \mathcal{I}(\mathscr{O}_2)$ in the proof of Lemma 3.2. We only need to prove that σ and σ^{-1} are continuous.

On the First, for $x = \sigma(a) \circ^{-1} \mathcal{I}(\mathscr{O}_2) \in \frac{(\mathscr{S}_{G_2};\mathscr{O}_2)}{(\mathcal{I}(\widetilde{O}_2);\widetilde{O}_2)}|_{(R_2,\widetilde{P}_2)}$ let $\widehat{U}$ be a neighborhood of $\sigma^{-1}(x)$ in the space $\frac{(\mathscr{S}_{G_1};\mathscr{O}_1)}{(\widetilde{\mathrm{Ker}\omega};\mathscr{O}_1)}|_{(R_1,\widetilde{P}_1)}$, where $\widehat{U}$ is a union of $a \circ \mathrm{Ker}\omega$ for a in an opened set U and $\circ \in \widetilde{P}_1$. Since ω is opened, there is a neighborhood $\widehat{V}$ of x such that $\omega(U) \supset \widehat{V}$, which enables us to find that $\sigma^{-1}(\widehat{V}) \subset \widehat{U}$. In fact, let $\widehat{y} \in \widehat{V}$. Then there exists $y \in U$ such that $\omega(y) = \widehat{y}$. Whence, $\sigma^{-1}(\widehat{y}) = y \circ \mathrm{Ker}\omega \in \widehat{U}$. Therefore, σ^{-1} is continuous.

On the other hand, let $\widehat{V}$ be a neighborhood of $\sigma(x) \circ^{-1} \mathcal{I}(\mathscr{O}_2)$ in the space $\frac{(\mathscr{S}_{G_2};\mathscr{O}_2)}{(\mathcal{I}(\widetilde{O}_2);\widetilde{O}_2)}|_{(R_2,\widetilde{P}_2)}$ for $x \circ \mathrm{Ker}\omega$. By the continuity of ω, we know that there is a neighborhood U of x such that $\omega(U) \subset \widehat{V}$. Denoted by $\widehat{U}$ the union of all sets $z \circ \mathrm{Ker}\omega$ for $z \in U$. Then $\sigma(\widehat{U}) \subset \widehat{V}$ because of $\omega(U) \subset \widehat{V}$. Whence, σ is also continuous. Combining the continuity of σ and its inverse σ^{-1}, we know that σ is also a homeomorphism from topological spaces $\frac{(\mathscr{S}_{G_1};\mathscr{O}_1)}{(\widetilde{\mathrm{Ker}\omega};\mathscr{O}_1)}|_{(R_1,\widetilde{P}_1)}$ to $\frac{(\mathscr{S}_{G_2};\mathscr{O}_2)}{(\mathcal{I}(\widetilde{O}_2);\widetilde{O}_2)}|_{(R_2,\widetilde{P}_2)}$. $\square$

Corollary 3.1 *Let $\omega : (\mathscr{S}_G;\mathscr{O}) \to (\mathscr{A};\circ)$ be a onto homomorphism from a topological multi-group $(\mathscr{S}_G;\mathscr{O})$ to a topological group $(\mathscr{A};\circ)$. Then there are representation pairs $(R, \widetilde{P})$, $\widetilde{P} \subset \mathscr{O}$ such that*

$$\frac{(\mathscr{S}_G;\mathscr{O})}{(\widetilde{\mathrm{Ker}\omega};\mathscr{O})}|_{(R,\widetilde{P})} \cong (\mathscr{A};\circ).$$

Particularly, if $\mathscr{O} = \{\bullet\}$, i.e., $(\mathscr{S}_G;\bullet)$ is a topological group, then

$$\mathscr{S}_G/\mathrm{Ker}\omega \cong (\mathscr{A};\circ).$$

§4. Topological multi-fields

Definition 4.1 *A distributive multi-system* $(\widetilde{\mathscr{A}};\mathscr{O}_1 \hookrightarrow \mathscr{O}_2)$ *with* $\widetilde{\mathscr{A}} = \bigcup\limits_{i=1}^{m} \mathscr{H}_i$, $\mathscr{O}_1 = \bigcup\limits_{i=1}^{m} \{\cdot_i\}$ *and* $\mathscr{O}_2 = \bigcup\limits_{i=1}^{m} \{+_i\}$ *is called a topological multi-ring if*

(i) $(\mathscr{H}_i;+_i,\cdot_i)$ *is a ring for each integer* i, $1 \leq i \leq m$, *i.e.,* $(\mathscr{H},\mathscr{O}_1 \hookrightarrow \mathscr{O}_2)$ *is a multi-ring;*

(ii) $\widetilde{\mathscr{A}}$ *is a combinatorially topological space* $\mathscr{S}_G$*;*

(iii) the mappings $(a,b) \to a \cdot_i b^{-1}$, $(a,b) \to a +_i (-_i b)$ *are continuous for* $\forall a,b \in \mathscr{H}_i$, $1 \leq i \leq m$.

Denoted by $(\mathscr{S}_G;\mathscr{O}_1 \hookrightarrow \mathscr{O}_2)$ a topological multi-ring. A topological multi-ring $(\mathscr{S}_G;\mathscr{O}_1 \hookrightarrow \mathscr{O}_2)$ is called a *topological divisible multi-ring* or *multi-field* if the previous condition (i) is replaced by $(\mathscr{H}_i;+_i,\cdot_i)$ *is a divisible ring or field for each integer* $1 \leq i \leq m$. Particularly, if $m = 1$, then a topological multi-ring, divisible multi-ring or multi-field is nothing but a topological ring, divisible ring or field. Some examples of topological fields are presented in the following.

Example 4.1 A 1-dimensional Euclidean space $\mathbf{R}$ is a topological field since $\mathbf{R}$ is itself a field under operations additive $+$ and multiplication $\times$.

Example 4.2 A 2-dimensional Euclidean space $\mathbf{R}^2$ is isomorphic to a topological field since for $\forall(x,y) \in \mathbf{R}^2$, it can be endowed with a unique complex number $x+iy$, where $i^2 = -1$. It is well-known that all complex numbers form a field.

Example 4.3 A 4-dimensional Euclidean space $\mathbf{R}^4$ is isomorphic to a topological field since for each point $(x,y,z,w) \in \mathbf{R}^4$, it can be endowed with a unique quaternion number $x+iy+jz+kw$, where

$$ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j,$$

and

$$i^2 = j^2 = k^2 = -1.$$

We know all such quaternion numbers form a field.

For topological fields, we have known a classification theorem following.

Lemma 4.1([12]) *A locally compacted topological field is isomorphic to one of the following:*

(i) *Euclidean real line* $\mathbf{R}$, *the real number field;*

(ii) *Euclidean plane* $\mathbf{R}^2$, *the complex number field;*

(iii) *Euclidean space* $\mathbf{R}^4$, *the quaternion number field.*

Applying Lemma 4.1 and the definition of combinatorial Euclidean space, we can determine these topological multi-fields underlying any connected graph G following.

Theorem 4.1 *For any connected graph G, a locally compacted topological multi-field $(\mathscr{S}_G;\mathscr{O}_1 \hookrightarrow \mathscr{O}_2)$ is isomorphic to one of the following:*

(i) Euclidean space $\mathbf{R}$, $\mathbf{R}^2$ or $\mathbf{R}^4$ endowed respectively with the real, complex or quaternion number for each point if $|G| = 1$;

(ii) combinatorially Euclidean space $\mathscr{E}_G(2,\cdots,2,4,\cdots,4)$ with coupling number, i.e., the dimensional number $l_{ij} = 1, 2$ or 3 of an edge $(\mathbf{R}^i,\mathbf{R}^j) \in E(G)$ only if $i = j = 4$, otherwise $l_{ij} = 1$ if $|G| \geq 2$.

Proof By the definition of topological multi-field $(\mathscr{S}_G;\mathscr{O}_1 \hookrightarrow \mathscr{O}_2)$, for an integer i, $1 \leq i \leq m$, $(\mathscr{H}_i;+_i,\cdot_i)$ is itself a locally compacted topological field. Whence, $(\mathscr{S}_G;\mathscr{O}_1 \hookrightarrow \mathscr{O}_2)$ is a topologically combinatorial multi-field consisting of locally compacted topological fields. According to Lemma 4.1, we know there must be

$$(\mathscr{H}_i;+_i,\cdot_i) \cong \mathbf{R},\ \mathbf{R}^2,\ \text{or}\ \mathbf{R}^4$$

for each integer i, $1 \leq i \leq m$. Let the coordinate system of $\mathbf{R},\mathbf{R}^2,\mathbf{R}^4$ be $x,(y_1,y_2)$ and (z_1,z_2,z_3,z_4). If $|G| = 1$, then it is just the classifying in Theorem 6.4.4. Now let $|G| \geq 2$. For $\forall(\mathbf{R}^i,\mathbf{R}^j) \in E(G)$, we know that $\mathbf{R}^i \setminus \mathbf{R}^j \neq \emptyset$ and $\mathbf{R}^j \setminus \mathbf{R}^i \neq \emptyset$ by the definition of combinatorial space. Whence, $i, j = 2$ or 4. If $i = 2$ or $j = 2$, then $l_{ij} = 1$ because of $1 \leq l_{ij} < 2$, which means $l_{ij} \geq 2$ only if $i = j = 4$. This completes the proof. □

References

[1] R.Abraham, J.E.Marsden and T.Ratiu, *Manifolds, Tensors Analysis and Applications*, Addison-Wesley Publishing Company, Inc., 1983.

[2] H.Iseri, *Smarandache manifolds*, American Research Press, Rehoboth, NM,2002.

[3] L.Kuciuk and M.Antholy, An Introduction to Smarandache Geometries, *JP Journal of Geometry and Topology*, 5(1), 2005,77-81.

[4] Linfan Mao, *Smarandache Multi-Space Theory*, Hexis, Phoenix,American 2006.

[5] Linfan Mao, On algebraic multi-group spaces, *Scientia Magna*, Vol.2,No.1(2006), 64-70.

[6] Linfan Mao, On multi-metric spaces, *Scientia Magna*, Vol.2,No.1(2006), 87-94.

[7] Linfan Mao, Combinatorial speculation and combinatorial conjecture for mathematics, *International J.Math. Combin.* Vol.1(2007), No.1, 1-19.

[8] Linfan Mao, Geometrical theory on combinatorial manifolds, *JP J.Geometry and Topology*, Vol.7, No.1(2007),65-114.

[9] Linfan Mao, Extending homomorphism theorem to multi-systems, *International J.Math. Combin.* Vol.3(2008), 1-27.

[10] Linfan Mao, Action of multi-groups on finite multi-sets, *International J.Math. Combin.* Vol.3(2008), 111-121.

[11] W.S.Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York, etc.(1977).

[12] L.S.Pontrjagin, *Topological Groups*, 2nd ed, Gordon and Breach, New York, 1966.

[13] L.S.Pontrjagin, Über stetige algebraische Körper, *Ann. of Math.*, 33(1932), 163-174.

[14] F.Smarandache, A Unifying Field in Logics. Neutrosopy: Neturosophic Probability, Set, and Logic, American research Press, Rehoboth, 1999.

[15] F.Smarandache, Mixed noneuclidean geometries, *eprint arXiv: math/0010119*, 10/2000.

[16] V.S.Varadarajan, *Lie Groups, Lie Algebras, and Their Representations*, Springer-Verlag New York Inc., (1984).

International J.Math. Combin. Vol.1 (2009), 18-26

Shortest Co-cycle Bases of Graphs

Han Ren

(Department of Mathematics,East China Normal University Shanghai 200062,P.R.China)

E-mail: hren@math.ecnu.edu.cn

Jing Ren

(Economics and Management School of Wuhan University Wuhan 430072,P.R.China)

Abstract In this paper we investigate the structure of the shortest co-cycle base(or SCB in short) of connected graphs, which are related with *map geometries*, i.e., *Smarandache* 2-*dimensional manifolds*. By using a Hall type theorem for base transformation, we show that the shortest co-cycle bases have the same structure (there is a 1-1 correspondence between two shortest co-cycle bases such that the corresponding elements have the same length). As an application in surface topology, we show that in an embedded graph on a surface any nonseparating cycle can't be generated by separating cycles. Based on this result, we show that in a 2-connected graph embedded in a surface, there is a set of surface nonseparating cycles which can span the cycle space. In particular, there is a shortest base consisting surface nonseparating cycle and all such bases have the same structure. This extends a Tutte's result [4].

Key Words: Shortest co-cycle base, nonseparating cycle, map geometries, Smarandache 2-dimensional manifolds.

MR(2000): 05C30.

§1. Introduction

In this paper, graphs are finite, undirected, connected. Used terminology is standard and may be found in $[1]-[2]$. Let A and B be nonempty(possibly overlapping) subsets of $V(G)$. The set $[A, B]$ is a subset of $E(G)$, namely,

$$[A, B] = \{(a, b) \in E(G) | a \in A, b \in B\}.$$

Then the edge set between S and $\overline{S}$ is a *co-cycle*(or a *cut*), denoted by $[S, \overline{S}]$, where S is a nonempty subset of $V(G)$ and $\overline{S} = V(G) - S$. Particularly, for any vertex u, $[u] = \{(u, v) | v \in V(G)\}$ is called a *vertical co-cycle*(or a *vertical cut*). Let X and Y be a pair of sets of edges of G. Then the following operations on co-cycles defined as

$$X \oplus Y = X \cup Y - X \cap Y,$$

[1]Received October 30, 2008. Accepted January 10, 2009.

[2]Supported by NNSF of China under the granted number 10671073

will form a linear vector space $\mathcal{C}^*$, called *co-cycle space* of G. It's well known that the dimension of co-cycle space of a graph G is $n-1$, where n is the number of vertices of G.

The length of a co-cycle c, denoted by $\ell(c)$, is the number of edges in c. The length of a base $\mathcal{B}$, denoted by $\ell(\mathcal{B})$, is the sum of the lengths of its co-cycles. A shortest base is that having the least number of edges.

Let $A, B \subseteq E(G)$. Then we may define an inner product denoted by (A, B) as

$$(A,B) = \sum_{e \in A \cap B} | e |, \qquad |e| = 1.$$

Since any cycle C has even number edges in any co-cycle, i.e., for any cycle C and a co-cycle $[S, \overline{S}]$

$$(C, [S, \overline{S}]) = 0$$

we have that C is *orthogonal* to $[S, \overline{S}]$, i.e.,

Theorem 1 *Let $\mathcal{C}$ and $\mathcal{C}^*$ be, respectively, the cycle space and co-cycle space of a graph G. Then $\mathcal{C}^*$ is just the orthogonal space of $\mathcal{C}$, i.e., $\mathcal{C}^\perp = \mathcal{C}^*$, which implies that*

$$\dim \mathcal{C} + \dim \mathcal{C}^* = |E(G)|.$$

There are many results on cycle space theory. But not many results have ever been seen in co-cycle spaces theory. Here in this paper we investigate the shortest co-cycle bases in a co-cycle space. We first set up a Hall Type theorem for base transformation and then give a sufficient and necessary condition for a co-cycle base to be of shortest. This implies that there exists a 1-1 correspondence between any two shortest co-cycle bases and the corresponding elements have the same length. As applications, we consider embedded graphs in a surface. By the definition of geometric dual multigraph, we show that a nonseparating cycle can't be generated by a collection of separating cycles. So there is a set of surface nonseparating cycles which can span the cycle space. In particular, there is a shortest base consisting surface nonseparating cycle and all such bases have the same structure. This extends a Tutte's result [4].

§2. Main results

Here in this section we will set up our main results. But first we have to do some preliminary works. Let $A = (A_1, A_2, \cdots, A_n)$ be a set of finite sets. A *distinct representatives(SDR)* is a set of $\{a_1, a_2, \cdots, a_n\}$ of n elements such that $a_i \in A_i$ for $i = 1, 2, \cdots, n$. The following result is the famous condition of Hall for the existence of SDR.

Hall's Theorem([3]) *A family $(A_1, \cdots, A_n)$ of finite sets has a system of distinct representatives(SDR) if and only if the following condition holds:*

$$\left| \bigcup_{\alpha \in J} A_\alpha \right| \geq |J|, \qquad \forall J \subseteq \{1, \cdots, n\}.$$

Let G be a connected graph with a co-cycle base $\mathcal{B}$ and c a co-cycle. We use $\mathrm{Int}(c,\mathcal{B})$ to represent the co-cycles in $\mathcal{B}$ which span c.

Another Hall Type Theorem *Let G be a connected graph with $\mathcal{B}_1$ and $\mathcal{B}_2$ as two co-cycle bases. Then the system of sets $A=\{\mathrm{Int}(c,\mathcal{B}_1)\mid c\in\mathcal{B}_2\}$, has a SDR.*

Proof What we need is to show that the system must satisfy the Hall's condition:

$$\forall J\subseteq\mathcal{B}_2\Rightarrow\left|\bigcup_{c\in J}\mathrm{Int}(c,\mathcal{B}_1)\right|\geq\big|J\big|.$$

Suppose the contrary. Then $\exists J\subseteq\mathcal{B}_2$ such that $\left|\bigcup\limits_{c\in J}\mathrm{Int}(c,\mathcal{B}_1)\right|<\big|J\big|$. Now the set of linear independent elements $\{c\mid c\in J\}$ is spanned by at most $\big|J\big|-1$ vectors in $\mathcal{B}_1$, a contradiction as desired. □

Theorem 2 *Let $\mathcal{B}$ be a co-cycle base of G. Then $\mathcal{B}$ is shortest if and only if for any co-cycle c,*

$$\forall\alpha\in\mathrm{Int}(c,\mathcal{B})\Rightarrow\ell(c)\geq\ell(\alpha).$$

Remark This result shows that in a shortest co-cycle base, a co-cycle can't be generated by shorter vectors.

Proof Let $\mathcal{B}$ be a co-cycle base of G. Suppose that there is a co-cycle c such that $\exists\alpha\in\mathrm{Int}(c),\ell(c)<\ell(\alpha)$, then $\mathcal{B}-c+\alpha$ is also a co-cycle base of G, which is a shorter co-cycle base, a contradiction as desired.

Suppose that $\mathcal{B}=\{\alpha_1,\alpha_2,\cdots,\alpha_{n-1}\}$ is a co-cycle base of G such that for any co-cycle $c,\ell(c)\geq\ell(\alpha),\forall\alpha\in\mathrm{Int}(c)$, but $\mathcal{B}$ is not a shortest co-cycle base. Let $\mathcal{B}^*=\{\beta_1,\beta_2,\cdots,\beta_{n-1}\}$ be a shortest co-cycle base. By Hall Type Theorem, $A=(\mathrm{Int}(\beta_1,\mathcal{B}),\mathrm{Int}(\beta_2,\mathcal{B}),\cdots,\mathrm{Int}(\beta_{n-1},\mathcal{B}))$ has an SDR $(\alpha_1',\alpha_2'\cdots,\alpha_{n-1}')$ such that $\alpha_i'\in\mathrm{Int}(\beta_i,\mathcal{B}),\ell(\beta_i)\geq\ell(\alpha_i')$. Hence $\ell(\mathcal{B}^*)=\sum\limits_{i=1}^{n-1}\ell(\beta_i)\geq\sum\limits_{i=1}^{n-1}\ell(\alpha_i')=\ell(\mathcal{B})$, a contradiction with the definition of $\mathcal{B}$. □

The following results say that some information about short co-cycles is contained in a shorter co-cycle base.

Theorem 3 *If $\{c_1,c_2,\cdots,c_k\}$ is a set of linearly independent shortest co-cycles of connected graph G, then there must be a shortest co-cycle base containing $\{c_1,c_2,\cdots,c_k\}$.*

Proof Let $\mathcal{B}$ be the shortest co-cycle base such that the number of co-cycles in $\mathcal{B}\cap\{c_1,c_2,\cdots,c_k\}$ is maximum. Suppose that $\exists c_i\notin\mathcal{B},1\leq i\leq k$. Then $\mathrm{Int}(c_i,\mathcal{B})\backslash\{c_1,\cdots,c_k\}$ is not empty, otherwise $\{c_1,c_2,\cdots,c_k\}$ is linear dependent. So there is a co-cycle $\alpha\in\mathrm{Int}(c_i,\mathcal{B})\backslash\{c_1,\cdots,c_k\}$ such that $\ell(c_i)\geq\ell(\alpha)$. Then $\ell(c_i)=\ell(\alpha)$, since c_i is the shortest co-cycle. Hence $\mathcal{B}^*=\mathcal{B}-\alpha+c_i$ is a shortest co-cycle base containing more co-cycles in $\{c_1,c_2,\cdots,c_k\}$ than $\mathcal{B}$. A contradiction with the definition of $\mathcal{B}$. □

Corollary 4 *If c is a shortest co-cycle, then c is in some shortest co-cycle base.*

Theorem 5 *Let $\mathcal{B}, \mathcal{B}^*$ be two different shortest co-cycle bases of connected graph G, then exists a one-to-one mapping $\varphi : \mathcal{B} \to \mathcal{B}^*$ such that $\ell(\varphi(\alpha)) = \ell(\alpha)$ for all $\alpha \in \mathcal{B}$.*

Proof Let $\mathcal{B} = \{\alpha_1, \alpha_2, \cdots, \alpha_{n-1}\}, \mathcal{B}^* = \{\beta_1, \beta_2, \cdots, \beta_{n-1}\}$. By Hall Type Theorem, $A = (\mathrm{Int}(\alpha_1, \mathcal{B}^*), \mathrm{Int}(\alpha_2, \mathcal{B}^*), \cdots, \mathrm{Int}(\alpha_{n-1}, \mathcal{B}^*))$ has a SDR $(\beta_{\sigma(1)}, \beta_{\sigma(2)}, \cdots, \beta_{\sigma(n-1)})$, where σ is a permutation of $\{1, 2, \cdots, n-1\}$. Since $\mathcal{B}^*$ is a SCB, by Theorem 2, we have $\ell(\alpha_i) \geq \ell(\beta_{\sigma(i)}), \forall i = 1, \ldots, n-1$. On the other hand, $\mathcal{B}$ and $\mathcal{B}^*$ are both shortest, i.e. $\ell(\mathcal{B}) = \ell(\mathcal{B}^*)$. So $\ell(\alpha_i) = \ell(\beta_{\sigma(i)}), \forall i = 1, \ldots, n-1$. Let $\varphi(\alpha_i) = \beta_{\sigma(i)}, \forall i = 1, \ldots n-1$. Then φ is a one-to-one mapping such that $\ell(\varphi(\alpha)) = \ell(\alpha)$ for all $\alpha \in \mathcal{B}$. □

Since a co-cycle can't be generated by longer ones in a shortest co-cycle base, we have

Corollary 6 *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be a pair of shortest co-cycle bases in a graph G. Then their parts of shortest co-cycles are linearly equivalent.*

Example 1 The length of the SCB of complete graph K_n is $(n-1)^2$.

Example 2 The length of the SCB of complete graph $K_{a,b}$ $(a \leq b)$ is $2ab - b$.

Example 3 The length of the SCB of a tree with n vertex T_n is $n-1$.

Example 4 The length of the SCB of a Halin graph with n vertex is $3(n-1)$.

Proof of Examples By theorem 1, for any vertex v, the vertical co-cycle $[v]$ is the shortest co-cycle of K_n. Clearly the set of $n-1$ vertical co-cycles is a SCB. So there're n SCBs with length $(n-1)^2$.

The proof for examples $2, 3$ and 4 is similar. □

§3. Application to surface topology

In this section we shall apply the results obtained in Section 1 to surface topology. Now we will introduce some concepts and terminologies in graph embedding theory, which are related with *map geometries*, i.e., *Smarandache 2-dimensional manifolds.*

Let G be a connected multigraph. An *embedding* of G is a pair $\Pi = (\pi, \lambda)$ where $\pi = \{\pi_v \mid v \in V(G)\}$ is a rotation system and λ is a signature mapping which assigns to each edge $e \in E(G)$ a sign $\lambda(e) \in \{-1, 1\}$. If e is an edge incident with $v \in V(G)$, then the cyclic sequence $e, \pi_v(e), \pi_v^2(e), \cdots$ is called the Π-*clockwise ordering* around v(or the local rotation at v). Given an embedding Π of G we say that G is Π-*embedded.*

We define the Π-*facial walks* as the closed walks in G that are determined by the face traversal procedure. The edges that are contained(twice) in only one facial walk are called *singular*.

A cycle C of a Π-embedded graph G is Π-*onesided* if it has an odd number of edges with negative sign. Otherwise C is Π-*twosided.*

Let H be a subgraph of G. An H-*bridge* in G is a subgraph of G which is either an edge not in H but with both ends in H, or a connected component of $G - V(H)$ together with all edges which have one end in this component and other end in H.

Let $C = v_0e_1v_1e_2\cdots v_{l-1}e_lv_l$ be a Π-twosided cycle of a Π-embedded graph G. Suppose that the signature of Π is positive on C. We define the *left graph* and the *right graph* of C as follows. For $i = 1, \cdots, l$, if $e_{i+1} = \pi_{v_i}^{k_i}(e_i)$, then all edges $\pi_{v_i}(e_i), \pi_{v_i}^2(e_i), \cdots, \pi_{v_i}^{k_i-1}(e_i)$ are said to be on the left side of C. Now, the left graph of C, denoted by $G_l(C,\Pi)$(or just $G_l(C)$), is defined as the union of all C-bridges that contain an edge on the left side of C. The right graph $G_r(C,\Pi)$(or just $G_r(C)$) is defined analogously. If the signature is not positive on C, then there is an embedding Π' equivalent to Π whose signature is positive on C(since C is Π-twosided). Now we define $G_l(C,\Pi)$ and $G_r(C,\Pi)$ as the left and the right graph of C with respect to the embedding Π'. Note that a different choice of Π' gives rise to the same pair $\{G_l(C,\Pi), G_r(C,\Pi)\}$ but the left and the right graphs may interchange.

A cycle C of a Π-embedded graph G is Π-*separating* if C is Π-twosided and $G_l(C,\Pi)$ and $G_r(C,\Pi)$ have no edges in common.

Given an embedding $\Pi = (\pi, \lambda)$ of a connected multigraph G, we define the geometric dual multigraph G^* and its embedding $\Pi^* = (\pi^*, \lambda^*)$, called the *dual embedding* of Π,as follows. The vertices of G^* correspond to the Π-facial walks. The edges of G^* are in bijective correspondence $e \longmapsto e^*$ with the edges of G, and the edge e^* joins the vertices corresponding to the Π-facial walks containing e.(If e is singular, then e^* is a loop.) If $W = e_1, \cdots, e_k$ is a Π-facial walk and w its vertex of G^*, then $\pi_w^* = (e_1^*, \cdots, e_k^*)$. For $e^* = ww'$ we set $\lambda^*(e^*) = 1$ if the Π-facial walks W and W' used to define π_w^* and $\pi_{w'}^*$ traverse the edge e in opposite direction; otherwise $\lambda^*(e^*) = -1$.

Let H be a subgraph of G. H^* is the union of edges e^* in G^*, where e is an edge of H.

Lemma 7 *Let G be a Π-embedded graph and G^* its geometric dual multigraph. C is a cycle of G. Then C is a Π-separating cycle if and only if C^* is a co-cycle of G^*, where C^* is the set of edges corresponding those of C.*

Proof First, we prove the necessity of the condition. Since C is a Π-separating cycle, C is Π-twosided and $G_l(C,\Pi)$ and $G_r(C,\Pi)$ have no edges in common. Assume that $C = v_0e_1v_1e_2\cdots v_{l-1}e_lv_l$, and$\lambda(e_i) = 1, i = 1, \cdots, l$. We divide the vertex set of G^* into two parts V_l^* and V_r^*, such that for any vertex w in $V_l^*(V_r^*)$, w corresponds to a facial walk W containing an edge in $G_l(C)(G_r(C))$.

Claim 1. $V_l^* \cap V_r^* = \Phi$, i.e. each Π-facial walk of G is either in $G_l(C) \cup C$ or in $G_r(C) \cup C$.

Otherwise, there is a Π-facial walk W of G, such that W has some edges in $G_l(C)$ and some in $G_r(C)$. Let $W = P_1Q_1\cdots P_kQ_k$, where P_i is a walk in which none of the edges is in $C(i = 1, \cdots, k)$,and Q_i is a walk in which all the edges are in $C(j = 1, \cdots, k)$.Since each P_i is contained in exactly one C-bridge, there exist $t \in \{1, \cdots, k\}$ such that $P_t \subseteq G_l(C), P_{t+1} \subseteq G_r(C)$(Note $P_{t+1} = P_1$). Let $Q_t = v_pe_{p+1}\cdots e_qv_q$. Then $W = \cdots e^tv_pe_{p+1}\cdots e_qv_qe^{t+1}\cdots$, where $e^t \in P_t, e^{t+1} \in P_{t+1}$. Since e^t and e^{t+1} are, respectively, on the left and right side of C, $\pi_{v_p}(e^t) = e_{p+1}$ and $\pi_{v_q}(e^{t+1}) = e_q$. As W is a Π-facial walk, there exist an edge e in Q_t such that $\lambda(e) = -1$, a contradiction with the assumption of C.

Next we prove that $[V_l^*, V_r^*] = C^*$.

Let $e^* = w_1w_2$ be an edge in G^*, where w_1 and w_2 are, respectively, corresponding to the

Π-facial walks W_1 and W_2 containing e in common.

If $e^* \in [V_l^*, V_r^*]$ where $w_1 \in V_l^*, w_2 \in V_r^*$. Then $W_1 \subseteq G_l(C) \cup C$ and $W_2 \subseteq G_r(C) \cup C$. As $G_l(C, \Pi)$ and $G_r(C, \Pi)$ have no edges in common, we have $e \in C$ i.e. $e^* \in C^*$. So $[V_l^*, V_r^*] \subseteq C^*$.

Claim 2. If $e^* = w_1 w_2 \in C^*$, i.e., $e \in C$, then $W_1 \neq W_2$, and W_1, W_2 can't be contained in $G_l(C) \cup C$(or $G_r(C) \cup C$) at the same time.

Suppose that $W_1 = W_2$. Let $W_1 = u_0 e u_1 \widetilde{e_1} u_2 \widetilde{e_2} \cdots u_k \widetilde{e_k} u_1 e u_0 \cdots$. Clearly, $\{\widetilde{e_1}, \cdots, \widetilde{e_k}\}$ is not a subset of $E(C)$, otherwise C isn't a cycle. So we may assume that $\widetilde{e_s} \notin C, \widetilde{e_t} \notin C, (1 \leq s \leq t \leq k)$ such that $\widetilde{e_i} \in C, i = 1, \cdots, s-1$ and $\widetilde{e_j} \in C, j = t+1, \cdots, k$. Let $C = u_0 e u_1 \widetilde{e_1} \cdots \widetilde{e_{s-1}} \cdots = u_0 e u_1 \widetilde{e_k} u_k \cdots \widetilde{e_{t+1}} \cdots$. Since W_1 is a Π-facial walk, assume that $\widetilde{e_1} = \pi_{u_1}(e)$ and $\widetilde{e_k} = \pi_{u_1}^{-1}(e)$. As the sign of edges on C is 1,we get $\widetilde{e_s} = \pi_{u_s}(\widetilde{e_{s-1}})$ and $\widetilde{e_t} = \pi_{u_{t+1}}^{-1}(\widetilde{e_{t+1}})$. So $\widetilde{e_s} \in G_l(C)$ and $\widetilde{e_t} \in G_r(C)$, a contradiction with **Claim 1**.

Suppose $W_1 \neq W_2$ and $W_1, W_2 \subseteq G_l(C) \cup C$. Let $W_1 = v_0 e v_1 e_1^1 v_2^1 e_2^1 \cdots v_0$ and $W_2 = v_0 e v_1 e_1^2 v_2^2 e_2^2 \cdots v_0$. Assume that $e_1^1 \neq e_1^2$, otherwise we consider e_2^1 and e_2^2.

Case 1. $e_1^1 \in C$ and $e_1^2 \in C$. Then $e_1^1 = e_1^2$.

Case 2. $e_1^1 \notin C$ and $e_1^2 \notin C$. By claim 1, $\pi_{v_1}(e) = e_1^1$ and $\pi_{v_1}(e) = e_1^2$, then $e_1^1 = e_1^2$.

Case 3. $e_1^1 \notin C$ and $e_1^2 \in C$. By claim 1, $\pi_{v_1}(e) = e_1^1$. As $e_1^1 \neq e_1^2$, we get $\pi_{v_1}^{-1}(e) = e_1^2$. Let $e_t^2 \notin C$, and $e_1^2, \cdots, e_{t-1}^2 \in C$. Since $\lambda(e_i^2) = 1$, $\pi_{v_{i+1}^2}^{-1}(e_i^2) = e_{i+1}^2$ $(i = 1, \cdots, t-1)$. Then $\pi_{v_t^2}^{-1}(e_{t-1}^2) = e_t^2$,i.e. $e_t^2 \in G_r(C)$. So $W_2 \subseteq G_r(C) \cup C$, a contradiction with **Claim 1**.

Case 4. $e_1^1 \in C$ and $e_1^2 \notin C$.Like case 3,it's impossible.

So claim 2 is valid. And by claim 2, $C^* \subseteq [V_l^*, V_r^*]$.

Summing up the above discussion, we get that C^* is a co-cycle of G^*.

Next, we prove the sufficiency of the condition. Since C^* is a co-cycle of G^*, let $C^* = [V_l^*, V_r^*]$, where $V_l^* \cap V_r^* = \Phi$. Then all the Π-facial walks are divided into two parts F_l and F_r, where for any Π-facial walk W in $F_l(F_r)$ corresponding to a vertex w in $V_l^*(V_r^*)$. Firstly, we prove that C is twosided.Let $C = v_0 e_1 v_1 e_2 \cdots v_{l-1} e_l v_l$. Supposed that C is onesided, with $\lambda(e_1) = -1$ and $\lambda(e_i) = 1, i = 2 \cdots, l$. Then $\lambda^*(e_1^*) = -1$ and $\lambda^*(e_i^*) = 1, i = 2 \cdots, l$. Let $e_1^* = \widetilde{w_1}\widetilde{w_2}$, where $\widetilde{w_1} \in V_l^*, \widetilde{w_2} \in V_r^*$. Suppose that $\widetilde{w_1}$ and $\widetilde{w_2}$ are, respectively, corresponding to the Π-facial walks $\widetilde{W_1}$ and $\widetilde{W_2}$ containing e_1.Then $\widetilde{W_1} \in F_l, \widetilde{W_2} \in F_r$. Since $\widetilde{W_1}$ is a Π-facial walk, there must be another edge $\widetilde{e_2}$ with negative sign appearing once in $\widetilde{W_1}$. We change the signature of $\widetilde{e_2}$ into 1.(Here we don't consider the embedding) Suppose W_2 is the other Π-facial walk containing $\widetilde{e_2}$. Like $\widetilde{W_1}$, there must be an edge $\widetilde{e_3}$ with negative sign appearing once in W_2.Then change the signature of $\widetilde{e_3}$ into 1. So similarly we got a sequence $\widetilde{W_1}, \widetilde{e_2}, W_2, \widetilde{e_3}, W_3, \cdots$,where the signature of $\widetilde{e_2}, \widetilde{e_3}, \cdots$ in Π are -1, and $W_2, W_3, \cdots$ are all in W_l. Since the number of edges with negative sign is finite, $\widetilde{W_2}$ must in the sequence, a contradiction with $V_l^* \cap V_r^* = \Phi$.

Secondly, we prove that $G_l(C)$ and $G_r(C)$ have no edge in common.

Let $C = v_0 e_1 v_1 e_2 \cdots v_{l-1} e_l v_l$, and $\lambda(e_i) = 1, i = 1, \cdots, l$. Let $\pi_{v_1} = (e_1^1, e_2^1, \cdots, e_s^1)$ and $\pi_{v_2} = (e_1^2, e_2^2, \cdots, e_t^2)$, where $e_1^1 = e_1, e_p^1 = e_2 (1 < p \leq s)$ and $e_1^2 = e_2, e_q^2 = e_3 (1 < q \leq t)$. Then we have some Π-facial walks $W_i^1 = e_i^1 v_1 e_{i+1}^1 \cdots$ $(i = 1, \cdots, s)$ and $W_j^2 = e_j^2 v_2 e_{j+1}^2 \cdots$

$(j = 1, \cdots, t)$.Note that $W^1_{p-1} = W^2_1 = e^1_{p-1}v_1e_2v_2e^2_2\cdots$. Suppose that $W^1_1 \in F_l$. Then $W^1_i \in F_l$, by $e^1_i \notin C(i = 2, \cdots, p-1)$. Further more $W^1_p \in F_r$, as $e^1_p \in C$. Then $W^1_j \in F_r$, since $e^1_j \notin C(j = p+1, \cdots, s)$. Similarly, as $W^2_1 = W^1_{p-1} \in F_l$, we get $W^2_i \in F_l, i = 1, \cdots, q-1$ and $W^2_j \in F_r, j = q, \cdots, t$. And then consider $v_3, v_4, \cdots$. It's clearly that for any facial walk W, if W contain an edge on the left(right) side of C, then $W \in F_l(F_r)$.

Let $V_l = V(F_l) - V(C)$ and $V_r = V(F_r) - V(C)$.

Claim 3. $V_l \cap V_r = \Phi$. If $v \notin C$, let $\pi_v = (e^1, e^2, \cdots, e^k)$, $W^i = e^ive^{i+1}\cdots$ be a Π-facial walk$(i = 1, \cdots, k)$, where $e^{k+1} = e^1$. Suppose $W^1 \in F_l$, then $W^i \in F_l$, since $e^i \notin C$ $(i = 2, \cdots, k)$. So we say $v \in V_l$. Similarly, if all the Π-facial walks are in F_r, we say $v \in V_r$.

Suppose B is a C-bridge containing an edge in $G_l(C)$ and an edge in $G_r(C)$. Then $V(B) \cap V_l \neq \Phi$ and $V(B) \cap V_r \neq \Phi$ On the other hand, since B is connected there is an edge v_lv_r, where $v_l \in V_l$ and $v_r \in V_r$. Clearly $v_lv_r \notin C$, then $v_lv_r \in F_l$(or $v_lv_r \in F_r$). So $V_l \cap V_r \neq \Phi$, a contradiction with claim 3. This completes the proof of lemma 7. □

Lemma 8 *Let C be a cycle in a Π-embedded graph G which is generated by a collection of separating cycles(i.e., $C = C_1 \oplus C_2 \oplus \cdots \oplus C_k$).Then the edge set C^* which is determined by edges in C is generated by $\{C^*_1, C^*_2, \cdots, C^*_k\}$ i.e.$C^* = C^*_1 \oplus C^*_2 \oplus \cdots \oplus C^*_k$, where C^*_i corresponds to C_i in G^*.*

Proof For any edge e^* in C^*,$e \in C = C_1 \oplus C_2 \oplus \cdots \oplus C_k$. So there are odd number of C_i containing e, i.e. there are odd number of C^*_i containing e^*.So $e^* \in C^*_1 \oplus C^*_2 \oplus \cdots \oplus C^*_k$.Thus $C^* \subseteq C^*_1 \oplus C^*_2 \oplus \cdots \oplus C^*_k$.

For any edge e^* in $C^*_1 \oplus C^*_2 \oplus \cdots \oplus C^*_k$,$e^*$ appears odd times in $\{C^*_1, C^*_2, \cdots, C^*_k\}$, i.e. e appears odd times in $\{C_1, C_2, \cdots, C_k\}$.So $e \in C_1 \oplus C_2 \oplus \cdots \oplus C_k = C$.Then $e^* \in C^*$.Thus $C^*_1 \oplus C^*_2 \oplus \cdots \oplus C^*_k \subseteq C^*$. □

Lemma 9 *Let $[S, \overline{S}]$ and $[T, \overline{T}]$ be a pair of co-cycle of G. Then $[S, \overline{S}] \oplus [T, \overline{T}]$ is also a co-cycle of G.*

Proof Let $A = S \cap T, B = S \cap \overline{T}, C = \overline{S} \cap T, D = \overline{S} \cap \overline{T}$. Then

$$\begin{aligned} & [S, \overline{S}] \oplus [T, \overline{T}] \\ = \; & ([A, C] \oplus [A, D] \oplus [B, C] \oplus [B, D]) \oplus ([A, B] \oplus [A, D] \oplus [C, B] \oplus [C, D]) \\ = \; & [A, C] \oplus [B, D] \oplus [A, B] \oplus [C, D] \\ = \; & [A \cup D, B \cup C] = [A \cup D, \overline{A \cup D}] \end{aligned}$$

So $[S, \overline{S}] \oplus [T, \overline{T}]$ is also a co-cycle. □

Theorem 10 *Separating cycles can't span any nonseparating cycle.*

Proof Let G be a connected Π-embedded multigraph and G^* its geometric dual multigraph. Suppose $C = C_1 \oplus \cdots \oplus C_k$ is a nonseparating cycle of G, where $C_1, \cdots, C_k$ are separating cycles. Then $C^* = C^*_1 \oplus \cdots \oplus C^*_k$,where C^* and C^*_i are, respectively, the geometric dual graph of C and C_i, for any $i = 1, \cdots, k$. By lemma 1, C^* isn't a co-cycle while C_i is a nonseparating cycle of G. Thus, some co-cycles could span a nonco-cycle, a contradiction with lemma 3. $\square$

A cycle of a graph is *induced* if it has no chord. A famous result in cycle space theory is due to W.Tutte which states that in a 3-connected graph, the set of induced cycles (each of which can't separated the graph) generates the whole cycle space[4]. If we consider the case of embedded graphs, we have the following

Theorem 11 *Let G be a 2-connected graph embedded in a nonspherical surface such that its facial walks are all cycles. Then there is a cycle base consists of induced nonseparating cycles.*

Remark Tutte's definition of nonseparating cycle differs from ours. The former defined a cycle which can't separate the graph, while the latter define a cycle which can't separate the surface in which the graph is embedded. So, Theorem 11 and Tutte's result are different. From our proof one may see that this base is determined simply by shortest nonseparating cycles. As for the structure of such bases, we may modify the condition of Theorem 2 and obtain another condition for bases consisting of shortest nonseparating cycles.

Proof Notice that any cycle base consists of two parts: the first part is determined by nonseparating cycles while the second part is composed of separating cycles. So, what we have to do is to show that any facial cycle may be generated by nonseparating cycles. Our proof depends on two steps.

Step 1. Let x be a vertex of G. Then there is a nonseparating cycle passing through x.

Let C' be a nonseparating cycle of G which avoids x. Then by Menger's theorem, there are two inner disjoint paths P_1 and P_2 connecting x and C'. Let $P_1 \cap C' = \{u\}$,$P_2 \cap C' = \{v\}$. Suppose further that $u\overrightarrow{C'}v$ and $v\overrightarrow{C'}u$ are two segments of C', where $\overrightarrow{C}$ is an orientation of C. Then there are three inner disjoint paths connecting u and v:

$$Q_1 = u\overrightarrow{C}v, \qquad Q_2 = v\overrightarrow{C}u, \qquad Q_3 = P_1 \cup P_2.$$

Since $C' = Q_1 \cup Q_2$ is non separating, at least one of cycles $Q_2 \cup Q_3$ is nonseparating by Theorem 10.

Step 2. Let ∂f be any facial cycle. Then there exist two nonseparating cycles C_1 and C_2 which span ∂f.

In fact, we add a new vertex x into the inner region of ∂f(i.e. $\mathrm{Int}(\partial f)$) and join new edges to each vertex of ∂f. Then the resulting graph also satisfies the condition of Theorem 11. By Step 1, there is a nonseparating C passing through x. Let u and v be two vertices of $C \cap \partial f$. Then $u\overrightarrow{C}v$ together with two segments of ∂f connecting u and v forms a pair of nonseparating cycles. $\square$

Theorem 12 *Let G be a 2-connected graph embedded in a nonspherical surface such that all of its facial walks are cycles. Let $\mathcal{B}$ be a base consisting of nonsepareting cycles. Then $\mathcal{B}$ is* *shortest iff for every nonseparating cycle C,*

$$\forall\alpha \in \mathrm{Int}(C) \Rightarrow |C| \geq |\alpha|,$$

where $\mathrm{Int}(C)$ *is the subset of cycles of* $\mathcal{B}$ *which span* C.

Theorem 13 *Let G be a 2-connected graph embedded in some nonspherical surface with all its facial walks are cycles. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be a pair of shortest nonseparating cycle bases. Then there exists a 1-1 correspondence φ between elements of $\mathcal{B}_1$ and $\mathcal{B}_2$ such that for every element $\alpha \in \mathcal{B}_1 : |\alpha| = |\varphi(\alpha)|$.*

Proof: It follows from the proving procedure of Theorems 2 and 5. □

References

[1] J.A.Bondy and U.S.R.Murty, *Graph Theory with Applications*, The Macmilan Press Ltd, 1976.
[2] B.Mohar and C.Thomassen, *Graphs on Surfaces*, The John Hopkins University Press, 2001.
[3] Phillip Hall, On representatives of subsets, *London Math.Soc.*,10(1935), 26-30.
[4] W.Tutte, How to draw a graph, *Proc.Lodon Math.Soc.*13(1963), 743-768.

International J.Math. Combin. Vol.1 (2009), 27-37

On Involute and Evolute Curves of Spacelike Curve with a Spacelike Principal Normal in Minkowski 3-Space

Bahaddin Bukcu

(Department of Mathematics of Gazi Osman Pasa University, Tokat-Turkey)

Murat Kemal Karacan

(Department of Mathematics of Usak University, Eylul Campus,64200, Usak-Turkey)

E-mail: murat.karacan@usak.edu.tr,mkkaracan@yahoo.com

Abstract In this study, we have generalized the involute and evolute curves of the spacelike curve α with a spacelike principal normal in Minkowski 3-Space. Firstly, we have shown that, the length between the spacelike curves α and β is constant. Furthermore, the Frenet frame of the involute curve β has been found as depend on curvatures of the curve α. We have determined the curve α is planar in which conditions. Secondly, we have found transformation matrix between the evolute curve β and the curve α. Finally, we have computed the curvatures of the evolute curve β.

Key Words: Spacelike curve, involutes, evolutes, Minkowski 3-space.

AMS(2000): 53A04,53B30,53A35.

§1. Preliminaries

Let $R^3 = \{(x_1, x_2, x_3)|x_1, x_2, x_3 \in R\}$ be a 3-dimensional vector space, and let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two vectors in IR^3. The Lorentz scalar product of x and y is defined by

$$\langle x, y\rangle_L = -x_1y_1 + x_2y_2 + x_3y_3,$$

$E_1^3 = \left(R^3, \langle x, y\rangle_L\right)$ is called 3-dimensional Lorentzian space, Minkowski 3-Space or 3- dimensional semi-euclidean space. The vector x in IE_1^3 is called a spacelike vector, null vector or a timelike vector if $\langle x, x\rangle_L > 0$ or $x = 0$, $\langle x, x\rangle_L = 0$ or $\langle x, x\rangle_L < 0$, respectively. For $x \in E_1^3$, the norm of the vector x defined by $\|x\|_L = \sqrt{|\langle x, x\rangle_L|}$, and x is called a unit vector if $\|x\|_L = 1$. For any x, $y \in E_1^3$, Lorentzian vectoral product of x and y is defined by

$$x \wedge_L y = (x_3y_2 - x_2y_3, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

We denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the curve $\alpha(s)$. Then $T(s), N(s)$ and $B(s)$ are tangent, the principal normal and the binormal vector of the curve

[1]Received November 24, 2008. Accepted January 12, 2009.

$\alpha(s)$, respectively. Depending on the causal character of the curve α, we have the following Frenet-Serret formulae:

If α is a spacelike curve with a spacelike principal normal N,

$$T' = \kappa N,\ N = -\kappa T + \tau B,\ B' = \tau N \tag{1.1}$$

$$\langle T,T\rangle_L = \langle N,N\rangle_L = 1, \langle B,B\rangle_L = -1, \langle T,N\rangle_L = \langle N,B\rangle_L = \langle T,B\rangle_L = 0.$$

If α is a spacelike curve with a timelike principal normal N,

$$T' = \kappa N,\ N = \kappa T + \tau B,\ B' = \tau N \tag{1.2}$$

$$\langle T,T\rangle_L = \langle B,B\rangle_L = 1, \langle N,N\rangle_L = -1, \langle T,N\rangle_L = \langle N,B\rangle_L = \langle T,B\rangle_L = 0.$$

If α is a timelike curve and finally,

$$T' = \kappa N,\ N = \kappa T + \tau B,\ B' = -\tau N \tag{1.3}$$

$$\langle T,T\rangle_L = -1, \langle B,B\rangle_L = \langle N,N\rangle_L = 1, \langle T,N\rangle_L = \langle N,B\rangle_L = \langle T,B\rangle_L = 0.$$

known in [2]. If the curve α is non-unit speed, then

$$\kappa(t) = \frac{\left\|\alpha^{'}(t) \wedge_L \alpha^{''}(t)\right\|_L}{\left\|\alpha^{'}(t)\right\|_L^3},\ \ \tau(t) = \frac{\det\left(\alpha^{'}(t), \alpha^{''}(t), \alpha^{'''}(t)\right)}{\left\|\alpha^{'}(t) \wedge_L \alpha^{''}(t)\right\|_L^2}. \tag{1.4}$$

If the curve α is unit speed, then

$$\kappa(s) = \left\|\alpha''(s)\right\|_L,\ \tau(s) = \left\|B'(s)\right\|_L. \tag{1.5}$$

§2. The involute of spacelike curve with a spacelike principal normal

Definition 2.1 *Let unit speed spacelike curve $\alpha : I \longrightarrow E_1^3$ with a principal normal and spacelike curve β :$I \longrightarrow E_1^3$ with a spacelike principal normal be given. For $\forall s \in I$, then the curve β is called the involute of the curve α, if the tangent at the point $\alpha(s)$ to the curve α passes through the tangent at the point $\beta(s)$ to the curve β and*

$$\langle T^*(s), T(s)\rangle_L = 0. \tag{2.1}$$

Let the Frenet-Serret frames of the curves α and β be $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$, respectively. In this case, the causal characteristics of the Frenet-Serret frames of the curves α and β must be of the form.

$$\{T \text{ spacelike, } N \text{ spacelike, } B \text{ timelike}\}$$

and

$$\{T^* \text{ spacelike, } N^* \text{ spacelike, } B^* \text{ timelike}\}.$$

Theorem 2.1 *Let the curve β be involute of the the curve α and let k be a constant real number. Then*

$$\beta(s) = \alpha(s) + (k-s)T(s). \tag{2.2}$$

Proof The curve $\beta(s)$ may be given as

$$\beta(s) = \alpha(s) + u(s)T(s) \tag{2.3}$$

If we take the derivative Eq. (2.3), then we have

$$\beta^{'}(s) = \left(1 + u^{'}(s)\right) T(s) + u(s)\kappa(s)N(s).$$

Since the curve β is involute of the curve α, $\langle T^{*}(s), T(s)\rangle_L = 0$. Then, we get

$$1 + u^{'}(s) = 0 \text{ or } u(s) = k - s. \tag{2.4}$$

Thus we get

$$\beta(s) - \alpha(s) = (k-s)\,T(s) \tag{2.5}$$

□

Corollary 2.2 *The distance between the curves β and α is $|k-s|$.*

Proof If we take the norm in Eq. (2.5), then we get

$$\|\beta(s) - \alpha(s)\|_L = |k-s|\,. \tag{2.6}$$

□

Theorem 2.3 *Let the curve β be involute of the the curve α, then*

$$\begin{bmatrix} T^{*} \\ N^{*} \\ B^{*} \end{bmatrix} = \left(\left|\kappa^2 - \tau^2\right|\right)^{-1} \begin{bmatrix} 0 & 1 & 0 \\ \kappa & 0 & -\tau \\ -\tau & 0 & \kappa \end{bmatrix} \cdot \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

Proof If we take the derivative Eq. (2.5), we can write

$$\beta^{'}(s) = (k-s)\kappa(s)N(s)$$

and

$$\left\|\beta^{'}(s)\right\|_L = |(k-s)\kappa(s)|\,.$$

Furthermore, we get

$$T^{*}(s) = \frac{\beta^{'}(s)}{\|\beta^{'}(s)\|_L} = \frac{(k-s)\kappa(s)}{|(k-s)\kappa(s)|}N(s).$$

From the last equation, we must have

$$T^{*}(s) = N(s) \ \text{ or } T^{*}(s) = -N(s)\,.$$

We assume that $T^*(s) = N(s)$. Let's denote the coordinate function on IR by x. Then, for$\forall s \in IR$, $x(s) = s$, we get

$$\begin{aligned} \beta^{'}(s) &= (k-s)\kappa(s)N(s), \\ \beta^{'} &= (k-x)\kappa N. \end{aligned}$$

Thus, we have

$$\begin{aligned} \beta^{''} &= -\kappa N + (k-x)\kappa^{'} N + (k-x)\kappa(-\kappa T + \tau B) \\ \beta^{''} &= -(k-x)\kappa^2 T + \left[(k-x)\kappa^{'} - \kappa\right] N + (k-x)\kappa\tau B \end{aligned}$$

Hence, we have

$$\beta^{'} \wedge_L \beta^{''} = (k-x)^2\kappa^2\left(-\tau T + \kappa B\right)$$

and

$$\left\|\beta^{'} \wedge_L \beta^{''}\right\|_L = |k-x|^2\,\kappa^2\sqrt{|\tau^2-\kappa^2|}.$$

Furthermore, we get

$$B^* = \frac{\beta^{'} \wedge_L \beta^{''}}{\left\|\beta^{'} \wedge_L \beta^{''}\right\|} = \frac{(k-x)^2\kappa^2\left(-\tau T + \kappa B\right)}{(k-x)^2\kappa^2\sqrt{|\tau^2-\kappa^2|}} = \frac{-\tau T + \kappa B}{\sqrt{|\kappa^2-\tau^2|}}.$$

Since $N^* = B^* \wedge_L T^*$, then we obtain

$$N^* = \frac{\tau T - \kappa B}{\sqrt{|\tau^2-\kappa^2|}}.$$

□

Theorem 2.4 *Let the curve β be involute of the the curve α. Let the curvature and torsion of the curve β be κ^* and τ^*, respectively. Then*

$$\kappa^*(s) = \frac{\sqrt{\left|\left(\tau^2-\kappa^2\right)(s)\right|}}{|k-s|\,\kappa(s)}\ ,\ \tau^*(s) = \frac{\kappa(s)\tau^{'}(s) - \kappa^{'}(s)\tau(s)}{|k-s|\,\kappa(s)\sqrt{\left|\left(\tau^2-\kappa^2\right)(s)\right|}}.$$

Proof From Eq. (1.3) and Eq. (1.4), we have

$$\kappa^*(s) = \frac{|k-s|^2\,\kappa^2(s)}{|k-s|^3\,\kappa^3(s)} = \frac{\sqrt{\left|\left(\tau^2-\kappa^2\right)(s)\right|}}{\kappa(s)\,|k-s|}$$

and

$$\begin{aligned}
\beta^{'''} &= \left[\kappa^2 T - (k-x)2\kappa\kappa^{'} T - (k-x)\kappa^2(\kappa N)\right] \\
&\quad + \left[-\kappa^{'} - \kappa^{'} + (k-x)\kappa^{''}\right] N \\
&\quad + \left[-\kappa + (k-x)\kappa^{'})\right](-\kappa T + \tau B) \\
&\quad + \left[-\kappa\tau + (k-x)\kappa^{'}\tau + (k-x)\kappa\tau^{'}\right] B \\
&\quad + [(k-x)\kappa\tau](\tau N) \\
&= \left[2\kappa^2 - 3(k-x)\kappa\kappa^{'}\right] T \\
&\quad + \left[-(k-x)\kappa^3 - 2\kappa^{'} + (k-x)\kappa^{''} + (k-x)\kappa\tau^2\right] N \\
&\quad + \left(-2\kappa\tau + 2(k-x)\kappa^{'}\tau + (k-x)\kappa\tau^{'}\right) B.
\end{aligned}$$

Furthermore, since

$$\tau^*(s) = \frac{\det\left(\beta^{'}(s), \beta^{''}(s), \beta^{'''}(s)\right)}{\left\|\beta^{'}(s) \wedge_L \beta^{''}(s)\right\|_L^2},$$

we have

$$\begin{aligned}
\Delta &= -(k-x)^2\kappa^2 \begin{bmatrix} -\kappa & \tau \\ 2\kappa^2 - 3(k-x)\kappa\kappa^{'} & -2\kappa\tau + 2(k-x)\kappa^{'}\tau + (k-x)\kappa\tau^{'} \end{bmatrix} \\
&= -(k-x)^2\kappa^2\left[2\kappa^2\tau - 2(k-x)\kappa\kappa^{'}\tau - (k-x)\kappa^2\tau^{'} - 2\kappa^2\tau + 3(k-x)\kappa\kappa^{'}\tau\right] \\
&= (k-x)^3.\kappa^3\left(\kappa\tau^{'} - \kappa^{'}\tau\right) \\
\Delta &= \det\left(\beta^{'}, \beta^{''}, \beta^{'''}\right).
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\tau^*(s) &= \frac{\kappa^3(k-s)^3\left(\kappa(s)\tau^{'}(s) - \kappa^{'}(s)\tau(s)\right)}{\kappa^4|k-s|^4(\tau^2(s) - \kappa^2(s))}, \\
\tau^*(s) &= \frac{\kappa(s)\tau^{'}(s) - \kappa^{'}(s)\tau(s)}{\kappa(s)|k-x|(\tau^2(s) - \kappa^2(s))}.
\end{aligned}$$

□

From the last equation, we have the following corollaries:

Corollary 2.5 *If the curve α is planar, then its involute curve β is also planar.*

Corollary 2.6 *If the curvature $\kappa \neq 0$ and the torsion $\tau \neq 0$ of the curve α are constant, then the involute curve β is planar, i.e., if the curve α is a ordinary helix, then its the involute curve β is planar.*

Corollary 2.7 *If the curvature $\kappa \neq 0$ and the torsion $\tau \neq 0$ of the curve α are not constant but $\frac{\tau}{\kappa}$ is constant, then the involute curve β is planar, i.e. if the curve α is a general helix, then their the involute curve β is planar.*

Theorem 2.8 *Suppose that the planar curve $\alpha : I \longrightarrow E_1^3$ with arc-length parameter are given. Then, the locus of the center of the curvature of the curve α is the unique involute of the curve α which lies on the plane of the curve α.*

Proof The locus of the center of the curvature of the curve α is

$$C(s) = \alpha(s) - \frac{1}{\kappa(s)} N(s), \;\; \kappa(s) \neq 0$$

If we take the derivative in the above equation, then we have

$$\begin{aligned}
\frac{dC}{ds} &= T - \left(\frac{1}{\kappa}\right)^{'} N + \frac{1}{\kappa}\left(-\kappa T\right), \\
C^{'} &= -\left(\frac{1}{\kappa}\right)^{'} N, \\
\left\langle C^{'}, T\right\rangle_L &= -\left(\frac{1}{\kappa}\right)^{'} \langle N, T\rangle_L\,, \\
\left\langle C^{'}(s), T(s)\right\rangle_L &= 0.
\end{aligned}$$

Therefore, the evolute C of the spacelike curve α is the locus of the center of the curvature. Is the curve C planar ? If the torsion of the curve C is denoted by τ^*, then

$$\tau^*(s) = \frac{\left(\kappa^{'}\tau - \kappa\tau^{'}\right)(s)}{\kappa(t)\,|k - s|\,.\,(\tau^2(s) - \kappa^2(s))}.$$

If we take $\tau = 0$, then we have

$$\tau^*(s) = 0$$

Thus, the curve C is planar. □

§3. The evolute of spacelike curve with a spacelike principal normal

Definition 3.1 *Let the unit speed spacelike curve α with a spacelike principal normal and the spacelike curve β with the same interval be given. For $\forall s \in I$, the tangent at the point $\beta(s)$ to the curve β passes through the point $\alpha(s)$ and*

$$\langle T^*(s), T(s)\rangle_L = 0.$$

Then, β is called the evolute of the curve α. Let the Frenet-Serret frames of the curves α and β be (T, N, B) and (T^*, N^*, B^*), respectively.

Theorem 3.1 *Let the curve β be the evolute of the unit speed spacelike curve α , Then*

$$\beta(s) = \alpha(s) + \frac{1}{\kappa(s)} N(s) - \frac{1}{\kappa(s)}\left[\tanh\left(\varphi(s) + c\right)\right] B(s), \tag{3.1}$$

where $c \in IR$ and $\varphi(s) + c = \int \tau(s)ds$. Furthermore, in the normal plane of the point $\alpha(s)$ the measure of directed angle between $\beta(s) - \alpha(s)$ and $N(s)$ is

$$\varphi(s) + c.$$

Proof The tangent of the curve β at the point $\beta(s)$ is the line constructed by the vector $T^*(s)$. Since this line passes through the point $\alpha(s)$, the vector $\beta(s)-\alpha(s)$ is perpendicular to the vector $T(s)$. Then

$$\beta(s)-\alpha(s)=\lambda N(s)+\mu B(s). \tag{3.2}$$

If we take the derivative of Eq. (3.2), then we have

$$\beta^{'}(s)=\alpha^{'}(s)+\lambda^{'}N+\lambda(-\kappa T+\tau B)+\mu^{'}B(s)+\mu(\tau N)$$

$$\beta^{'}(s)=(1-\lambda\kappa)\,T+\left(\lambda^{'}+\mu\tau\right)N+\left(\lambda\tau+\mu^{'}\right)B. \tag{3.3}$$

According to the definition of the evolute, since $\langle T^*(s),T(s)\rangle=0$, from Eq. (3.3), we get

$$\lambda=\frac{1}{\kappa}\ , \tag{3.4}$$

and

$$\beta^{'}=\left(\lambda^{'}+\mu\tau\right)N+\left(\lambda\tau+\mu^{'}\right)B. \tag{3.5}$$

From the Eq. (3.2) and Eq. (3.5), the vector field $\beta^{'}$is parallel to the vector field $\beta-\alpha$. Then we have

$$\frac{\lambda^{'}+\mu\tau}{\lambda}=\frac{\lambda\tau+\mu^{'}}{\mu}.$$

After that, we have

$$\begin{aligned}\tau &= \frac{\lambda^{'}\mu-\lambda\mu^{'}}{\lambda^2-\mu^2}\\ \tau &= -\frac{\left(\frac{\mu}{\lambda}\right)^{'}}{1-\left(\frac{\mu}{\lambda}\right)^2}.\end{aligned}$$

If we take the integral the last equation, we get

$$\varphi(s)+c=-\arg\tanh\left(\frac{\mu(s)}{\lambda(s)}\right).$$

Hence, we find

$$\mu(s)=-\lambda(s)\tanh\left(\varphi(s)+c\right). \tag{3.6}$$

If we substitute Eq. (3.4) and Eq. (3.6) into Eq. (3.2), we have

$$\begin{aligned}\beta(s) &= \alpha(s)+\frac{1}{\kappa(s)}N(s)-\frac{1}{\kappa(s)}\left[\tanh\left(\varphi(s)+c\right)\right]B(s)\\ \beta(s) &= M(s)-\frac{1}{\kappa(s)}\tanh\left[\varphi(s)+c\right]B(s).\end{aligned}$$

Then, we obtain an evolute curve for each $c\in IR$. Since

$$\left\langle \overrightarrow{M(s)\beta(s)},\overrightarrow{M(s)\ \alpha(s)}\right\rangle_L=0,$$

in the Lorentzian triangle which have corners $\beta(s)$, $M(s)$ and $\alpha(s)$, the angle M is right angle in the Lorentzian mean. In the same triangle, the tangent of the angle $\alpha(s)$ is

$$\frac{\frac{1}{\kappa(s)}\tanh\left[\varphi(s)+c\right]}{\frac{1}{\kappa(s)}}=\tanh\left[\varphi(s)+c\right]. \tag{3.7}$$

Then, the measure of the angle between the vectors $\beta(s)-\alpha(s)$ and $N(s)$ is $\varphi(s)+c$. □

Theorem 3.2 *Let the spacelike curve $\beta : I \longrightarrow E_1^3$ be evolute of the unit speed spacelike curve $\alpha : I \longrightarrow E_1^3$. If the Frenet-Serret vector fields of the curve β are T^* (spacelike), N^* (space), B^* (timelike), then*

$$\begin{bmatrix} T^* \\ N^* \\ B^* \end{bmatrix}=\begin{bmatrix} 0 & \cosh(\varphi+c) & -\sinh(\varphi+c) \\ -1 & 0 & 0 \\ 0 & -\sinh(\varphi+c) & \cosh(\varphi+c) \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{3.8}$$

Proof Since the Frenet-Serret vector fields of the curve β are T^*, N^*, B^* and

$$\beta=\alpha+\rho N-\rho\tanh\left(\varphi+c\right)B,$$

we have

$$\begin{aligned}
\beta^{'}(s) &= \alpha^{'}+\rho^{'}N+\rho\left(-\kappa T+\tau B\right) \\
&\quad -\left[\rho^{'}\tanh\left(\varphi+c\right)B+\rho\varphi^{'}\operatorname{sec h}^2\left(\varphi+c\right)B+\rho\tanh\left(\varphi+c\right)\tau N\right] \\
&= \left(1-\rho\kappa\right)T+\left(\rho^{'}-\rho\tau\tan\left(\varphi+c\right)\right)N \\
&\quad +\left[\left(\rho\tau-\rho\varphi^{'}\right)-\rho^{'}\tanh\left(\varphi+c\right)+\rho\varphi^{'}\tanh^2\left(\varphi+c\right)\right]B \\
&= \left[\rho^{'}-\rho\tau\tanh\left(\varphi+c\right)\right]N+\left[-\rho^{'}+\rho\tau\tanh\left(\varphi+c\right)\right]B\tanh\left(\varphi+c\right) \\
&= \left[\rho^{'}-\rho\tau\tanh(\varphi+c)\right]\left[N-\tanh\left(\varphi+c\right)B\right]
\end{aligned}$$

$$\beta^{'}(s)=\left[\frac{\rho^{'}-\rho\tau\tanh(\varphi+c)}{\cosh\left(\varphi+c\right)}\right]\left[\cosh\left(\varphi+c\right)N-\sinh\left(\varphi+c\right)B\right]. \tag{3.9}$$

If we take the norm in the Eq. (3.9), then we obtain

$$\begin{aligned}
\left\|\beta^{'}(s)\right\|_L &= \frac{\left|\rho^{'}-\rho\tau\tanh(\varphi+c)\right|}{\cosh\left(\varphi+c\right)} \\
&= \frac{\left|-\frac{\kappa^{'}}{\kappa^2}-\frac{1}{\kappa}\tau\frac{\sinh(\varphi+c)}{\cosh(\varphi+c)}\right|}{\cosh\left(\varphi+c\right)} \\
&= \frac{\left|\kappa^{'}\cosh\left(\varphi+c\right)+\kappa\tau\sinh(\varphi+c)\right|}{\kappa^2\cosh\left(\varphi+c\right)}.
\end{aligned}$$

Since $T^*=\frac{\beta^{'}}{\left\|\beta^{'}\right\|_L}$, then we get

$$T^*=\cosh\left(\varphi+c\right)N-\sinh\left(\varphi+c\right)B. \tag{3.10}$$

Therefore, we have obtained Eq. (3.9). The curve β is not a unit speed curve. If we take the derivative of Eq. (3.10) with respect to s, we find

$$\begin{aligned}(T^*)' &= \left(\tau-\varphi'\right)\left[B\cosh(\varphi+c)+N\sinh(\varphi+c)\right]-\kappa T\cosh(\varphi+c)\\ &= -\kappa T\cosh(\varphi+c)\end{aligned}$$

Since $T'=\left\|\alpha'\right\|_L \kappa N$, we have

$$(T^*)'=\left\|\beta'\right\|_L \kappa^* N^*.$$

Thus

$$\left\|\beta'\right\|_L \kappa^* N^* = -\kappa\cosh(\varphi+c)\,T.$$

Since the vectors N^* and T have the unit length, we get $N^*=-T$ or $N^*=T$. Since $B^*=N^*\wedge_L(-T^*)$, we have

$$B^*=-\sinh(\varphi+c)N+\cosh(\varphi+c)B. \tag{3.11}$$

Thus, the proof is completed. □

Theorem 3.3 *Let $\beta: I\longrightarrow E_1^3$ be the evolute of the unit speed spacelike curve $\alpha: I\longrightarrow E_1^3$. Let the Frenet vector fields, curvature and torsion of the curve β be T^*, N^*, B^*, κ^* and τ^*, respectively. Then*

$$\begin{aligned}\kappa^* &= \frac{\kappa^3\cosh^3(\varphi+c)}{|\kappa\tau\sinh(\varphi+c)+\kappa'\cos(\varphi+c)|},\ \kappa>0\\ |\tau^*| &= \frac{\kappa^3\cosh^2(\varphi+c)\,|\sinh(\varphi+c)|}{|\kappa\tau\sinh(\varphi+c)+\kappa'\cosh(\varphi+c)|}.\end{aligned}$$

Proof Since N^* and T have unit length, then taking norm from equility $\left\|\beta'\right\|_L \kappa^* N^* = -\kappa\cosh(\varphi+c)\,T$ .We can write have

$$|\kappa^*|=\frac{\kappa\cosh(\varphi+c)}{\|\beta'\|_L} \tag{3.12}$$

$$=\kappa\cosh(\varphi+c):\frac{\left|\kappa'\cosh(\varphi+c)+\kappa\tau\sinh(\varphi+c)\right|}{\kappa^2\cosh(\varphi+c)},$$

$$|\kappa^*|=\frac{\kappa^3\cosh^3(\varphi+c)}{\kappa'\cosh(\varphi+c)+\kappa\tau\sinh(\varphi+c)}$$

If we take the derivative Eq. (3.11) with respect to s, then we have

$$\begin{aligned}(B^*)' &= \left(\varphi'-\tau\right)\left[N\cosh(\varphi+c)-B\sinh(\varphi+c)\right]+\kappa T\sinh(\varphi+c)\\ &= \kappa T\sin(\varphi+c).\end{aligned}$$

Since $(B^*)'=\left\|\beta'\right\|_L \tau^* N^*$, we get

$$\left\|\beta'\right\|_L \tau^* N^* = \kappa T\sin(\varphi+c).$$

From the last equation, we must have

$$T^*(s) = N(s) \ \text{ or } T^*(s) = -N(s)\,.$$

We assume that $T^*(s) = -N(s)$ then we find that

$$\begin{aligned} |\tau^*| &= \frac{\kappa\,|\sinh(\varphi + c)|}{\|\beta'\|} \qquad (3.13) \\ &= \kappa\,|\sinh(\varphi + c)| : \frac{\left|\kappa' \cosh(\varphi + c) + \kappa\tau \sinh(\varphi + c)\right|}{\kappa^2 \cosh(\varphi + c)}, \\ |\ \tau^*| &= \frac{\kappa^3 \cosh^2(\varphi + c)\,|\sinh(\varphi + c)|}{|\kappa' \cosh(\varphi + c) + \kappa\tau \sinh(\varphi + c)|}. \end{aligned}$$

□

Theorem 3.4 *Let $\beta : I \longrightarrow E_1^3$ be the evolute of the unit speed spacelike curve $\alpha : I \longrightarrow E_1^3$. Let the curvature and torsion of the curve β be κ^* and τ^*, respectively. Then*

$$\left|\frac{\tau^*}{\kappa^*}\right| = |\tanh(\varphi + c)|\,. \qquad (3.14)$$

Furthermore, we denote by $\beta^{(1)}$ and $\beta^{\ (2)}$, the evolute curves obtained by using c_1 and c_2 instead of c, respectively. The tangents of the curves $\beta^{(1)}$ and $\beta^{\ (2)}$ at the points $\beta^{(1)}(s)$ and $\beta^{(2)}(s)$ intersect at the point $\alpha(s)$. The measure of the angle between the tangents is $c_1 - c_2$.

Proof The Eq. (3.14) is obtained easily by using Eq. (3.12) and Eq. (3.13), i.e.,

$$\begin{aligned} \left|\frac{\tau^*}{\kappa^*}\right| &= \frac{\kappa\,|\sinh(\varphi + c)|}{\|\beta'\|_L} : \frac{\kappa \cosh(\varphi + c)}{\|\beta'\|_L} \\ &= |\tanh(\varphi + c)|\,. \end{aligned}$$

The measure of the angle between the vectors $\overrightarrow{\alpha(s)\beta^{(1)}(s)}$ and $V_2(s)$, and between the vectors $\overrightarrow{\alpha(s)\beta^{(2)}(s)}$ and $N(s)$ are $\varphi(s)+c_1$ and $\varphi(s)+c_2$, respectively. The vector $\overrightarrow{\alpha(s)\beta^{(1)}(s)}$ is parallel to the tangent of the curve $\beta^{(1)}$ at the point $\beta^{(1)}(s)$. The vector $\overrightarrow{\alpha(s)\beta^{(2)}(s)}$ is parallel to the tangent of the curve $\beta^{(2)}$ at the point $\beta^{(2)}(s)$. Furthermore, since $\overrightarrow{\alpha(s)\beta^{(1)}(s)}$, $\overrightarrow{\alpha(s)\beta^{(1)}(s)}$ and $\overrightarrow{N}$ are perpendicular to the vector $T(s)$, these three vectors are planar. Then, the measure of the angle between the tangents of the curves $\beta^{(1)}$ and $\beta^{(2)}$ at the points $\beta^{(1)}(s)$ and $\beta^{(2)}(s)$ is

$$\varphi(s) + c_1 - [\varphi(s) + c_2] = c_1 - c_2.$$

So, the proof is completed. □

Theorem 3.5 *Suppose that, two different evolutes of the spacelike curve a spacelike principal normal curve α are given. Let the points on the evolutes of the curve α corresponding to the point P be P_1 and P_2. Then the angle $\widehat{P_1 P\, P_2}$ is constant.*

Proof Let the evolutes of the curve α be β and γ. Let the arc-length parameters of the α, β and γ be s, s^* and $\widehat{s}$, respectively. Let the curvatures of the curves α, β and γ be k, k^* and

$\widehat{k}$ respectively. And let the Frenet vectors of the curves α, β and γ be $\{T, N, B\}, \{T^*, N^*, B^*\}$ and $\left\{\widehat{T}, \widehat{N}, \widehat{B}\right\}$. Then

$$T = N^*, T = \widehat{N}. \tag{3.15}$$

Since the curves β and γ are evolute, then

$$\langle T, T^* \rangle_L = \left\langle T, \widehat{T} \right\rangle_L = 0 \tag{3.16}$$

Therefore, if $f(s) = \left\langle T^*, \widehat{T} \right\rangle_L$, then we have

$$\begin{aligned}
(f)^{'}(s) &= \left\langle (T^*)^{'}, \widehat{T} \right\rangle_L + \left\langle T^*, \left(\widehat{T}\right)^{'} \right\rangle_L \\
&= \left\langle \kappa^* N^* \frac{ds^*}{ds}, \widehat{T} \right\rangle_L + \left\langle T^*, \widehat{\kappa} \widehat{N} \frac{d\widehat{s}}{ds} \right\rangle_L \\
&= \kappa^* \frac{ds^*}{ds} \left\langle N^*, \widehat{T} \right\rangle_L + \widehat{\kappa} \frac{d\widehat{s}}{ds} \left\langle T^*, \widehat{N} \right\rangle_L \\
&= \kappa^* \frac{ds^*}{ds} \left\langle T, \widehat{T} \right\rangle_L + \widehat{\kappa} \frac{d\widehat{s}}{ds} \langle T^*, N^* \rangle_L \\
&= \kappa^* \frac{ds^*}{ds}.0 + \widehat{\kappa} \frac{d\widehat{s}}{ds}.0 \\
(f)^{'}(s) &= 0.
\end{aligned}$$

Therefore, we have $f(s) = \theta$ =constant. Hence, $m\left(\widehat{P_1 P P_2}\right) = m\left(T^*, \widehat{T}\right) = \theta$ =constant. $\square$

References

[1] Hacısalihoglu, H.H, *Solutions of The Differential Geometry Problems*, Ankara University, Faculty of Science, 1995.

[2] Petrovic, M. and Sucurovic, E., Some Characterizations of The spacelike,The Timelike and The Null Curves on The Pseudohyperbolic Space H_0^2 in E_1^3, *Kragujevac J.Math.*, Vol 22(2000), 71-82.

[3] Sabuncuoğlu, A., *Differential Geometry*, Nobel Yayın Dağıtım, Ankara, 2004.

[4] Bukcu, B and Karacan M.K, On the Involute and Evolute Curves Of Spacelike Curve with a Spacelike Binormal in Minkowski 3-Space, *Int. J. Contemp. Math. Sciences*, Vol. 2, 2007, No.5, 221 - 232.

International J.Math. Combin. Vol.1 (2009), 38-41

Notes on the Curves in Lorentzian Plane $\mathbf{L}^2$

Süha Yılmaz

(Dokuz Eylül University,Buca Educational Faculty, Department of Mathematics, 35160 Buca,Izmir, Turkey)

E-mail: suha.yilmaz@yahoo.com

Abstract In this study, position vector of a Lorentzian plane curve (space-like or time-like, i.e.) is investigated. First, a system of differential equation whose solution gives the components of the position vector on the Frenet axis is constructed. By means of solution of mentioned system, position vector of all such curves according to Frenet frame is obtained. Thereafter, it is proven that, position vector and curvature of a Lorentzian plane curve satisfy a vector differential equation of third order. Moreover, using this result, position vector of such curves with respect to standard frame is presented. By this way, we present a short contribution to *Smarandache geometries*.

Key Words Classical differential geometry, Smarandache geometries, Lorentzian plane, position vector.

AMS(2000): 53B30, 51B20.

§1. Introduction

In recent years, the theory of degenerate submanifolds is treated by the researchers and some of classical differential geometry topics are extended to Lorentzian manifolds. For instance in [1], author deeply studies theory of the curves and surfaces and also presents mathematical principles about theory of Relativitiy. Also, T. Ikawa [4] presents some characterizations of the theory of curves in an indefinite-Riemannian manifold.

F. Smarandache in [2], defined a geometry which has at least one Smarandachely denied axiom, i.e., an axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways and a Smarandache n-manifold is a n- manifold that support a Smarandache geometry.

Since, following these constructions, nearly all existent geometries, such as those of Euclid geometry, Lobachevshy- Bolyai geometry, Riemann geometry, Weyl geometry, K a hler geometry and Finsler geometry, ...,etc., are their sub-geometries (further details, see [3].

In the presented paper, we have determined position vector of a Lorentzian plane curve. First, using Frenet formula, we have constructed a system of differential equation. Solution of it yields components of the position vector on Frenet axis. Thereafter, again, using Frenet equations, we have constructed a vector differential equation with respect to position vector. Moreover, its solution has given us position vector the curve according to standard Euclidean frame. Since, we get a short contribution about Smarandache geometries.

[1]Received November 24, 2008. Accepted January 12, 2009.

§2. Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the Lorentzian plane are briefly presented (A more complete elementary treatment can be found in [1], [4], [5]).

Let L^2 be the Lorentzian plane with metric

$$g = dx_1^2 - dx_2^2, \tag{1}$$

where x_1 and x_2 are rectangular coordinate system. A vector a of L^2 is said to be space-like if $g(a,a) > 0$ or $a = 0$, time-like if $g(a,a) < 0$ and null if $g(a,a) = 0$ for $a \neq 0$. A curve x is a smooth mapping $x : I \rightarrow \mathrm{L}^2$ from an open interval I onto L^2. Let s be an arbitrary parameter of x. By $x = (x_1(s), x_2(s))$, we denote the orthogonal coordinate representation of x. The vector

$$\frac{dx}{ds} = \left(\frac{dx_1}{ds}, \frac{dx_2}{ds}\right) = t \tag{2}$$

is called the tangent vector field of the curve $x = x(s)$. If tangent vector field t of $x(s)$ is a space-like, time-like or null, then, the curve $x(s)$ is called space-like, time-like or null, respectively.

In the rest of the paper, we shall consider non-null curves. When the tangent vector field t is non-null, we can have the arc length parameter s and have the Frenet formula

$$\begin{bmatrix} \dot{t} \\ \dot{n} \end{bmatrix} = \begin{bmatrix} 0 & \kappa \\ \kappa & 0 \end{bmatrix} \begin{bmatrix} t \\ n \end{bmatrix} \tag{3}$$

where $\kappa = \kappa(s)$ is the curvature of the unit spped curve $x = x(s)$. The vector field n is called the normal vector field of the curve $x(s)$. Remark that, we have the same representation of the Frenet formula regardless of whether the curve is space-like ot time-like. And, if $\phi(s)$ is the slope angle of the curve, then we have

$$\frac{d\phi}{ds} = \kappa(s). \tag{4}$$

§3. Position vector of a Lorentzian plane curve

Let $x = x(s)$ be an unit speed curve on the plane L^2. Then, we can write position vector of $x(s)$ with respect to Frenet frame as

$$x = x(s) = \delta t + \lambda n \tag{5}$$

where δ and λ are arbitrary functions of s. Differentiating both sides of (5) and using Frenet equations, we have a system of ordinary differential equations as follows:

$$\begin{array}{c} \frac{d\delta}{ds} + \lambda\kappa - 1 = 0 \\ \frac{d\lambda}{ds} + \delta\kappa = 0 \end{array} . \tag{6}$$

Using $(6)_1$ in $(6)_2$, we write

$$\frac{d}{ds}\left[\frac{1}{\kappa}\left(1-\frac{d\delta}{ds}\right)\right]+\delta\kappa=0. \tag{7}$$

This differential equation of second order, according to δ, is a characterization for the curve $x=x(s)$. Using an exchange variable $\phi=\int\limits_0^s \kappa ds$ in (7), we easily arrive

$$\frac{d^2\delta}{d\phi^2}-\delta=\frac{d\rho}{d\phi}, \tag{8}$$

where $\kappa=\frac{1}{\rho}$. By the method of variation of parameters and hyperbolic functions, solution of (8) yields

$$\delta=\cosh\phi\left[A-\int\limits_0^\phi \rho\sinh\phi d\phi\right]+\sinh\phi\left[B+\int\limits_0^\phi \rho\cosh\phi d\phi\right]. \tag{9}$$

Here $A, B\in R$. Rewriting the exchange variable, that is,

$$\delta=\cosh\int\limits_0^s \kappa ds\left[A-\int\limits_0^\phi\left(\sinh\int\limits_0^s\kappa ds\right)ds\right]+\sinh\int\limits_0^s\kappa ds\left[B+\int\limits_0^\phi\left(\cosh\int\limits_0^s\kappa ds\right)ds\right]. \tag{10}$$

Denoting differentation of equation (10) as $\frac{d\delta}{ds}=\xi(s)$, we have

$$\lambda=\rho(\xi(s)-1). \tag{11}$$

Since, we give the following theorem.

Theorem 3.1 *Let $x=x(s)$ be an arbitrary unit speed curve (space-like or time-like, i.e.) in Lorentzian plane. Position vector of the curve $x=x(s)$ with respect to Frenet frame can be composed by the equations* (10) *and* (11).

§4. Vector differential equation of third order characterizes Lorentzian plane curves

Theorem 4.1 *Let $x=x(s)$ be an arbitrary unit speed curve (space-like or time-like, i.e.) in Lorentzian plane. Position vector and curvature of it satisfy a vector differential equation of third order.*

Proof Let $x=x(s)$ be an arbitrary unit speed curve (space-like or time-like, i.e.) in Lorentzian plane. Then formula (3) holds. Using $(3)_1$ in $(3)_2$, we easily have

$$\frac{d}{ds}\left(\frac{1}{\kappa}\frac{dt}{ds}\right)-\kappa t=0, \tag{12}$$

where $\frac{dx}{ds}=t=\dot{x}$. Consequently, we write

$$\frac{d}{ds}\left(\frac{1}{\kappa}\frac{d^2x}{ds^2}\right)-\kappa\frac{dx}{ds}=0. \tag{13}$$

Formula (13) completes the proof.

Let us solve equation (12) with respect to t. Here, we know, $t = (t_1, t_2) = (\dot{x}_1, \dot{x}_2)$. Using the exchange variable $\phi = \int_0^s \kappa ds$ in (), we obtain

$$\frac{d^2 t}{d\phi^2} - t = 0 \tag{14}$$

or in parametric for

$$\begin{array}{l} \frac{d^2 t_1}{d\phi^2} - t_1 = 0 \\ \frac{d^2 t_2}{d\phi^2} - t_2 = 0 \end{array} . \tag{15}$$

It follows that

$$\begin{array}{l} t_1 = \varepsilon_1 e^{\phi} - \varepsilon_2 e^{-\phi} \\ t_2 = \varepsilon_3 e^{\phi} - \varepsilon_4 e^{-\phi} \end{array} , \tag{16}$$

where $\varepsilon_i \in R$ for $1 \leq i \leq 4$. Therefore, we get

$$\begin{array}{l} t_1 = \gamma_1 \cosh \int_0^s \kappa ds + \gamma_2 \sinh \int_0^s \kappa ds \\ t_2 = \gamma_3 \cosh \int_0^s \kappa ds + \gamma_4 \sinh \int_0^s \kappa ds \end{array} . \tag{17}$$

Finally, we give the following theorem.

Theorem 4.2 *Let $x = x(s)$ be an arbitrary unit speed curve (space-like ot time-like, i.e.) in Lorentzian plane. Position vector of it with respect to standard frame can be expressed as*

$$x = x(s) = \begin{pmatrix} \int_0^s \left\{ \gamma_1 \cosh \int_0^s \kappa ds + \gamma_2 \sinh \int_0^s \kappa ds \right\} ds, \\ \int_0^s \left\{ \gamma_3 \cosh \int_0^s \kappa ds + \gamma_4 \sinh \int_0^s \kappa ds \right\} ds \end{pmatrix} \tag{18}$$

for the real numbers $\gamma_1, ..., \gamma_4$. □

References

[1] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York, 1983.

[2] F. Smarandache, Mixed noneuclidean geometries, *eprint arXiv: math/0010119*, 10/2000.

[3] L.F. Mao, Differential geometry on Smarandache n-manifolds, *Scientific Elements*, Ser. Vol.1, 01-17, 2007.

[4] T. Ikawa, On curves and submanifolds in an indefinite-Riemannian manifold, *Tsukuba J. Math.* 9 353-371. 1985.

[5] T. Ikawa, Euler-Savary's formula on Minkowski geometry, *Balkan J. Geo. Appl.* **8** 2, 31-36, 2003.

International J.Math. Combin. Vol.1 (2009), 42-45

Cycle-Complete Graph Ramsey Numbers $r(C_4, K_9), r(C_5, K_8) \leq 33$

M.M.M. Jaradat

(Department of Mathematics, Yarmouk University, Irbid-Jordan)

(Department of Mathematics and Physics, Qatar University, Doha-Qatar)

Email: mmjst4@yu.edu.jo, mmjst4@qu.edu.qa

B.M.N. Alzaleq

(Department of Mathematics, Yarmouk University, Irbid-Jordan)

Email: alzaleq@yahoo.com

Abstract For an integer $k \geq 1$, a cycle-complete graph Smarandache-Ramsey number $r_{s^k}(C_m, K_n)$ is the smallest integer N such that every graph G of order N contains k cycles, C_m, on m vertices or the complement of G contains k complete graph, K_n, on n vertices. If $k = 1$, then the Smarandache-Ramsey number $r_{s^k}(C_m, K_n)$ is nothing but the classical Ramsey number $r(C_m, K_n)$. Radziszowski and Tse proved that $r(C_4, K_9) \geq 30$. Also, By considering the known graph $G = 7K_4$, we have that $r(C_5, K_8) \geq 29$. In this paper we give an upper bound of $r(C_4, K_9)$ and $r(C_5, K_8)$.

Key Words: (Smarandache-)Ramsey number; independent set; cycle; complete graph.

AMS(2000): 05C55, 05C35.

§1. Introduction

Through out this paper we adopt the standard notations, a cycle on m vertices will be denoted by C_m and the complete graph on n vertices by K_n. The minimum degree of a graph G is denoted by $\delta(G)$. An independent set of vertices of a graph G is a subset of $V(G)$ in which no two vertices are adjacent. The independence number of a graph $G, \alpha(G)$, is the size of the largest independent set.

For an integer $k \geq 1$, a Smarandache-Ramsey number $r_{s^k}(H, F)$ is the smallest integer N such that every graph G of order N contains k graph H, or the complement of G contains k graph F. If $k = 1$, then the Smarandache-Ramsey number $r_{s^k}(H, F)$ is nothing but the classical Ramsey number $r(H, F)$. $r(C_m, K_n)$ is called the cycle-complete graph Ramsey number. In one of the earliest contributions to graphical Ramsey theory, Bondy and Erdős [3] proved that for all $m \geq n^2 - 2, r(C_m, K_n) = (m-1)(n-1)+1$. The restriction in the above result was improved by Nikiforov [10] when he proved the equality for $m \geq 4n + 2$. Erdős et al. [5] conjectured that $r(C_m, K_n) = (m-1)(n-1)+1$, for all $m \geq n \geq 3$ except $r(C_3, K_3) = 6$. The conjectured were

[1]Received December 12, 2008. Accepted January 16, 2009.

confirmed for some $n = 3, 4, 5$ and 6 (see [2], [6], [12], and [14]). Moreover, in [7] and [8] the conjecture was proved for $m = n = 8$, and $m = 8$ with $n = 7$. Also, the case $n = m = 7$ was proved independently by Baniabedalruhman and Jaradat [1] and Cheng et al. [4].

In a related work, Radziszowski and Tse [11] showed that $r(C_4, K_7) = 22$, $r(C_4, K_8) = 26$ and $r(C_4, K_9) \geq 30$. Also, In [8] Jayawardene and Rousseau proved that $r(C_5, K_6) = 21$. Recently, Schiermeyer [13] and Cheng et al. [4] proved that $r(C_5, K_7) = 25$ and $r(C_6, K_7) = 25$, respectively. In this article we prove the following Theorems:

Theorem A *The complete-cycle Ramsey number $r(C_4, K_9) \leq 33$.*

Theorem B *The complete-cycle Ramsey number $r(C_5, K_8) \leq 33$.*

In the rest of this work, $N(u)$ stands for the neighbor of the vertex u which is the set of all vertices of G that are adjacent to u and $N[u] = N(u) \cup \{u\}$. For a subgraph R of the graph G and $U \subseteq V(G)$, $N_R(U)$ is defined as $(\cup_{u \in U} N(u)) \cap V(R)$. Finally, $\langle V_1 \rangle_G$ stands for the subgraph of G whose vertex set is $V_1 \subseteq V(G)$ and whose edge set is the set of those edges of G that have both ends in V_1 and is called the subgraph of G induced by V_1.

§2. Proof of Theorem A

We prove our result using the contradiction. Suppose that G is a graph of order 33 which contains neither C_4 nor a 9-element independent set. Then we have the following:

1. $\delta(G) \geq 7$. Assume that u is a vertex with $d(u) \leq 6$. Then $|V(G) - N[u]| \geq 33 - 7 = 26$. But $r(C_4, K_8) = 26$. Hence, $G - N[u]$ contains an 8-element independent set. This set with u form a 9-element independent set. That is a contradiction.

2. G contains no K_3. Suppose that G contains K_3. Let $\{u_1, u_2, u_3\}$ be the vertex set of K_3. Also, let $R = G - \{u_1, u_2, u_3\}$ and $U_i = N(u_i) \cap V(R)$. Then $U_i \cap U_j = \varnothing$ because otherwise G contains C_4. Also, for each $x \in U_i$ and $y \in U_i$, we have that $xy \notin E(G)$ because otherwise G contains C_4. Now, since $\delta(G) \geq 7$, $|U_i| \geq 5$. Since $r(P_3, K_3) = 5$, as a result either $\langle U_i \rangle_G$ contains P_3 for some $i = 1, 2, 3$ and so G contains C_4 or $\langle U_i \rangle_G$ does not contains P_3 for each $i = 1, 2, 3$ and so each of which contains a 3-element independent set, Thus, three independent set of each consists a 9-element independent set. This is a contradiction.

Now, let u be a vertex of G. Let $N(u) = \{u_1, u_2, \ldots, u_r\}$ where $r \geq 7$. Since G contains no K_3, as a result $\langle N(u) \cup \{u\} \rangle_G$ forms a star. And so,$\{u_1, u_2, \ldots, u_r\}$ is independent. Now, let $N(u_1) = \{v_1, v_2, \ldots, v_k, u\}$ where $k \geq 6$. For the same reasons, $\langle N(u_1) \cup \{u_1\} \rangle_G$ forms a star and so $\{v_1, v_2, \ldots, v_k\}$ is independent. Since G contains no K_3 and no C_4. Then $\{u_2, \ldots, u_r, v_1, v_2, \ldots, v_k\}$ is an independent set. That is a contradiction. The proof is complete. $\square$

§3. Proof of Theorem B

We prove our result by using the contradiction. Assume that G is a graph of order 33 which contains neither C_5 nor an 8-element independent set. By an argument similar to the one in Theorem A and by noting that $r(C_5, K_8) = 25$, we can show that $\delta(G) \geq 8$. Now, we have the following:

1. G contains K_3. Suppose that G does not contain K_3. Let $u \in V(G)$ and $r = |N(u)|$. Then the induced subgraph $< N(u) >_G$ does not contain P_2. Hence $< N(u) >_G$ is a null graph with r vertices. Since $\alpha(G) \leq 7$, as a result $r \leq 7$. Therefore, $8 \leq \delta(G) \leq r \leq 7$. That is a contradiction.

2. G contains $K_4 - e$. Let $U = \{u_1, u_2, u_3\}$ be the vertex set of K_3. Let $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \leq i \leq 3$. Since $\delta(G) \geq 8$, $|U_i| \geq 6$ for all $1 \leq i \leq 3$. Now we have the following two cases:

Case 1: $U_i \cap U_j \neq \varnothing$ for some $1 \leq i < j \leq 3$, say $w \in U_i \cap U_j$. Then it is clear that G contains $K_4 - e$. In fact, the induced subgraph $\langle U \cup \{w\} \rangle_G$ contains $K_4 - e$.

Case 2 : $U_i \cap U_j = \varnothing$ for each $1 \leq i < j \leq 3$. Then $\alpha(\langle U_i \rangle_G) \leq 2$, for some $1 \leq i \leq 3$. To see that suppose that $\alpha(\langle U_i \rangle_G) \geq 3$ for each $1 \leq i \leq 3$. Since between any two vertices of U there is a path of order 3, as a result for any $x \in U_i$ and $y \in U_j$, we have $xy \notin E(G), 1 \leq i < j \leq 3$ because otherwise G contains C_5. Therefore, $\alpha(\langle U_1 \cup U_2 \cup U_3 \rangle_G) \geq 3 + 3 + 3 = 9$. and so $\alpha(G) \geq 9$, which is a contradiction.
Now, since $|U_i| \geq 6$ and $\alpha(\langle U_i \rangle_G) \leq 2$, for some $1 \leq i \leq 3$ and since $r(K_3, K_3) = 6$ as a result the induced subgraph $\langle U_i \rangle_G$ contains K_3. And so $\langle U_i \cup \{u_i\} \rangle_G$ contains K_4. Hence, G contains $K_4 - e$.

3. G contains K_4. Let $U = \{u_1, u_2, u_3, u_4\}$ be the vertex set of $K_4 - e$, where the induced subgraph of $\{u_1, u_2, u_3\}$ is isomorphic to K_3. Without loss of generality we may assume that $u_1u_4, u_2u_4 \in E(G)$. We consider the case where $u_3u_4 \notin E(G)$ because otherwise the result is obtained. Let $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \leq i \leq 4$. Then as in **2**, $|U_i| \geq 5$ for $i = 1, 2$ and $|U_i| \geq 6$ for $i = 3, 4$. To this end, we have that $U_i \cap U_j = \varnothing$ for all $1 \leq i < j \leq 4$ except possibly for $i = 1$ and $j = 2$ (To see that suppose that $w \in U_i \cap U_j$ for some $1 \leq i < j \leq 4$ with $i \neq 1$ or $j \neq 2$. Then we consider the following cases:

(1) $i = 3$ and $j = 4$. Then $u_3wu_4u_1u_2u_3$ is a cycle of order 5, a contradiction.

(2) $i = 3$ and $j = 2$. Then $u_3wu_2u_4u_1u_3$ is a cycle of order 5, a contradiction.

(3) i, j are not as in the above cases. Then by similar argument as in (2) G contains a C_5. This is a contradiction.

Now, By arguing as in Case 2 of 2, $\alpha(\langle U_2 \rangle_G) \leq 1$ or $\alpha(\langle U_i \rangle_G) \leq 2$, for $i = 3$ or 4. And so, the induced subgraph $\langle U_i \rangle_G$ contains K_3 for some $2 \leq i \leq 4$. Thus, G contains K_4.
To this end, let $U = \{u_1, u_2, u_3, u_4\}$ be the vertex set of K_4. Let $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \leq i \leq 4$. Since $\delta(G) \geq 8$, $|U_i| \geq 5$ for all $1 \leq i \leq 4$. Since there is a path of order 4 joining any two vertices of U, as a result $U_i \cap U_j = \varnothing$ for all $1 \leq i < j \leq 4$ (since otherwise, if $w \in U_i \cap U_j$ for some $1 \leq i < j \leq 4$, then the concatenation of the u_i-u_j path of order 4 with u_iwu_j is a cycle of order 5, a contradiction). Similarly, since there is a path of order 3 joining any two vertices of U, as a result for all $1 \leq i < j \leq 4$ and for all $x \in U_i$ and $y \in U_j, xy \notin E(G)$ (otherwise, if there are $1 \leq i < j \leq 4$ such that $x \in U_i$ and $y \in U_j$, and $xy \in E(G)$, then the concatenation of the u_i-u_j path of order 3 with u_ixyu_j is a cycle of

order 5, a contradiction). Also, since there is a path of order 2 joining any two vertices of U, as a result $N_R(U_i) \cap N_R(U_j) = \varnothing, 1 \leq i < j \leq 4$ (otherwise, if there are $1 \leq i < j \leq 4$ such that $w \in N_R(U_i) \cap N_R(U_j)$, then the concatenation of the u_i-u_j path of order 2 with $u_i x w y u_j$ where $x \in U_i$ and $y \in U_j$, and $xw, yw \in E(G)$ is a cycle of order 5, a contradiction). Therefore, $|U_i \cup N_R(U_i) \cup \{u_i\}| \geq \delta(G) + 1$. Thus, $|V(G)| \geq 4(\delta(G) + 1) \geq 4(8 + 1) = 4.9 = 36$. That contradicts the fact that the order of G is 33. □

References

[1] A. Baniabedalruhman and M.M.M. Jaradat, The cycle-complete graph Ramsey number $r(C_7, K_7)$. *Journal of combinatorics, information & system sciences* (Accepted).

[2] B. Bollobás, C. J. Jayawardene, Z. K. Min, C. C. Rousseau, H. Y. Ru, and J. Yang, On a conjecture involving cycle-complete graph Ramsey numbers, *Australas. J. Combin.*, 22 (2000), 63-72.

[3] J.A. Bondy and P. Erdős, Ramsey numbers for cycles in graphs, *Journal of Combinatorial Theory, Series B*, 14 (1973), 46-54.

[4] T.C. E. Cheng, Y. Chen, Y. Zhang and C.T. Ng, The Ramsey numbers for a cycle of length six or seven versus a clique of order seven, *Discrete Mathematics*, 307 (2007), 1047-1053.

[5] P. Erdős, R. J. Faudree, C. C. Rousseau, and R. H. Schelp, On cycle-complete graph Ramsey numbers, *J. Graph Theory*, 2 (1978), 53-64.

[6] R.J. Faudree and R. H. Schelp, All Ramsey numbers for cycles in graphs, *Discrete Mathematics*, 8 (1974), 313-329.

[7] M.M.M. Jaradat and B. Alzaliq, The cycle-complete graph Ramsey number $r(C_8, K_8)$. *SUT Journal of Mathematics* 43 (2007), 85-98.

[8] M.M.M. Jaradat and A.M.M. Baniabedalruhman, The cycle-complete graph Ramsey number $r(C_8, K_7)$. *International Journal of pure and applied mathematics*, 41 (2007), 667-677.

[9] C.J. Jayawardene and C. C. Rousseau, The Ramsey number for a cycle of length five versus a complete graph of order six, *J. Graph Theory*, 35 (2000), 99-108.

[10] V. Nikiforov, The cycle-complete graph Ramsey numbers, *Combin. Probab. Comput.* 14 (2005), no. 3, 349–370.

[11] S. P. Radziszowski and K.-K. Tse, A computational approach for the Ramsey numbers $r(C_4, K_n)$, *J. Comb. Math. Comb. Comput.*, 42 (2002), 195-207.

[12] I. Schiermeyer, All cycle-complete graph Ramsey numbers $r(C_n, K_6)$, *J. Graph Theory*, 44 (2003), 251-260.

[13] I. Schiermeyer, The cycle-complete graph Ramsey number $r(C_5, K_7)$, *Discussiones Mathematicae Graph Theory* 25 (2005) 129-139.

[14] Y.J. Sheng, H. Y. Ru and Z. K. Min, The value of the Ramsey number $r(C_n, K_4)$ is $3(n-1)+1$ $(n \geq 4)$, *Ausralas. J. Combin.*, 20 (1999), 205-206.

International J.Math. Combin. Vol.1 (2009), 46-49

Smarandache Breadth Pseudo Null Curves in Minkowski Space-time

Melih Turgut

(Dokuz Eylül University,Buca Educational Faculty, Department of Mathematics, 35160 Buca,Izmir, Turkey)

E-mail: melih.turgut@gmail.com

Abstract A regular curve with more than 2 breadths in Minkowski 3-space is called a *Smarandache Breadth Curve* [8]. In this short paper, we adapt notion of Smarandache breadth curves to Pseudo null curves in Minkowski space-time and study a special case of Smarandache breadth curves. Some characterizations of Pseudo null curves of constant breadth in Minkowski space-time are presented.

Key Words: Minkowski space-time, pseudo null curves, Smarandache breadth curves, curves of constant breadth.

AMS(2000): 51B20, 53C50.

§1. Introduction

Curves of constant breadth were introduced by L. Euler [4]. In [6], some geometric properties of plane curves of constant breadth are given. And, in another work [7], these properties are studied in the Euclidean 3-Space E^3. Moreover, M. Fujivara [5] had obtained a problem to determine whether there exist space curve of constant breadth or not, and he defined "*breadth*" for space curves and obtained these curves on a surface of constant breadth. In [1], this kind curves are studied in four dimensional Euclidean space E^4.

A regular curve with more than 2 breadths in Minkowski 3-space is called a *Smarandache Breadth Curve*. In this paper, we adapt Smarandache breadth curves to pseudo null curves in Minkowski space-time. We investigate position vector of simple closed pseudo null curves and give some characterizations in the case of constant breadth. We used the method of [7], [8].

§2. Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space E_1^4 are briefly presented (A more complete elementary treatment can be found in [2]).

Minkowski space-time E_1^4 is an Euclidean space E^4 provided with the standard flat metric given by

[1]Received January 5, 2009. Accepted February 6, 2009.

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system in E_1^4.

Since g is an indefinite metric, recall that a vector $v \in E_1^4$ can have one of the three causal characters; it can be space-like if $g(v, v) > 0$ or $v = 0$, time-like if $g(v, v) < 0$ and null (light-like) if $g(v, v)$=0 and $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_1^4 can be locally be space-like, time-like or null (light-like), if all of its velocity vectors $\alpha'(s)$ are respectively space-like, time-like or null. Also, recall the norm of a vector v is given by $\|v\| = \sqrt{|g(v, v)|}$. Therefore, v is a unit vector if $g(v, v) = \pm 1$. Next, vectors v, w in E_1^4 are said to be orthogonal if $g(v, w) = 0$. The velocity of the curve $\alpha(s)$ is given by $\|\alpha'(s)\|$. And $\alpha(s)$ is said to be parametrized by arclength function s, if $g(\alpha'(s), \alpha'(s)) = \pm 1$.

Denote by $\{T(s), N(s), B_1(s), B_2(s)\}$ the moving Frenet frame along the curve $\alpha(s)$ in the space E_1^4. Then T, N, B_1, B_2 are, respectively, the tangent, the principal normal, the first binormal and the second binormal vector fields. Recall that space-like curve with space-like first binormal and null principal normal with null second binormal is called a pseudo null curve in Minkowski space-time. Let $\alpha = \alpha(s)$ be a pseudo unit speed null curve in E_1^4. Then the following Frenet equations are given in [3]:

$\alpha = \alpha(s)$ is a pseudo null curve. Then we can write that

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ 0 & 0 & \tau & 0 \\ 0 & \sigma & 0 & -\tau \\ -\kappa & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} \tag{1}$$

where T, N, B_1 and B_2 are mutually orthogonal vectors satisfying equations

$$g(T, T) = 1, g(B_1, B_1) = 1, g(N, N) = g(B_2, B_2) = 0, g(N, B_2) = 1,$$
$$g(T, N) = g(T, B_1) = g(T, B_2) = g(N, B_1) = g(B_1, B_2) = 0.$$

And here, κ, τ and σ are first, second and third curvature of the curve α, respectively. And, a pseudo null curve's first curvature κ can take only two values: 0 when α is a straight line or 1 in all other cases. In the rest of the paper, we shall assume $\kappa = 1$ at every point.

In the same space, authors, in [3], gave a characterization with the following theorem.

Theorem 2.1 *Let $\alpha = \alpha(s)$ be a pseudo null unit speed curve with curvatures $\kappa = 1$, $\tau \neq 0$ and $\sigma \neq 0$ for each $s \in I \subset R$. Then, α lies on the hyperbolic sphere (H_0^3), if and only if $\dfrac{\sigma}{\tau} = constant < 0$.*

§3. Smarandache breadth pseudo null curves in $\mathbf{E}_1^4$

In this section, first, we adapt the notion of Smarandache breadth curves to the space E_1^4 with the following definition.

Definition 3.1 *A regular curve with more than 2 breadths in Minkowski space-time is called a Smarandache breadth curve.*

Let $\varphi = \varphi(s)$ be a Smarandache Breadth pseudo null curve. Moreover, let us suppose $\varphi = \varphi(s)$ simple closed pseudo null curve in the space E_1^4. These curves will be denoted by (C). The normal plane at every point P on the curve meets the curve at a single point Q other than P. We call the point Q the opposite point of P. We consider a curve in the class Γ as in [5] having parallel tangents T and T^* in opposite directions at the opposite points φ and φ^* of the curve. A simple closed pseudo null curve having parallel tangents in opposite directions at opposite points can be represented with respect to Frenet frame by the equation

$$\varphi^* = \varphi + m_1 T + m_2 N + m_3 B_1 + m_4 B_2, \tag{2}$$

where $m_i(s)$, $1 \leq i \leq 4$ are arbitrary functions and φ and φ^* are opposite points. Differentiating both sides of (2) and considering Frenet equations, we have

$$\begin{aligned} \frac{d\varphi^*}{ds} = T^* \frac{ds^*}{ds} = (\frac{dm_1}{ds} - m_4 + 1)T + (\frac{dm_2}{ds} + m_1 + m_3\sigma)N + \\ (\frac{dm_3}{ds} + m_2\tau - m_4\sigma)B_1 + (\frac{dm_4}{ds} - m_3\tau)B_2. \end{aligned} \tag{3}$$

We know that $T^* = -T$ and if we call ϕ as the angle between the tangent of the curve (C) at point $\varphi(s)$ with a given fixed direction and consider $\frac{d\phi}{ds} = \kappa = 1 = \frac{d\phi}{ds^*} = \kappa^*$, since $ds = ds^*$. Then, we get the following system of ordinary differential equations:

$$\begin{aligned} m_1' &= m_4 - 2 \\ m_2' &= -m_1 - m_3\sigma \\ m_3' &= m_4\sigma - m_2\tau \\ m_4' &= m_3\tau \end{aligned} \tag{4}$$

Using system (4), we have the following differential equation with respect to m_1 as

$$\frac{d}{ds}\left[\frac{1}{\tau}\frac{d}{ds}(\frac{1}{\tau}\frac{d^2 m_1}{ds^2})\right] - \frac{\sigma}{\tau}\frac{d^2 m_1}{ds^2} - \frac{d}{ds}\left[\frac{\sigma}{\tau}(\frac{dm_1}{ds} + 2)\right] - m_1 = 0. \tag{5}$$

Corollary 3.2 *The differential equation of fourth order with variable coefficients* (5) *is a characterization for* φ^*. *Via its solution, position vector of a simple closed pseudo null curve can be determined.*

However, a general solution of (5) has not yet been found. If the distance between opposite points of (C) and (C^*) is constant, then, due to null frame vectors, we may express

$$\|\varphi^* - \varphi\| = m_1^2 + 2m_2m_4 + m_3^2 = l^2 = constant. \tag{6}$$

Hence, we write

$$m_1\frac{dm_1}{ds} + m_2\frac{dm_4}{ds} + m_4\frac{dm_2}{ds} + m_3\frac{dm_3}{ds} = 0. \tag{7}$$

Considering system (4), we obtain

$$m_1 = 0. \tag{8}$$

Since, we have, respectively

$$
\begin{aligned}
m_2 &= s + c \\
m_3 &= 0 \\
m_4 &= 2
\end{aligned}
\tag{9}
$$

Using obtained equations and considering $(4)_2$, we have $\dfrac{\sigma}{\tau} = \dfrac{s+c}{2}$. Thus, we immediately arrive at the following results.

Corollary 3.3 *Let $\varphi = \varphi(s)$ be a pseudo null curve of constant breadth. Then;*
$i)$ There is a relation among curvature functions as

$$
\frac{\sigma}{\tau} = \frac{s+c}{2}. \tag{10}
$$

$ii)$ There are no spherical pseudo null curve of constant breadth in Minkowski space-time.
$iii)$ Position vector of a pseudo null curve of constant breadth can be expressed

$$
\varphi^* = \varphi + (s+c)N + 2B_2. \tag{11}
$$

References

[1] A. Mağden and Ö. Köse, On the curves of constant breadth, *Tr. J. of Mathematics*, **1**(1997), 277–284.

[2] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York,1983.

[3] C. Camci, K. Ilarslan and E. Sucurovic, On pseudohyperbolical curves in Minkowski space-time. *Turk J.Math.* **27** (2003) 315-328.

[4] L. Euler, De curvis trangularibis, *Acta Acad. Petropol* (1780), 3–30.

[5] M. Fujivara, On space curves of constant breadth, *Tohoku Math. J.* **5** (1914), 179-184.

[6] Ö. Köse, Some properties of ovals and curves of constant width in a plane, *Doga Mat.*,(8) **2** (1984), 119-126.

[7] Ö. Köse, On space curves of constant breadth, *Doga Math.* (10) **1** (1986), 11–14.

[8] S. Yılmaz and M. Turgut, On the time-like curves of constant breadth in Minkowski 3-space. *International J.Math. Combin.* Vol.3 (2008), 34-39.

International J.Math. Combin. Vol.1 (2009), 50-60

Smarandachely k-Constrained labeling of Graphs

ShreedharK[1], B. Sooryanarayana[2] and RaghunathP[3]

[1]Department of Mathematics, K.V.G.College of Engineering, Karnataka, INDIA, 574 327

[2]Department of Math.& Comput., Dr.Ambedkar Institute of Technology, Karnataka, INDIA, 560 056

[3]Dept. of Master of Computer Science, Reva Institute of Technology, Karnataka , INDIA, 560 064

Email: shreedhar.k@rediffmail.com, sooryanarayan.mat@dr-ait.org, p_raghunath1@yahoo.co.in

Abstract A *Smarandachely* $k-$ *constrained labeling* of a graph $G(V, E)$ is a bijective mapping $f : V \cup E \to \{1, 2, .., |V|+|E|\}$ with the additional conditions that $|f(u)-f(v)| \geq k$ whenever $uv \in E$, $|f(u)-f(uv)| \geq k$ and $|f(uv)-f(vw)| \geq k$ whenever $u \neq w$, for an integer $k \geq 2$. A graph G which admits a such labeling is called a *Smarandachely* $k-$ *constrained total graph*, abbreviated as $k-CTG$. The minimum number of isolated vertices required for a given graph G to make the resultant graph a $k-CTG$ is called the $k-$ *constrained number* of the graph G and is denoted by $t_k(G)$. Here we obtain $t_k(K_{1,n}) = n(k-2)$, for all $k \geq 3$ and $n \geq 4$ and also prove that wheels, cycles, paths, complete graphs and Cartesian product of any two non trivial graphs etc., are CTG's for some k .In addition we pose some open problems.

Key Words: Smarandachely k-constrained labeling, Smarandachely k-constrained total graph.

AMS(2000): 05C78

§1. Introduction

All the graphs considered in this paper are simple, finite and undirected. For standard terminology and notations we refer [2], [3]. There are several types of graph labelings studied by various authors. We refer [1] for the entire survey on graph labeling. Here we introduce a new labeling and call it as Smarandachely k-constrained labeling. Let $G = (V, E)$ be a graph. A bijective mapping $f : V \cup E \to \{1, 2, ..., |V| + |E|\}$ is called a *Smarandachely* $k-$ *constrained labeling* of G if it satisfies the following conditions for every $u, v, w \in V$:

(i) $|f(u) - f(v)| \geq k$ whenever $uv \in E$;
(ii) $|f(u) - f(uv)| \geq k$;
(iii) $|f(uv) - f(vw)| \geq k$ whenever $u \neq w$.

A graph G which admits such a labeling is called a *Smarandachely* k*-constrained total graph*, abbreviated as $k - CTG$. We note here that every graph G need not be a $k - CTG$ (e.g. the path P_2). However, with the addition of some isolated vertices, we can always make

[1]Received January 10, 2009. Accepted February 12, 2009.

the resultant graph a $k-CTG$. The minimum n such that the graph $G \cup \overline{K}_n$ is a $k-CTG$ is called *k-constrained number of the graph* G and denoted by $t_k(G)$, the corresponding labeling is called a *minimum k-constrained total labeling of* G. Further it follows from the definitions that if G is a $k-CTG$, then its total graph $T(G)$ is $k-chromatic$ (i.e. minimum span of $L(k,1)$ labeling of $T(G)$ is $|V(T(G))|$) and vice-versa.

If G and H are any two graphs, then $G \cup H$, $G+H$, and $G \times H$ respectively denote the Union, Sum and Cartesian product of G and H. For any real number n, $\lceil n \rceil$ and $\lfloor n \rfloor$ are respectively denote the smallest integer greater than or equal to n and the greatest integer less than or equal to n.

In this paper we obtain $t_k(K_{1,n}) = n(k-2)$, for all $k \geq 3$ and is $n(k-2)+1$ if $n=3$ or $k=2$, and also prove that wheels, cycles, paths, complete graphs and Cartesian product of any two non trivial graphs etc., are CTG's for some k. In addition we pose some open problems.

§2. Results and Open problems on 2-CTG

Observation 2.1 *Every totally disconnected graph is trivially a $k-CTG$, for all $k \geq 1$ and every graph is trivially a $1-CTG$.*

Observation 2.2 *No nontrivial connected $2-CTG$ of order less than 4, and P_4 is the smallest such connected graph.*

Observation 2.3 *If G_1 and G_2 are $k-CTG$'s, then their union is again a $k-CTG$.*

Theorem 2.4 *For a path P_n on n vertices,* $t_2(P_n)=\begin{cases} 2 \ \ if \ \ n=2, \\ 1 \ \ if \ \ n=3, \\ 0 \ \ else. \end{cases}$

Proof Let $V(P_n) = \{v_1, v_2, ..., v_n\}$ and $E(P_n) = \{v_i v_{i+1} | 1 \leq i \leq n-1\}$. Consider a total labeling $f : V \cup E \longrightarrow \{1, 2, 3, ..., 2n-1\}$ defined as $f(v_1) = 2n-3$; $f(v_2) = 2n-1$; $f(v_1v_2) = 2$; $f(v_2v_3) = 4$; and $f(v_k) = 2k-5$, $f(v_kv_{k+1}) = 2k$, for all $k \geq 3$. This function f serves as a Smarandachely 2-constrained labeling for P_n, for $n \geq 4$. Further, the cases $n = 2$ and $n = 3$ are easy to prove. □

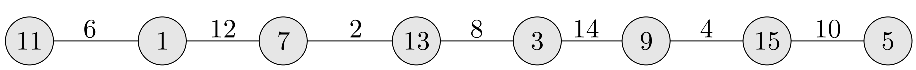


Figure 1: A 2-constrained labeling of a path P_7.

Corollary 2.5 *For every $n \geq 4$, the cycle C_n is a 2-CTG and when $n = 3$, $t_2(C_n) = 2$.*

Proof If $n \geq 4$, then the result follows immediately by joining end vertices of P_n by an edge v_1v_n , and, extending the total labeling f of the path as in the proof of the Theorem 2.4 above to include $f(v_1v_2) = 2n$.

Consider the case $n = 3$. If the integers a and $a+1$ are used as labels, then one of them is assigned for a vertex and other is to the edge not incident with that vertex. But then, $a+2$

can not be used to label the vertex or an edge in C_3. Therefore, for each three consecutive integers we should leave at least one integer to label C_3. Hence the span of any Smarandachely 2-constrained labeling of C_3 should be at least 8. So $t_2(C_3) \geq 2$. Now from the Figure 3 it is clear that $t_2(C_3) \leq 2$. Thus $t_2(C_3) = 2$.

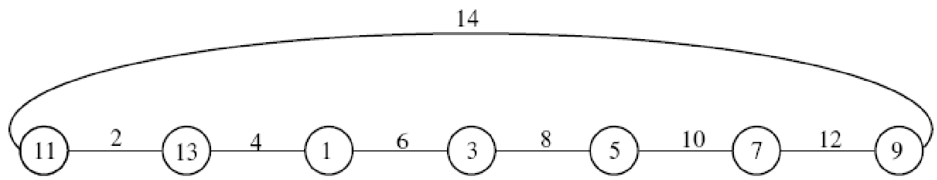


Figure 2: A 2-constrained labeling of a path C_7

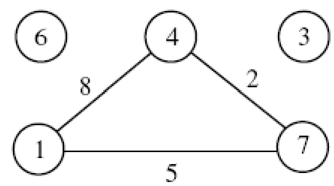


Figure 3: A 2-constrained labeling of a path $C_3 \cup \overline{K}_2$

Lemma 2.6 *For any integer* $n \geq 3$, $t_2(K_{1,n}) = 1$.

Proof Since each edge is incident with the central vertex and every other vertex is adjacent to the central vertex, no two consecutive integers can be used to label the central vertex and an edge (or a vertex) of the star. Hence $t_2(K_{1,n}) \geq 1$. Now to prove the opposite inequality, let $\acute{G} = K_{1,n} \cup K_1$, v_0 be the central vertex and $v_1, v_2, ..., v_n$ be the end vertices of the star $K_{1,n}$. Let v_{n+1} be the isolated vertex of $\acute{G}$.

We now define $f : \acute{G} \rightarrow \{1, 2, ..., 2n + 2\}$ as follows:

$f(v_0) = 2n + 2$; $f(v_1) = 2n - 1$; $f(v_{n+1}) = 2n + 1$; $f(v_k) = 2k - 3$ for all $k, 2 \leq k \leq n$; $f(v_0v_i) = 2i$, for all $i, 1 \leq i \leq n$.

The function f defined above is clearly a Smarandachely 2-constrained labeling of $\acute{G}$. So $t_2(K_{1,n}) \leq 1$. Hence the result. □

Lemma 2.7 *The graph* $K_{2,n}$ *is a 2-CTG if and only if* $n \geq 2$.

Proof When $n = 1$ or $n = 2$ the result follows respectively from Theorem 2.4 and Corollary 2.5. For $n \geq 3$, let $H_1 = \{v_1, v_2, ..., v_n\}$ and $H_2 = \{u_1, u_2\}$ be the bipartitions of the graph $K_{2,n}$. Define a total labeling f as follows:

$f(u_1) = 2n+1$; $f(u_2) = 2n+2$; $f(v_1) = 2n-1$; $f(v_{i+1}) = 2i-1$, for all $i, 1 \leq i \leq n-1$; and for all odd j, $f(u_1v_j) = 2(n + 1) + j$,$f(u_2v_j) = 2j$; and for all even j, $f(u_1v_j) = 2j$, $f(u_2v_j) = 2(n+1)+j$, $1 \leq j \leq n$. Since f assigns no two consecutive integers for the adjacent or incident pairs, it is a Smarandachely 2-constrained labeling with span $3n+2$. Hence $K_{2,n}$ is a 2-CTG. □

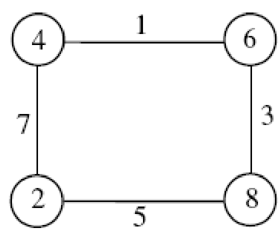


Figure 4: A 2-constrained total labeling of *l*

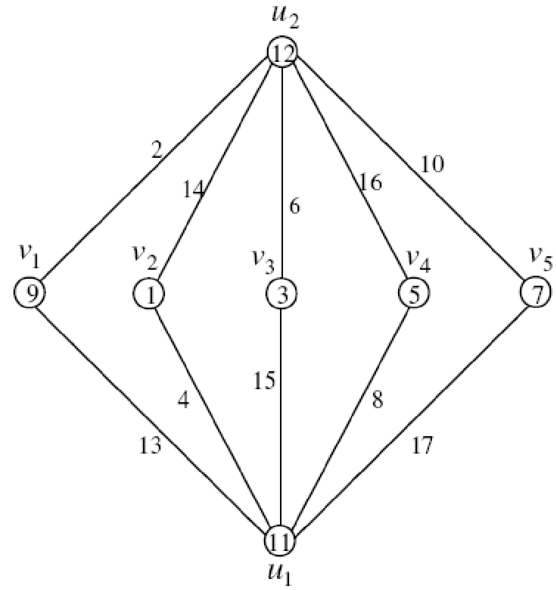


Figure 5: A 2-constrained total labeling of $K_{2,5}$

A function $f : E \to \{1, 2, ..., |E|\}$ is called a *k-constrained edge labeling of a graph $G(V, E)$* if $|f(e_1) - f(e_2)| \geq k$ whenever the edges e_1 and e_2 are adjacent in G. A graph G which admits a k-constrained edge labeling is called a *$k-$ constrained edge labeled graph* ($k - CEG$).

Lemma 2.8 *For any two positive integers $m, n \geq 3$, the complete bipartite graph $K_{m,n}$ is a 2-CEG.*

Proof Without loss of generality, we assume that $m \geq n$. Let $U = \{u_0, u_1, u_3, ..., u_{m-1}\}$ and $V = \{v_0, v_1, v_2, ..., v_{n-1}\}$ be the bipartitions of $K_{m,n}$.

Case(i): $m \not\equiv 2(modn)$

Define a function $f : E(K_{m.n}) \to \{1, 2, 3, ..., mn\}$, by

$f(u_i v_{i+k(mod\ \ n)}) = km + i + 1$, for all i and k, where $0 \leq i \leq m-1$ and $0 \leq k \leq n-1$.

The function f defined above is clearly a bijection. Further, the two distinct edges u_iv_j and u_lv_k are adjacent only if $i = l$ or $j = k$, but not both. So for $0 \leq j, k \leq n-1$, we have $|f(u_iv_j) - f(u_iv_k)| = |[(j-i)m+i+1] - [(k-i)m+i+1]| = |(j-k)m| = |(j-k)|m \geq m \geq 2$, whenever $j \neq k$. And if $j = k$, then $l \neq i$ and hence $|f(u_iv_j) - f(u_lv_j)| = |1+i+m(i-j) - 1-l-m(j-l)| = |(i-l)+m(j-i-j+l)| = |(i-l)(1-m)| = |m-1||l-i| \geq 2$ (since $m \geq 3$). Therefore the function f is a valid 2-constrained edge labeling.

Case(ii): $m \equiv 2(\text{mod } n)$

Relabel the vertices $v_0, v_1, v_2, ..., v_{n-1}$ in V respectively as $v_0, v_{n-1}, v_1, v_{n-2}, v_2, ..., v_{\lfloor \frac{n}{2} \rfloor}$.

Then the function f defined in the above case (i) serves again as a valid 2-constrained edge labeling. □

Theorem 2.9 *For the given positive integers m and n, with $m \geq n$*

$$t_2(K_{m,n})=\begin{cases} 2 & if \quad n=1 \quad and \quad m=1, \\ 1 & if \quad n=1 \quad and \quad m\geq 2, \\ 0 & else. \end{cases}$$

Proof For $n = 1$ and $m = 1$ or 2, the result follows from Theorem 2.4. And the case $n = 1$ and $m \geq 3$ follows from Lemma 2.6. We now take the case $n > 1$. When $n = 2$, $m \geq 2$, the result follows by Lemma 2.7. If $m, n \geq 3$, then by Lemma 2.8, there exists a 2-constrained edge labeling $f : E(K_{m,n}) \to \{1, 2, ..., mn\}$. Let $U = \{u_0, u_1, ..., u_{m-1}\}$ and $V = \{v_0, v_1, v_2, ..., v_{n-1}\}$ be the bipartitions of $K_{m,n}$. We now consider a function $g : V(K_{m,n}) \cup E(K_{m,n}) \to \{1, 2, 3, ..., m+n+mn\}$, defined as follows:

$g(u_i) = i + 1$,
$g(v_j) = mn + m + j + 1$, and
$g(u_iv_j) = f(u_iv_j) + m$,

for all i, j such that $0 \leq i \leq m-1$, $\;0 \leq j \leq n-1$.

The function g so defined is a Smarandachely 2-constrained labeling of $K_{m,n}$ for $m, n \geq 3$. Hence the result. □

Theorem 2.10 *If G_1 and G_2 are any two nontrivial connected graphs which are 2-CTG's, then $G_1 + G_2$ is a 2-CTG.*

Proof Let $G_1(V_1, E_1)$ be a graph of order m and size q_1 and $G_2(V_2, E_2)$ be a graph of order n and size q_2 . Let $u_0, u_1, ..., u_{m-1}$ be the vertices of G_1 and $v_0, v_1, v_2, ..., v_{n-1}$ be the vertices of G_2 . Since G_1 and G_2 are 2-CTG's, there exist Smarandachely 2-constrained labelings, $f_1 : V(G_1) \cup E(G_1) \to \{1, 2, 3, ..., m + q_1\}$, and $f_2 : V(G_2) \cup E(G_2) \to \{1, 2, 3, ..., n + q_2\}$ for G_1 and G_2 respectively.

Let $G = G_1 + G_2$ and G^* be the graph obtained from G by deleting all the edges of G_1 as well as G_2. Then G^* is a complete bipartite graph $K_{m,n}$ and $G = G_1 \cup G_2 \cup G^*$. Since both the graphs G_1 and G_2 are 2-CTG's, we have both m and n are at least 4, and hence by Lemma 2.8, there exists a 2-constrained edge labeling $g : E(G^*) \to \{1, 2, ..., mn\}$ for G^*. Since G_1 is Smarandachely 2-constrained total graph, the maximum label assigned to a vertex or edge is $m + q_1$. Let u_i be the vertex of G_1 such that $m + q_1$ is assigned for the vertex u_i or to an edge incident with the vertex u_i in G_1 by the function f_1. If g is not assigned 1 for the edge incident with u_i of G^* , then just super impose the vertex u_i of G_1 with the vertex u_i of G^* for all $i, 0 \leq i \leq m-1$. Else if g is assigned 1 for an edge incident with u_i then re-label the vertex u_i of G^* as $u_{i+1(mod\ m)}$ for every $i, 0 \leq i \leq m-1$, before the superimposition. Repeat the process of superimposition of the vertex v_i of G^* with the corresponding vertex v_i of G_2 in the similar manner depending on whether the largest assignment of g to an edge of G^* adjacent to the smallest assignment 1 of G_2 assigned by the function f_2 or not. Now extend these functions to the function $f : VG \cup E(G) \to \{1, 2, 3, ..., m + n + q_1 + q_2 + mn\}$, by defining it as follows:

$$f(x)=\begin{cases} f_1(x), & if \quad x \in V(G_1) \cup E(G_1), \\ f_2(x) + m(n + q_1), & if \quad x \in V(G_2) \cup E(G_2), \\ g(x) + m + q_1 & if \quad x = u_i v_j \quad for \quad all \quad i, j, \quad 0 \le i \le m-1, \quad 0 \le j \le n-1. \end{cases}$$

The function f defined above serves as a Smarandachely 2-constrained labeling. □

Corollary 2.11 *For every integer $n \ge 4$, the complete graph K_n is a 2-CTG.*

Proof Follows from the following four Figures 6 to 9 and by Theorem 2.10 (since every other complete graph is a successive sum of two or more of these graphs). □

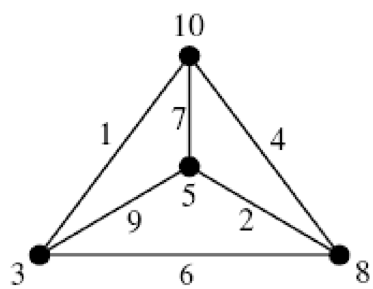


Figure 6: A 2-constrained labeling of K_4

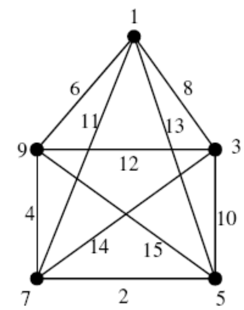


Figure 7: A 2-constrained labeling of K_5

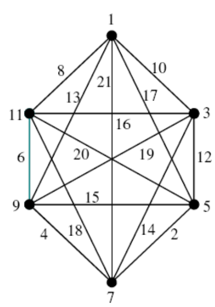


Figure 8: A 2-constrained labeling of K_6

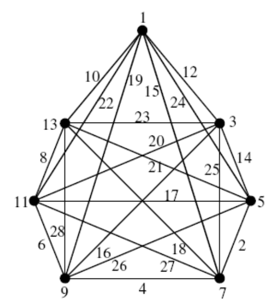


Figure 9: A 2-constrained labeling of K_7

Theorem 2.12 *For any integer $n \ge 3$, the wheel $W_{1,n}$ is a 2-CTG.*

Proof Let v_0 be the central vertex and $v_1, ..., v_n$ be the rim vertices of $W_{1,n}$. Define a total labeling f on $W_{1,n}$ as; (i)$f(v_0) = 3n + 1$; (ii) For all $i, 1 \le i \le n$, $f(v_i) = 2i - 3(mod\ 2n)$; (iii) $f(v_0v_i) = 2i$; and (iv) For all $l, 0 \le l \le n$, $f(v_{1+lk(mod\ n)}v_{2+lk(mod\ n)}) = 2n + l + 1$, where k is any integer such that $2 \le k < n - 1$ and $gcd(n, k) = 1$. The existence of such k for a given integer n is obvious for all n except $n = 3, 4$ and 6. For $n = 3$, the result follows by Corollary **??**. The required labeling for the special cases $n = 4$ and $n = 6$ are shown in Figures 10 and 11 below. □

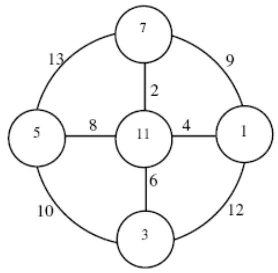


Figure 10: A 2-constrained labeling of $W_{1,4}$

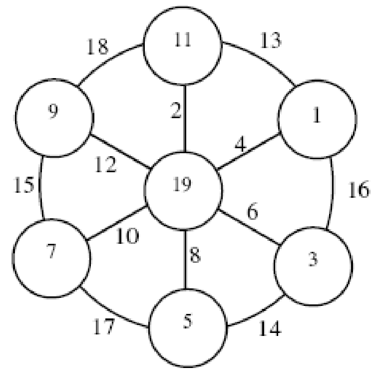
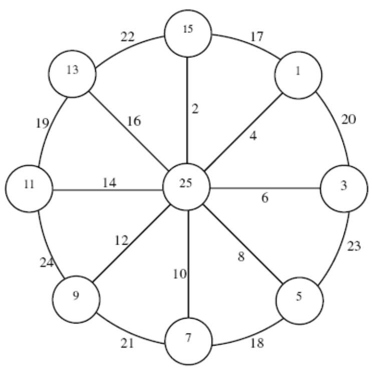


Figure 11:A 2-constrained labeling of $W_{1,6}$



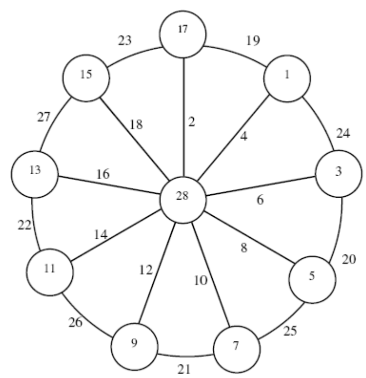


Figure 12:A 2-constrained labeling of $W_{1,8}$ Figure 13: A 2-constrained labeling of $W_{1,9}$

We end up this section with the following open problem.

Problem 2.13 *Determine the graph of order at least 4 which is not a 2-CTG?*

§3. Results on k-CTG

We now prove the results of previous sections for general cases and give some open problems.

Observation 3.1 *G is a k-CTG $\Rightarrow$ G is a $(k-1)$-CTG.*

Lemma 3.2 *If the path P_n on n vertices is a k-CTG for some $k \geq 2$, then $k \leq \frac{2n-3}{2}$.*

Proof The result is obvious for the case $n \leq 4$. In fact, if $n \leq 4$, $2n-3 \leq 5 \Rightarrow k = 1$ or 2, so the result follows by Theorem 2.4. Now assume that $n \geq 5$. Let f be any Smarandachely k-constrained labeling of the path P_n. Then the span of f is $2n-1$. Further f assigns the integer 1 to a vertex or an edge.

Case (i) $f(v_i) = 1$, for some $i, 1 \leq i \leq n$.

Subcase (i) $i \neq 1$ (or $i \neq n$)

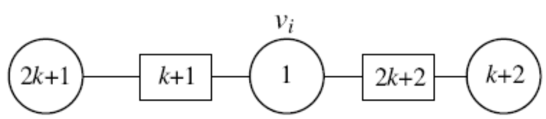


Figure 14: A minimum possible assignment for three consecutive vertices of a path.

The minimum assignment for the neighboring vertices of v_i is shown in the Figure 14. Since span of f is $2n-1$, we get $2k+2 \leq 2n-1$. Hence the result is true in this case.

Subcase (ii) $i = 1$ (or $i = n$)

In this case for the internal (other than the end vertex) vertex v_j, $f(v_j) \geq 2$, and hence for the minimum assignment for the neighboring vertices as well as the incident edges we get(again referring the same Figure 14 with label 1 as $f(v_j)$) $2k+f(v_j)+1 \leq 2n-1 \Rightarrow 2k \leq 2n-2-f(v_j) < 2n-3$.

Case (ii) $f(v_iv_{i+1}) = 1$, for $i, 1 \leq i \leq n-1$.

Result follows immediately by the Figure 14 treating rectangular boxes as vertices and circles as edges. □

The following theorem extends Theorem 2.9 up to certain k.

Theorem 3.3 *The path P_n on n vertices is a k-CTG whenever $2 \leq k \leq n - \lceil \frac{(n+1)}{3} \rceil$.*

Proof In view of observation 3.1, it suffices to define a total labeling f for $k = n - \lceil \frac{(n+1)}{3} \rceil$. Let us first denote the vertices and edges of the path simultaneously by the integers $1, 2, 3, ..., 2n-1$ as $v_1 = 1, v_1v_2 = 2, v_2 = 3, v_2v_3 = 4, v_3 = 5, ..., v_i = 2i-1, v_iv_{i+1} = 2i, v_{i+1} = 2i+1, ..., v_{n-1}v_n = 2(n-1), v_n = 2n-1$. Define an automorphism on $Z_{2n}/\{0\}$ as $f(1) = n+1+\lfloor \frac{(n-2)}{3} \rfloor, f(2) = n+1-\lceil \frac{(n+1)}{3} \rceil$, $f(3) = 1$ and for all $i, 4 \leq i \leq 2n-1$, $f(i) = f(i-3)+1$. The function f defined above is a Smarandachely $(n - \lceil \frac{(n+1)}{3} \rceil)$-constrained labeling for P_n. □

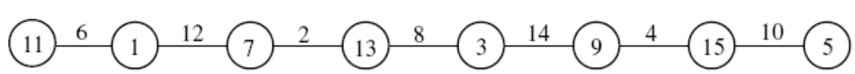


Figure 15: A 5-constrained total labeling of the path P_8.

Problem 3.4 *For any integers $n, k \geq 3$, determine the value of $t_k(P_n)$.*

Corollary 3.5 *The cycle C_n on n vertices are k-CTG's for every $2 \leq k \leq n - \lceil \frac{(n+1)}{3} \rceil$.*

Proof Let $v_0, v_1, \cdots, v_{n-1}$ be the vertices of C_n such that $v_iv_{i \oplus_n +1} \in V(C_n)$. Now for each $i, 0 \leq i \leq n-1$, denote the vertices and edges of C_n consecutively as $v_0 = 0, v_0v_1 = 1, v_1 =$

$2, v_1v_2 = 3, v_2 = 4, \cdots, v_{i-1} = 2(i-1), v_{i-1}v_i = 2i-1, v_i = 2i, \cdots, v_{n-2}v_{n-1} = 2n-3, v_{n-1} = 2n-2, v_{n-1}v_0 = 2n-1$. We now define a function f as follows:

Case (i) $3 \nmid (n-2)$.

Define: $f(0) = 1, f(1) = n+1-\lfloor \frac{n}{3} \rfloor, f(2) = n+1+\lceil \frac{n}{3} \rceil, f(i) = f(i-3)+1$, for all $i, 3 \leq i \leq 2n-1$. The function f is a Smarandachely $(n - \lceil \frac{(n+1)}{3} \rceil)$-constrained labeling of C_n.

Case (ii) $3 \mid (n-2)$.

Define: $f(0) = 1, f(1) = n+1+\lceil \frac{n}{3} \rceil, f(2) = n+1-\lfloor \frac{n}{3} \rfloor, f(i) = f(i-3)+1$, for all $i, 3 \leq i \leq 2n-1$. The function f is again a Smarandachely $(n - \lceil \frac{(n+1)}{3} \rceil)$-constrained labeling of C_n. □

Problem 3.6 *For any integers $n, k \geq 3$, determine the value of $t_k(C_n)$.*

Observation 3.7 *We are not sure about the range of k, that is, k may exceed $(n - \lceil \frac{(n+1)}{3} \rceil)$ for some path or cycle on n vertices. However achieving the maximum value of k may be tedious for a general graph (even for a path itself).*

Problem 3.8 *For a given integer $k \geq 2$, determine the bounds for a graph G to be a k-CTG.*

Problem 3.9 *For given positive integers m, n and k, does there exist a connected graph G with n vertices such that $t_k(G) = m$?*

Following theorem is a partial answer to the above Problem 3.9, which is also an extension of Lemma 2.6.

Theorem 3.10 *If $k \geq 3$ is any integer and $n \geq 3$, then,*

$$t_k(K_{1,n}) = \begin{cases} 3k-5, & if \quad n = 3, \\ n(k-2), & otherwise. \end{cases}$$

Proof For any Smarandachely k-constrained labeling f of a star $K_{1,n}$, the span of f, after labeling an edge by the least positive integer a is at least $a+nk$. Further, the span is minimum only if $a = 1$. Thus, as there are only $n+1$ vertices and n edges, for any minimum total labeling we require at least $1+nk-(2n+1) = n(k-2)$ isolated vertices if $n \geq 4$ and at least $1+nk-2n = n(k-2)+1$ if $n = 3$. In fact, for the case $n = 3$, as the central vertex is incident with each edge and edges are mutually adjacent, by a minimum k-constrained total labeling, the edges as well the central vertex can be labeled only by the set $\{1, 1+k, 1+2k, 1+3k\}$. Suppose the label 1 is assigned for the central vertex, then to label the end vertex adjacent to edge labeled $1+2k$ is at least $(1+3k)+1$ (since it is adjacent to 1, it can not be less than $1+k$). Thus at most two vertices can only be labeled by the integers between 1 and $1+3k$. Similar argument holds for the other cases also.

Therefore, $t(K_{1,n}) \geq n(k-2)$ for $n \geq 4$ and $t(K_{1,n}) \geq n(k-2)+1$ for $n = 3$.

To prove the reverse inequality, we define a k-constrained total labeling for all $k \geq 3$, as follows:

(1) When $n = 3$, the labeling is shown in the Figure 16 below

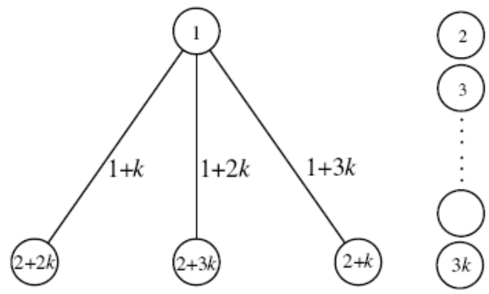


Figure 16: A k-constrained total labeling of $K_{1,3} \cup \overline{K}_{3k-5}$.

(2) When $n \geq 4$, define a total labeling f as $f(v_0v_j) = 1 + (j-1)k$ for all $j, 1 \leq j \leq n$. $f(v_0) = 1 + nk$, $f(v_1) = 2 + (n-2)k$, $f(v_2) = 3 + (n-2)k$,and for $3 \leq i \leq (n-1)$,

$$f(v_{i+1}) = \begin{cases} f(v_i) + 2, & if \quad f(v_i) \equiv 0(mod \quad k), \\ f(v_i) + 1, & otherwise. \end{cases}$$

and the rest all unassigned integers between 1 and $1 + nk$ to the $n(k-2)$ isolated vertices, where v_0 is the central vertex and $v_1, v_2, v_3, ..., v_n$ are the end vertices.

The function so defined is a Smarandachely k-constrained labeling of $K_{1,n} \cup \bar{K}_{n(k-2)}$, for all $n \geq 4$. $\square$

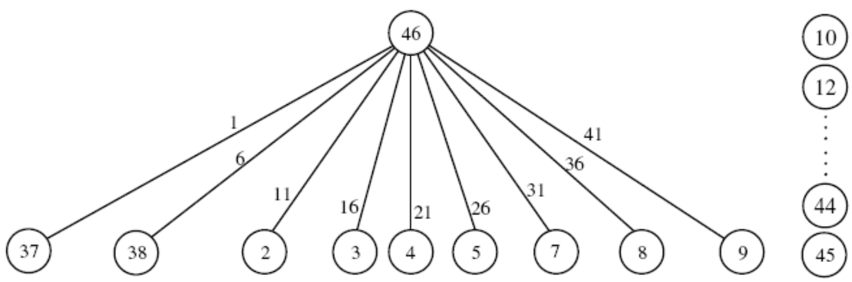


Figure 17: A 5-constrained total labeling of $K_{1,9} \cup \overline{K}_{27}$.

Theorem 3.11 *Let G_1 and G_2 be any two connected non-trivial graphs of order m and n respectively. Then their Cartesian product graph $G_1 \times G_2$ is a k-CTG for every $k \leq min\{m, n\}$.*

Proof Let $u_1, u_2, ..., u_m$ be the vertices of G_1 and $v_1, v_2, ..., v_n$ be the vertices of G_2 . Let $G = G_1 \times G_2$. Define a total labeling f on G as follows:

If $u_iu_j \in E(G_1)$, then label the corresponding edge $\{(u_i, v_1), (u_j, v_1)\}$ in G by the integer 1, the edge $\{(u_i, v_2), (u_j, v_2)\}$ by the integer 2, . . . so on, the edge $\{(u_i, v_l), (u_j, v_l)\}$ by the integer l, for all $l, 1 \leq l \leq n$. Label the vertex (u_i, v_l) by $n + l$ and the vertex (u_j, v_l) by $2n + l$ for all $l, 1 \leq l \leq n$. Next choose the new edge (if it exists) incident with either u_i or u_j, label the corresponding edges to this edge in $G_1 \times G_2$ by next n integers respectively as above and then continue the labeling for the corresponding unlabeled end vertices of these edges (if they exist). Repeat the process until all the edges as well as the vertices of each copy of G_1 in $G_1 \times G_2$ is labeled.

Since G_2 is connected, for each $s, 1 \leq s \leq m$, there exists an edge $\{(u_s, v_1), (u_s, v_i)\}$, for some $i, 1 \leq i \leq n$. Label the edge $\{(u_1, v_1), (u_1, v_i)\}$ by $n(m + q_1) + 1$ and then the parallel edges $\{(u_s, v_1), (u_s, v_i)\}$ by $n(m+q_1)+s$, for each $s, 2 \leq s \leq m$. Repeat the process of labeling by the next integers for each possible i, then repeat for next s. Continue this process for the possible edges $\{(u_s, v_2), (u_s, v_i)\}$, $2 \leq i \leq n$, then to $\{(u_s, v_3), (u_s, v_i)\}$, $3 \leq i \leq n$, . . . so on $\{(u_s, v_{n-1}), (u_s, v_n)\}$ (if no such edge exists at any stage then skip that step). Since the difference between two adjacent edges (as well as adjacent vertices and incident pairs) is at least $min\{m, n\}$, f is a Smarandachely $Min\{m, n\}$-constrained labeling of G. $\square$

The illustration of the proof of the theorem is shown in the following figure.

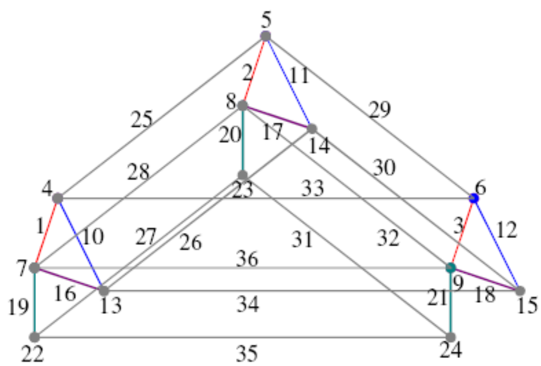


Figure 18: A 3-constrained total labeling of Cartesian product of graphs.

Problem 3.12 *Determine $t_k(K_{m,n})$, for any integer $k \geq 3$.*

Problem 3.13 *For any integer $n \geq 4$, determine $t_k(K_n)$.*

Problem 3.14 *Determine $t_k(W_{1,n})$, for any integer $k \geq 3$.*

Acknowledgment

We are very much thankful to the Principals, Prof. Jnanesh N.A., K.V.G. College of Engineering, Prof. Martin Jebaraj, Dr. Ambedkar Institute of Technology and Prof. Rana Prathapa Reddy, Reva Institute of Technology for their constant support and encouragement during the preparation of this paper.

References

[1] J. A. Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, # DS6,16(2009),1-219.

[2] Buckley F and Harary F, *Distance in Graphs*, Addison-Wesley, 1990.

[3] Hartsfield Gerhard and Ringel, *Pearls in Graph Theory*, Academic Press, USA, 1994.

International J.Math. Combin. Vol.1 (2009), 61-76

Equiparity Path Decomposition Number of a Graph

K. Nagarajan

(Department of Mathematics, Sri S.R.N.M.College, Sattur - 626 203, Tamil Nadu, India)

A. Nagarajan

(Department of Mathematics, V.O.C.College, Tuticorin - 628 008, Tamil Nadu, India)

I. Sahul Hamid

(Department of Mathematics, The Madura College, Madurai - 625 001, Tamil Nadu, India

Email: k_nagarajan_srnmc@yahoo.co.in, nagarajan_voc@gmail.com, sahulmat@yahoo.co.in

Abstract A *decomposition* of a graph G is a collection ψ of edge-disjoint subgraphs $H_1, H_2, \ldots, H_n$ of G such that every edge of G belongs to exactly one H_i. If each H_i is a path in G, then ψ is called a *path partition* or *path cover* or *path decomposition* of G. Various types of path covers such as Smarandache path k-cover, simple path covers have been studied by several authors by imposing conditions on the paths in the path covers . Here we impose parity condition on lengths of the paths and define an equiparity path cover as follows. An *equiparity path decomposition* of a graph G is a path cover ψ of G such that the lengths of all the paths in ψ have the same parity. The minimum cardinality of a equiparity path decomposition of G is called the *equiparity path decomposition number* of G and is denoted by $\pi_P(G)$. In this paper we initiate a study of the parameter π_P and determine the value of π_P for some standard graphs. Further, we obtain some bounds for π_P and characterize graphs attaining the bounds.

Key words: Odd parity path decomposition, even parity path decomposition, equiparity path decomposition, equiparity path decomposition number, Smarandache path k-cover.

AMS(2000): 05C35, 05C38.

§1. Introduction

By a graph, we mean a finite, undirected, non-trivial, connected graph without loops and multiple edges. The order and size of a graph are denoted by p and q respectively. For terms not defined here we refer to Harary [6].

Let $P = (v_1, v_2, \ldots, v_n)$ be a path in a graph $G = (V, E)$. The vertices $v_2, v_3, \ldots, v_{n-1}$ are called *internal vertices* of P and v_1 and v_n are called *external vertices* of P. The length of a path is denoted by $l(P)$. If the length of the path is odd(even) then we say that it is an *odd(even) path*.

A *subdivision graph* $S(G)$ of a graph G is obtained by subdividing each edge of G only

[1]Received January 8, 2008. Accepted February 14, 2009.

once. Two graphs are said to be *homeomorphic* if both can be obtained from the same graph by a sequence of subdivision of edges. A cycle with exactly one chord is called a *θ-graph*. The length of a largest cycle of a graph is called *circumference* of a graph and it is denoted by c. For vertices x and y in a connected graph G, the *detour distance* $D(x, y)$ is the length of a longest $x-y$ path in G. The *detour diameter* D of G is defined to be $D = max\{D(x, y) : x, y \in V(G)\}$. An *$(n, t)$-kite* consists of a cycle of length n with a t-edge path(called tail) attached to one vertex of the cycle. An *$(n, 1)$-kite* is called a kite with tail length 1.

A *decomposition* of a graph G is a collection of edge-disjoint subgraphs $H_1, H_2, \ldots, H_r$ of G such that every edge of G belongs to exactly one H_i. If each $H_i \cong H$, then the decomposition is called *isomorphic decomposition* and we also say that G is H decomposable. If each H_i is a path, then ψ is called a *path partition* or *path cover* or *path decomposition* of G. The minimum cardinality of a path partition of G is called the path partition number of G and is denoted by $\pi(G)$ and any path partition ψ of G for which $|\psi| = \pi(G)$ is called a *minimum path partition* or *π-cover* of G. The parameter π was studied by Harary and Schwenk [7], Peroche [9], Stanton *et.al.,* [10] and Arumugam and Suresh Suseela [4].

A more general definition on graph covering using paths is given as follows.

Definition 1.1([2]) *For any integer $k \geq 1$, a Smarandache path k-cover of a graph G is a collection ψ of paths in G such that each edge of G is in at least one path of ψ and two paths of ψ have at most k vertices in common.*

Thus if $k = 1$ and every edge of G is in exactly one path in ψ, then a Smarandache path k-cover of G is a simple path cover of G.

Consider the following path decomposition theorems.

Theorem 1.2([5]) *For any connected graph $(p, q)-$graph G, if q is even, then G has a P_3-decomposition.*

Theorem 1.3([10]) *If G is a 3-regular $(p, q)-$graph, then G is P_4 decomposable and*

$$\pi(G) = \frac{q}{3} = \frac{p}{2}.$$

Theorem 1.4([10]) *A complete graph K_{2n} is hamilton path decomposable of length $2n-1$. The path partition number π of a complete graphs are given by a) $\pi(K_{2n}) = n$ and (b) $\pi(K_{2n+1}) = n + 1$.*

The Theorems 1.2, 1.3 and 1.4(a) give the path decomposition in which all the paths are of even (odd) length. The above results give the isomorphic path decomposition in which all the paths are of same parity. This observation motivates the following definition for non-isomorphic path decomposition also.

Definition 1,5 *An equiparity path decomposition(EQPPD) of a graph G is a path cover ψ of G such that the lengths of all the paths in ψ have the same parity.*

Since for any graph G, the edge set $E(G)$ is an equiparity path decomposition, the collection $\mathcal{P}_P$ of all equiparity path decompositions of G is non-empty. Let $\pi_P(G) = min\,|\psi|$. Then $\pi_P(G)$ is called the *equiparity path decomposition number* of G and any equiparity path decomposition ψ of G for which $|\psi| = \pi_P(G)$ is called a *minimum equiparity path decomposition* of G or *π_P-cover* of G.

If the lengths of all the paths in ψ are even(odd) then we say that ψ is an *even (odd) parity path decomposition*, shortly *EPPD (OPPD)*.

Remark 1.6 Let $\psi = \{P_1, P_2, \ldots, P_n\}$ be an EQPPD of a $(p, q)-$graph G such that $l(P_1) \leq l(P_2) \leq, \ldots, l(P_n)$. Since every edge of G is in exactly one path P_i , we have $\sum\limits_{i=1}^{n} l(P_i) = q$ and hence every EQPPD of G gives rise to a partition of an integer q into same parity,

Remark 1.7 If G is a graph of odd size, then any equiparity path decomposition ψ of a graph G is an odd parity path decomposition and consequently $\pi_P(G)$ is odd.

Remark 1.8 If an equiparity path decomposition ψ of a graph G is an even equiparity path decomposition, then q is even.

Various types of path decompositions and corresponding parameters have been studied by several authors by imposing conditions on the paths in the decomposition. Some such path decomposition parameters are graphoidal covering number [1], simple path covering number [2], simple graphoidal covering number [3], simple acyclic graphoidal covering number [3] and 2-graphoidal path covering number [8].

In this paper we initiate a study of the parameter π_P and determine the value of π_P for some standard graphs. Further, we obtain bounds for π_P and characterize graphs attaining the bounds.

§2. Main results

We first present a general result which is useful in determining the value of π_p.

Theorem 2.1 *For any EQPPD ψ of a graph G, let $t_\psi = \sum\limits_{P \in \psi} t(P)$, where $t(P)$ denotes the number of internal vertices of P and let $t = \max t_\psi$, where the maximum is taken over all equiparity path decompositions ψ of G. Then $\pi_p(G) = q - t$.*

Proof Let ψ be any EQPPD of G. Then

$$\begin{aligned} q &= \sum_{P \in \psi} |E(P)| = \sum_{P \in \psi} (t(P) + 1) \\ &= \sum_{P \in \psi} t(P) + |\psi| = t_\psi + |\psi|. \end{aligned}$$

Hence $|\psi| = q - t_\psi$ so that $\pi_p = q - t$. □

Next we will find some bounds for π_P. First, we find a simple bound for π_P in terms of the size of G.

Theorem 2.2 *For any graph G of even size,$\pi_P(G) \leq \frac{q}{2}$.*

Proof It follows from Theorem **??** that G has a P_3-decomposition, which is an EPPD and hence $\pi_P(G) \leq \frac{q}{2}$. □

Remark 2.3 The bound given in Theorem 2.2 is sharp. For the cycle C_4 and the star $K_{1,n}$, where n is even, $\pi_P = \frac{q}{2}$.

The following problem naturally arises.

Problem 2.4 *Characterize graphs of an even size for which $\pi_P = \frac{q}{2}$.*

Now, we characterize graphs attaining the extreme bounds.

Theorem 2.5 *For a graph G, $1 \leq \pi_P(G) \leq q$. Then $\pi_P(G) = 1$ if and only if G is a path and $\pi_P(G) = q$ if and only if G is either K_3 or $K_{1,q}$ where q is odd.*

Proof The inequalities are trivial. Further , it is obvious that $\pi_P(G) = 1$ if and only if G is a path.

Now, suppose $\pi_P(G) = q > 1$. Then it follows from Theorem 2.2 that q is odd. Let P be a path of length greater than one in G. If the length of P is odd, then $\psi = \{P\} \bigcup \{E(G) \setminus E(P)\}$ is an OPPD of G so that $\pi_P(G) < q$, which is a contradiction. Thus every path of length greater than one is even and consequently every path in G is of length 1 or 2. Hence any two edges in G are adjacent, so that G is either a triangle or a star. Converse is obvious. □

The following theorem gives the lower and upper bounds for π_P in terms of π.

Theorem 2.6 *For any graph G, $\pi(G) \leq \pi_P(G) \leq 2\pi(G) - 1$.*

Proof Since every equiparity path decomposition is a path cover, we have $\pi(G) \leq \pi_P(G)$.

Let ψ be a $\pi-$cover of G and let m and n be the number of even and odd paths in ψ respectively, Then $1 \leq m, n \leq \pi - 1$ and $m + n = \pi$. Then the path decomposition ψ_1 obtained from ψ by splitting each even path in ψ into two odd paths is an OPPD and hence

$$\pi_P(G) \leq |\psi_1| = 2m + n = m + (m + n) \leq \pi - 1 + \pi = 2\pi - 1.$$

□

Corollary 2.7 *For a graph G of odd size, if $\pi(G)$ is even, then $\pi(G) + 1 \leq \pi_P(G)$.*

Proof Since $\pi(G)$ is even and q is odd, we have, $\pi(G) \neq \pi_P(G)$ and from Theorem **??** , we have $\pi(G) + 1 \leq \pi_P(G)$. □

The above bounds will be very useful to find the value of π_P for some standard graphs.

Remark 2.8 It is obvious that $\pi_P(G) = \pi(G)$ if and only if there exists a π-cover of G in which lengths of all the paths have the same parity. Further, if $\pi_P(G) = 2\pi(G) - 1$, then every π-cover of G contains only one path of odd length.

From the above bounds the following problems will naturally arise.

Problem 2.9 *Characterize the class of graphs for which* $\pi_P(G) = \pi(G)$.

Problem 2.10 *Characterize the class of graphs for which* $\pi_P(G) = 2\pi(G) - 1$.

Problem 2.11 *Characterize the class of graphs for which* $\pi_P(G) = \pi(G) + 1$.

Corollary 2.12 *For a graph G, if q is even, then* $\pi_P(G) \leq q - 1$.

Proof From Theorem 1.2, it follows that $\pi(G) \leq \frac{q}{2}$. Then from Theorem 2.6, it follows that $\pi_P(G) \leq q - 1$. □

Now, we characterize graphs attaining the above bound.

Theorem 2.13 *For any graph G, $\pi_P(G) = q - 1$ if and only if $G \cong P_3$.*

Proof Suppose $\pi_P(G) = q - 1$. If G has a path P of length 3, then the path P together with the remaining edges form an OPPD ψ of G so that $\pi_P(G) \leq |\psi| = q - 2 < q - 1$, which is a contradiction. Thus every path in G is of length at most 2. Hence any two edges in G are adjacent , so that G is either a triangle or a star. From Theorem 2.5, it follows that G is neither a triangle nor a star of odd size. Thus G is a star of even size. Then clearly, $\pi_P(G) = \frac{q}{2}$. Thus $q = 2$ and hence $G \cong P_3$. The converse is obvious. □

Next we solve the following realization problem.

Theorem 2.14 *If a is a positive integer and for every odd b with $a \leq b \leq 2a - 1$, then there exists a connected graph G such that $\pi(G) = a$ and $\pi_P(G) = b$.*

Proof Now, suppose a is a positive integer and for every odd b with $a \leq b \leq 2a - 1$.

Case (i) a is odd.

We now construct a graph $G_r, r = 0, 1, 2, \ldots, \frac{r-1}{2}$ as follows. Let G_0 be a star graph with $v_1, v_2, \ldots, v_{2a-2}, v_{2a-1}$ as pendant vertices and v_{2a} as central vertex. Let G_r be a graph obtained from G_0 by subdividing $2r$ edges $v_1v_{2a}, v_2v_{2a}, \ldots, v_{2r}v_{2a}$ of G_0 once by the vertices $v'_1, v'_2, \ldots, v'_{2r}$, where $r = 1, 2, \ldots, \frac{a-1}{2}$ (Fig.1). Note that $p = 2a + 2r$ and $q = 2a - 1 + 2r$.

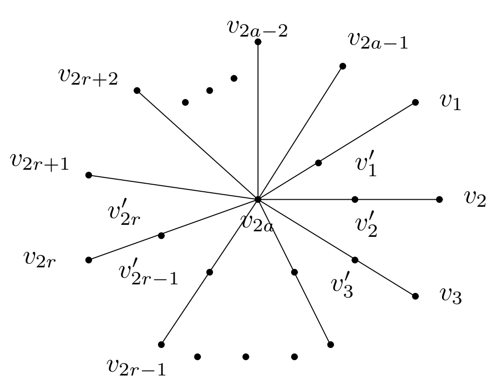


Fig.1

First we prove that $\pi(G_r) = a, (r = 0, 1, 2, \ldots, \frac{r-1}{2})$. Since every odd degree vertex of G_r is an end vertex of a path in any path cover of G_r, we have $\pi(G_r) \geq \frac{2a}{2} = a$. Now the paths $(v_i, v_i', v_{2a}, v_{2r+i})$, $1 \leq i \leq 2r$, $(v_{4r+1}, v_{2a}, v_{4r+2})$, $(v_{4r+3}, v_{2a}, v_{4r+4})$, $\cdots$, $(v_{2a-3}, v_{2a}, v_{2a-2})$, $(v_{2a-1} v_{2a})$ form a path cover for G_r so that $\pi(G_r) \leq 2r + \frac{(2a-1)-(4r+1)}{2} + 1 = a$. Hence $\pi(G_r) = a$.

Next we prove that $\pi_P(G_r) = b$, where $a \leq b \leq 2a-1$. Now the paths $P_i = (v_i, v_i', v_{2a}, v_{2r+i}), 1 \leq i \leq 2r$ and the remaining edges form an OPPD ψ of G_r such that $\pi_P(G_r) \leq |\psi| = 2r+(2a-1+2r-6r) = 2a-(2r+1)$. Now let ψ be any minimum EQPPD of G_r. Since q is odd, ψ is an OPPD. Now it is clear that any OPPD ψ of G_r contains either all the edges of G_r or paths of length 3 together with the remaining edges. Hence it follows that $|\psi| \geq 2r+(2a-1+2r-6r) = 2a-(2r+1)$ so that $\pi_P(G_r) \geq 2a-(2r+1)$. Thus $\pi_P(G_r) = 2a-(2r+1)$ where $r = 0, 1, 2, \ldots, \frac{a-1}{2}$. Let $b = 2a-(2r+1), r = 0, 1, 2, \ldots, \frac{a-1}{2}$. Then $a \leq b \leq 2a-1$. Thus $\pi_P(G_r) = b$, where $a \leq b \leq 2a-1$.

Case (ii) a is even.

Since b is odd, we have $a+1 \leq b \leq 2a-1$. Let G_0 be a star graph with $v_1, v_2, \ldots, v_{2a-1}, v_{2a}$ as pendant vertices and v_{2a+1} as central vertex with a subdivision of the edge $v_1 v_{2a+1}$ by a vertex v_1' . Let G_r be a graph obtained from G_0 by subdividing $2r$ edges $v_2 v_{2a+1}, v_3 v_{2a+1}, \ldots, v_{2r} v_{2a+1}$, $v_{2r+1} v_{2a+1}$ of G_0 once by the vertices $v_2', v_3', \ldots, v_{2r}', v_{2r+1}'$, where $r = 1, 2, \ldots, \frac{a-2}{2}$ (Fig. 2). Note that $p = 2a+2r+2$ and $q = 2a+2r+1$.

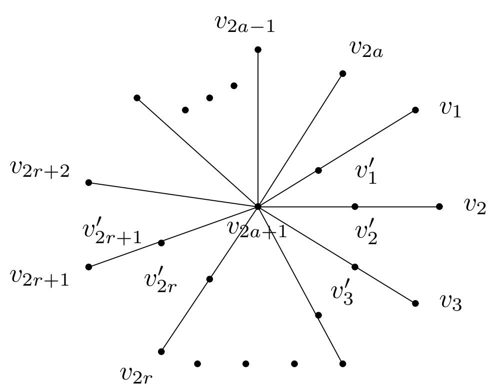


Fig.2

First we prove that $\pi(G_r) = a, (r = 0, 1, 2, \ldots, \frac{r-1}{2})$. Since every odd degree vertex of G_r is an end vertex of a path in any path cover of G_r, we have $\pi(G_r) \geq \frac{2a}{2} = a$. Now the paths $(v_i, v_i', v_{2a+1}, v_{2r+1+i})$, $1 \leq i \leq 2r+1$, $(v_{4r+3}, v_{2a+1}, v_{4r+4})$, $(v_{4r+5}, v_{2a+1}, v_{4r+5})$, $\cdots$, $(v_{2a-1}, v_{2a+1}, v_{2a})$ form a path cover for G_r so that $\pi(G_r) \leq 2r+1+\frac{(2a-1)-(4r+3)}{2}+1 = a$. Hence $\pi(G_r) = a$.

Next we prove that $\pi_P(G_r) = b$, where $a+1 \leq b \leq 2a-1$. Now the paths $P_i = (v_i, v_i', v_{2a+1}, v_{2r+1+i}), 1 \leq i \leq 2r+1$ and the remaining edges form an OPPD ψ of G_r such that $\pi_P(G_r) \leq |\psi| = 2r+1+(2a+2r+1-6r-3) = 2a-(2r+1)$. Now let ψ be any minimum EQPPD of G_r. Since q is odd, ψ is an OPPD. Now it is clear that any OPPD ψ of G_r contains either all the edges of G_r or paths of length 3 together with the remaining edges. Hence it follows that $|\psi| \geq 2r+1+(2a+2r+1-6r-3) = 2a-(2r+1)$ so that $\pi_P(G_r) \geq 2a-(2r+1)$. Thus $\pi_P(G_r) = 2a-(2r+1)$ where $r = 0, 1, 2, \ldots, \frac{a-2}{2}$. Let $b = 2a-(2r+1), r = 0, 1, 2, \ldots, \frac{a-2}{2}$. Then $a+1 \leq b \leq 2a-1$. Thus $\pi_P(G_r) = b$, where $a+1 \leq b \leq 2a-1$. $\square$

For the even number b, we make a problem as follows.

Problem 2.15 *If a is a positive integer and for every even b with $a \leq b \leq 2a-1$, then there exists a connected graph G such that $\pi(G) = a$ and $\pi_P(G) = b$.*

The following theorem gives the lower bound for π_P in terms of detour diameter D.

Theorem 2.16 *For any graph G, $\pi_P(G) \geq \lceil \frac{q}{D} \rceil$ where D is the detour diameter of G.*

Proof Let ψ be a minimum π_P-cover of G. Since every edge of G is in exactly one path in ψ we have $q = \sum\limits_{P \in \psi} |E(P)|$. Also $|E(P)| \leq D$ for each P in ψ. Hence $q \leq \pi_p D$ so that $\pi_P(G) \geq \lceil \frac{q}{D} \rceil$. $\square$

The following theorem shows that the path covering number π of a graph G is same as the equiparity path decomposition numberπ_P of a subdivision graph of G.

Theorem 2.17 *For any graph G, $\pi(G) = \pi_P(S(G))$, where $S(G)$ is the subdivision graph of G.*

Proof As G and $S(G)$ are homeomorphic, $\pi(G) = \pi(S(G))$ and hence by Theorem 2.6, $\pi(G) \leq \pi_P(S(G))$. Now let $\psi = \{P_1, P_2, \cdots, P_\pi\}$ be a π-cover of G. Let $P_i^{'}, 1 \leq i \leq \pi$, be the path obtained from P_i by subdividing each edge P_i exactly once. Then $\psi' = \{P_1', P_2' \cdots, P_\pi'\}$ is an EPPD of $S(G)$ and hence $\pi_P(S(G)) \leq \pi(G)$. Thus $\pi(G) = \pi_P(S(G))$. $\square$

In the following theorems we determine the value of the equiparity path decomposition number of several classes of graphs such as cycle, wheel, cubic graphs and complete graphs.

Theorem 2.18 *For a cycle C_p,*

$$\pi_P(C_p) = \begin{cases} 2 & if\ n\ is\ even, \\ 3 & if\ n\ is\ odd. \end{cases}$$

Proof Let $C = (v_1, v_2, \ldots, v_p, v_1)$.

If p even, then $\psi = \{(v_1, v_2, \ldots, v_{\frac{p}{2}}), (v_{\frac{p}{2}}, v_{\frac{p}{2}+1}, \ldots, v_p, v_1)\}$ is an EPPD, so that $\pi_P(C_p) \leq |\psi| = 2$ and further $\pi_P(C_p) \geq 2$ and hence $\pi_P(C_p) = 2$.

If p odd, then $\psi = \{(v_1, v_2, \ldots, v_{p-1}), (v_{p-1}, v_p), (v_p, v_1)\}$ is an OPPD, so that $\pi_P(C_p) \leq |\psi| = 3$. Since q is odd, it follows that $\pi_P(C_p)$ is odd. Then we have $\pi_P(C_p) \geq 3$. Hence $\pi_P(C_p) = 3$. $\square$

Theorem 2.19 *For the wheel W_p on p vertices, we have $\pi_P(W_p) = \lfloor \frac{p}{2} \rfloor$.*

Proof Let $V(W_p) = \{v_1, v_2, \ldots, v_{p-1}, v_p\}$ and let $E(W_p) = \{v_i v_{i+1} : 1 \leq i \leq p-2\} \bigcup \{v_1 v_{p-1}\} \bigcup \{v_p v_i : 1 \leq i \leq p-1\}$. Let

$$\psi = \begin{cases} \{(v_{i+1}, v_i, v_p, v_{\frac{p-1}{2}+i}, v_{\frac{p+1}{2}+i}) : 1 \leq i \leq \frac{p-3}{2}\} \bigcup \{(v_{\frac{p+1}{2}}, v_{\frac{p-1}{2}}, v_p, v_{p-1}, v_1)\}, \text{if p is odd}, \\ \{(v_{i+1}, v_i, v_p, v_{\frac{p-2}{2}+i}, v_{\frac{p}{2}+i}) : 1 \leq i \leq \frac{p-2}{2}\} \bigcup \{(v_p, v_{p-1}, v_1)\}, \text{ if p iseven}, \end{cases}$$

then ψ is a EPPD with $|\psi| = \lfloor \frac{p}{2} \rfloor$ and hence $\pi_P(W_p) \leq \lfloor \frac{p}{2} \rfloor$. Since every odd degree vertex of W_p is an end vertex of a path in any path cover of W_p, we have $\pi_P(W_p) \geq \lfloor \frac{p}{2} \rfloor$. Then $\pi_P(W_p) = \lfloor \frac{p}{2} \rfloor$. □

Theorem 2.20 *For a 3-regular graph G, $\pi_P(G) = \frac{p}{2}$.*

Proof It follows from Theorem 1.3 that every 3-regular graph is P_4 decomposable and hence $\pi_P(G) \leq \frac{q}{3} = \frac{p}{2}$. Further, since every vertex of G is of odd degree, they are the end vertices of paths in any path cover of G. So, we have $\pi_P(G) \geq \frac{p}{2}$. Thus $\pi_P(G) = \frac{p}{2}$. □

Theorem 2.21 *For any $n \geq 1$, $\pi_P(K_{2n}) = n$.*

Proof From Theorems 1.4 and 2.6, it follows that $\pi_P(K_{2n}) \leq n$. Further, since every vertex of K_{2n} is of odd degree, they are the end vertices of paths in any path cover of K_{2n}. So, we have $\pi_P(K_{2n}) \geq n$ and hence $\pi_P(K_{2n}) = n$. □

Theorem 2.22 *For any $n \geq 1$,*

$$\pi_P(K_{2n+1}) = \begin{cases} n+1 & if\ n\ is\ even, \\ n+2 & if\ n\ is\ odd. \end{cases}$$

Proof Let $V(K_{2n+1}) = \{v_1, v_2, \cdots, v_{2n+1}\}$.

Case (i) n is even.

Consider paths following:

$P_1 = (v_{2n+1}, v_3, v_{2n}, v_4, v_{2n-1}, \ldots, v_n, v_{n+3}, v_{n+1}, v_{n+2}, v_1, v_2)$,
$P_2 = (v_2, v_4, v_{2n+1}, v_5, v_{2n}, \cdots, v_{n+1}, v_{n+4}, v_{n+2}, v_{n+3}, v_1, v_3)$,
$P_3 = (v_3, v_5, v_2, v_6, v_{2n+1}, \cdots, v_{n+2}, v_{n+5}, v_{n+3}, v_{n+4}, v_1, v_4)$,
$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$,
$P_n = (v_n, v_{n+2}, v_{n-1}, v_{n+3}, v_{n-2}, \cdots, v_{2n-1}, v_2, v_{2n}, v_{2n+1}, v_1, v_{n+1})$,
$P_{n+1} = (v_{2n+1}, v_2, v_3, v_4, v_5, cldots, v_{n-1}, v_n, v_{n+1})$.

The paths P_i $(1 \leq i \leq n)$ can be obtained from n hamiltonian cycles of K_{2n+1} by removing an edge from each cycle and the path P_{n+1} is obtained by joining the removed edges. It follows that the lengths of $P_i, 1 \leq i \leq n$ are $2n$ and the length of P_{n+1} is n, so that $\psi = \{P_1, P_2, \ldots, P_n, P_{n+1}\}$ is an EPPD and hence $\pi_{{}_P}(K_{2n+1}) \leq |\psi| = n+1$. From Theorems 1.4 and 2.6, it follows that $\pi_p(K_{2n+1}) \geq n+1$ and hence $\pi_P(K_{2n+1}) = n+1$.

Case (ii) n is odd.

Consider the hamilton cycles of K_{2n+1}

$C_1 = (v_1, v_2, v_{2n+1}, v_3, v_{2n}, v_4, v_{2n-1}, \ldots, v_n, v_{n+3}, v_{n+1}, v_{n+2}, v_1)$,
$C_2 = (v_1, v_3, v_2, v_4, v_{2n+1}, v_5, v_{2n}, \ldots, v_{n+1}, v_{n+4}, v_{n+2}, v_{n+3}, v_1)$,
$C_3 = (v_1, v_4, v_3, v_5, v_2, v_6, v_{2n+1}, \ldots, v_{n+2}, v_{n+5}, v_{n+3}, v_{n+4}, v_1)$,
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$,
$C_{\frac{n-1}{2}} = (v_1, v_{\frac{n+1}{2}}, v_{\frac{n-1}{2}}, v_{\frac{n+3}{2}}, v_{\frac{n-3}{2}}, \ldots, v_{\frac{3n-3}{2}}, v_{\frac{3n+3}{2}}, v_{\frac{3n-1}{2}}, v_{\frac{3n+1}{2}}, v_1)$,
$C_{\frac{n+1}{2}} = (v_1, v_{\frac{n+3}{2}}, v_{\frac{n+1}{2}}, v_{\frac{n+5}{2}}, v_{\frac{n-1}{2}}, \ldots, v_{\frac{3n-1}{2}}, v_{\frac{3n+5}{2}}, v_{\frac{3n+1}{2}}, v_{\frac{3n+3}{2}}, v_1)$.
$C_{\frac{n+3}{2}} = (v_1, v_{\frac{n+5}{2}}, v_{\frac{n+3}{2}}, v_{\frac{n+7}{2}}, v_{\frac{n+1}{2}}, \ldots, v_{\frac{3n+1}{2}}, v_{\frac{3n+7}{2}}, v_{\frac{3n+3}{2}}, v_{\frac{3n+5}{2}}, v_1)$,
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$,
$C_{n-1} = (v_1, v_n, v_{n-1}, v_{n+1}, v_{n-2}, v_{n+2}, v_{n-3}, \cdots, v_{2n-2}, v_{2n+1}, v_{2n-1}, v_{2n}, v_1)$,
$C_n = (v_1, v_{n+1}, v_n, v_{n+2}, v_{n-1}, v_{n+3}, v_{n-2}, \ldots, v_{2n-1}, v_2, v_{2n}, v_{2n+1}, v_1)$.

We will construct the following paths from the above hamilton cycles. Let
$P_1 = C_1 - (v_1, v_2, v_{2n+1})$,
$P_2 = C_2 - (v_{2n+1}, v_5, v_{2n})$,
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$,
$P_{\frac{n-1}{2}} = C_{\frac{n-1}{2}} - (v_{\frac{3n+7}{2}}, v_{\frac{3n-5}{2}}, v_{\frac{3n+5}{2}})$,
$P_{\frac{n+1}{2}} = C_{\frac{n+1}{2}} - (v_{\frac{3n+5}{2}}, v_{\frac{3n+1}{2}}, v_{\frac{3n+3}{2}})$,
$P_{\frac{n+3}{2}} = C_{\frac{n+3}{2}} - (v_{\frac{3n-1}{2}}, v_{\frac{3n+7}{2}}, v_{\frac{3n+1}{2}})$,
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$,
$P_{n-1} = C_{n-1} - (v_{n+2}, v_{n-3}, v_{n+3})$,
$P_n = C_n - (v_{n+1}, v_n, v_{n+2})$,
$P_{n+1} = (v_1, v_2, v_{2n+1}, v_5, v_{2n}, \ldots, v_{\frac{3n+7}{2}}, v_{\frac{3n-5}{2}}, v_{\frac{3n+5}{2}}, v_{\frac{3n+1}{2}})$,
$P_{n+2} = (v_{\frac{3n+3}{2}}, v_{\frac{3n+1}{2}}, v_{\frac{3n+7}{2}}, v_{\frac{3n-1}{2}}, \ldots, v_{n+3}, v_{n-3}, v_{n+2}, v_n, v_{n+1})$.

The paths P_i $(1 \leq i \leq n)$ can be obtained from n hamiltonian cycles of K_{2n+1} by removing two adjacent edges from each cycle and the paths P_{n+1} and P_{n+2} are obtained by joining the removed edges. It follows that the lengths of $P_i, 1 \leq i \leq n$ are $2n-1$ and the lengths of P_{n+1} and P_{n+2} are n, so that $\psi = \{P_1, P_2, \cdots, P_n, P_{n+1}\}$ is an OPPD and hence $\pi_P(K_{2n+1}) \leq |\psi| = n+2$. From Theorems 1.4 and T2.6 , it follows that $\pi_P(K_{2n+1}) \geq n+1$. Now , since n is odd, $q = n(2n+1)$ is odd. Thus $\pi_P(K_{2n+1})$ is odd, so that $\pi_P(K_{2n+1}) \geq n+2$ and hence $\pi_P(K_{2n+1}) = n+2$. □

We now proceed to obtain upper bounds for π_p involving circumference of a graph and characterize graphs attaining the bounds.

Theorem 2.23 *For a graph G, $\pi_P(G) \leq q-c+3$, where c is the circumference of G. Further, equality holds if and only if G is an odd cycle.*

Proof Let C be a longest cycle of length c. Let c be even. Then the path of length $c-1$, together with the remaining edges form an OPPD and hence $\pi_P(G) \leq q-(c-1)+1 = q-c+2$. Let c be odd. Then path of length $p-2$, together with the remaining edges form an OPPD and hence $\pi_P(G) \leq q-(c-2)+1 = q-c+3$. Thus from both the cases, it follows that $\pi_P(G) \leq q-c+3$.

Suppose G is a graph with $\pi_P(G) = q-c+3$. Let $C = (v_1, v_2, \ldots, v_c, v_1)$ be a longest cycle in G. If c is even, then as in the first paragraph of the proof, $\pi_P(G) \leq q-c+2$ and so c is odd.

Now, we claim that C has no chords. Suppose it is not. Let $e = v_1v_i$ be a chord in C. Let $P_1 = (v_1, v_2, \ldots, v_{c-1})$ and $P_2 = (v_{c-1}, v_c, v_1, v_i)$. Since c is odd, $\psi = \{P_1, P_2\} \bigcup S$ where S is the set of edges of G not covered by P_1, P_2 is an OPPD of G such that $|\psi| < q - c + 3$, which is a contradiction. Thus C has no chords.

Next, we claim that $V(G) = V(C)$. Suppose there exists a vertex v not on C adjacent to a vertex of C, say v_1. Let $P_1 = (v_1, v_2, \ldots, v_{c-1})$ and $P_2 = (v_{c-1}, v_c, v_1, v)$. Since c is odd, $\psi = \{P_1, P_2\} \bigcup S$ where S is the set of edges of G not covered by P_1, P_2 is an OPPD of G such that $|\psi| < q - c + 3$, which is a contradiction. Then it follows that $V(G) = V(C)$. Thus G is an odd cycle.

The converse is obvious. □

Theorem 2.24 *For a graph G, $\pi_P(G) = q - c + 2$ if and only if G is either an even cycle or a $\theta-$graph of odd size or a kite with tail length 1 of odd size.*

Proof Clearly, the result is true for $p = 3, 4$ and 5. So we assume that $p \geq 6$.

Suppose $\pi_P(G) = q - c + 2$. Let $C = (v_1, v_2, \ldots, v_c, v_1)$ be a longest cycle in G.

Claim 1 c is even.

Suppose c is odd. Since the value of π_P for an odd cycle is $q - c + 3$, it follows that $G \neq C$. Hence C has a chord, say $e = v_1v_i$ (Fig.3).

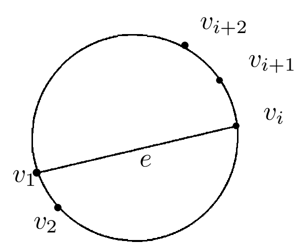


Fig. 3

Let $P_1 = (v_1, v_i, v_{i+1}, v_{i+2})$ and $P_2 = (v_{i+2}, v_{i+3}, \cdots, v_c, v_1, v_2, \ldots, v_i)$. Then $\psi = \{P_1, P_2\} \bigcup S$ where S is the set of edges of G not covered by P_1, P_2 is an OPPD of G such that $|\psi| < q - c + 2$, which is a contradiction.

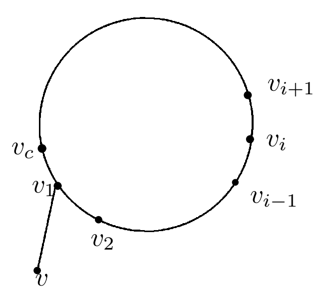


Fig. 4

Hence there is a vertex v not on C adjacent to a vertex of C, say v_1 (Fig.4). Let $P_1 = (v, v_1, v_2, v_3)$ and $P_2 = (v_3, v_4, \ldots, v_{c-1}, v_c, v_1)$. Since c is odd, $\psi = \{P_1, P_2\} \bigcup S$ where S is the set of edges of G not covered by P_1, P_2 is an OPPD of G such that $|\psi| < q - c + 2$, which is a contradiction. Thus c is even.

Case (i) $V(G) = V(C)$

We now prove that C has at most one chord.

Claim 2 No two chords of C are adjacent.

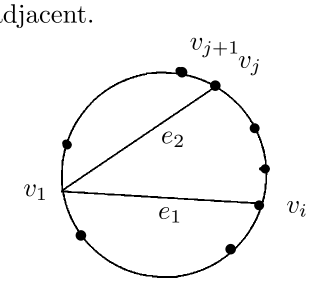


Fig.5

Suppose there exists two adjacent chords $e_1 = v_1v_i$ and $e_2 = v_1v_j$ $(1 < i < j)$ in C (Fig 5). Let P_1 be the (v_j, v_{j+1})-section of C containing v_1 and let $P_2 = (v_{j+1}, v_j, v_1, v_i)$. From Claim 1, it follows that $\psi = \{P_1, P_2\} \bigcup S$ where S is the set of edges of G not covered by P_1, P_2 is an OPPD of G such that $|\psi| < q - c + 2$, which is a contradiction. Thus no two chords of C are adjacent.

Next we define some sections of cycle.

A section C' of length greater than 1 of a cycle C is said to be of type 1 if the end vertices of C' are adjacent and no internal vertex of C' is an end vertex of a chord of C.

A section C' of a cycle C is said to be of type 2 if the end vertices of C' are the end vertices of two different chords of C and no internal vertex of C' is an end vertex of a chord of C.

Claim 3 The type 2 sections of C formed by any two nonadjacent chords are of even length.

Let e_1 and e_2 be two nonadjacent chords of C. Then the choices of e_1 and e_2 are as in the following figure (Fig.6).

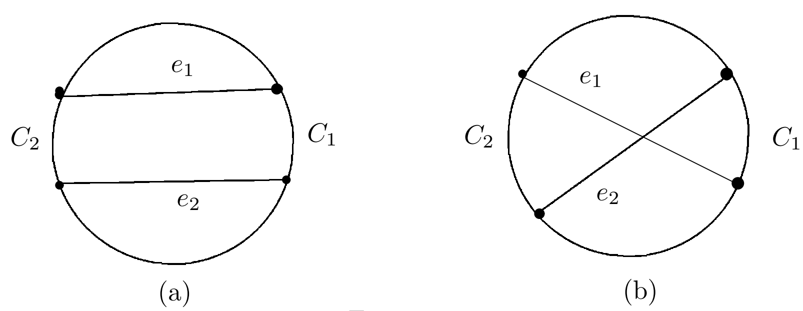


Fig. 6

Let C_1 and C_2 be the sections of C. We now claim that the section C_1 is of even length. Suppose not. Now, let $P_1 = e_1 \circ C_1 \circ e_2$ and $P_2 = C - C_1$. Then it follows from Claim 1 that

$\psi = \{P_1, P_2\} \bigcup S$ where S is the set of edges of G not covered by P_1, P_2 is an OPPD of G such that $|\psi| < q - c + 2$, which is a contradiction. Hence the section C_1 is of even length. Similarly, we can prove that the section C_2 is of even length.

Claim 4 Type 1 sections of C formed by three mutually disjoint chords are of even length.

Let e_1,e_2 and e_3 be three mutually disjoint chords of C. Let C_1 be a type 1 section of C formed by e_1. We now claim that C_1 is of even length. Suppose not. Then by claim 1, $C - C_1$ is of odd length. Now, since there are exactly six sections of either type 1 or type 2 formed by e_1, e_2 and e_3 in C and since $C - C_1$ is of odd length, it follows from claim 3 that there is a type 1 section C_2 of odd length formed either by e_2 or e_3, say e_2. then the chord e_3 is as in the following Fig. 7.

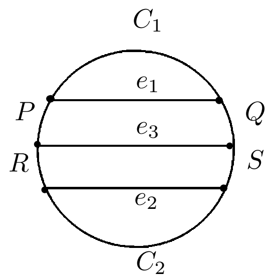


Fig.7

Let P, Q, R and S denote the remaining type 2 sections of C as in Fig. 7. Then it follows from claim 3 that the sections P, Q, R and S are of even length. Now, let $P_1 = e_2 \circ S \circ e_3 \circ P \circ e_1$, $P_2 = C_1 \circ Q$, $P_3 = C_2 \circ R$. Then $\psi = \{P_1, P_2, P_3\} \bigcup S$ where S is the set of edges of G not covered by P_1, P_2 and P_3 is an OPPD of G such that $|\psi| < q - c + 2$, which is a contradiction. Thus the type 1 sections of C formed by three mutually disjoint chords are of even length.

Claim 5 C has at most one chord.

Suppose C has exactly two chords, say e_1 and e_2. Then by Claim 2 the choices of e_1 and e_2 are as in Fig. 6. Also, there are exactly 4 sections of type 1 or type 2 in C, say C_1, C_2, C_3 and C_4. Suppose e_1 and e_2 are as in Fig. 6(b). Then the sections C_1, C_2, C_3 and C_4 are of type 2 and hence it follows from Claim 3 that each is of even length.

Now, let $P_1 = e_1 \circ C_2 \circ e_2$ and $P_2 = C_3 \circ C_1 \circ C_4$. Then $\psi = \{P_1, P_2\}$ is an EPPD of G and hence $\pi_P(G) = 2 < q - c + 2$, which is a contradiction.

Suppose e_1 and e_2 are as in Fig. 6(a). Then the sections C_1 and C_3 are of type 1 and the sections C_2 and C_4 are of type 2 and hence it follows from Claims 3 4 that each is of even length.

Now, let $P_1 = e_1 \circ C_2 \circ e_2$ and $P_2 = C_3 \circ C_1 \circ C_4$. Then $\psi = \{P_1, P_2\}$ is an EPPD of G and hence $\pi_P(G) = 2 < q - c + 2$, which is a contradiction.

Thus C does not have exactly two chords.

Suppose C has 3 chords, say e_1, e_2 and e_3. Then by Claim 2 the choices of e_1, e_2 and e_3 are as in Fig. 8.

Also, there are exactly 6 sections of types 1 or 2 in C, say C_1, C_2, C_3, C_4, C_5 and C_6. By Claim 3 and 4, any section of C formed by the chords is of even length and so C_1, C_2, C_3, C_4, C_5

and C_6 are of even length.

If e_1, e_2 and e_3 are as in Fig 8(a), let $P_1 = C_1 \circ C_6 \circ e_2$, $P_2 = C_5 \circ e_1$ and $P_3 = e_3 \circ C_2 \circ C_3 \circ C_4$. If e_1, e_2 and e_3 are as in Fig. 8(b), let $P_1 = C_1 \circ C_6 \circ e_2$, $P_2 = C_4 \circ C_3 \circ C_2 \circ e_3$ and $P_3 = C_5 \circ e_1$.

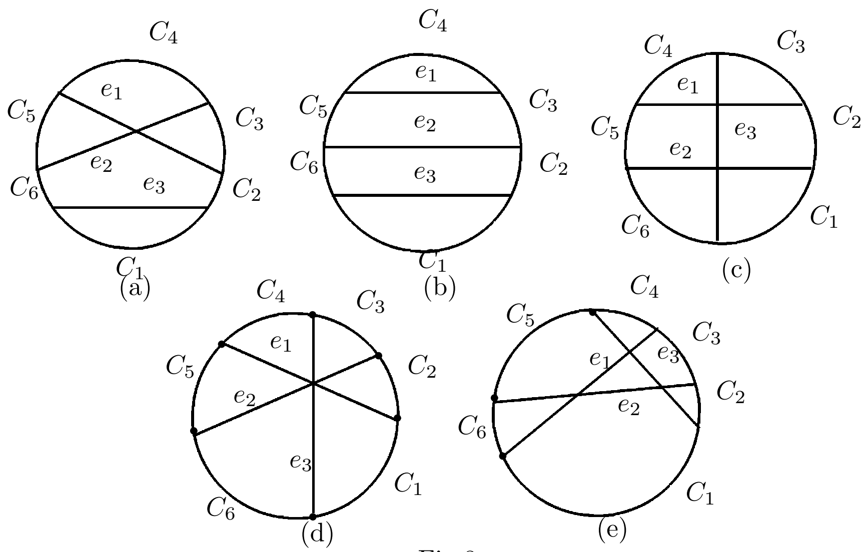


Fig.8

If e_1, e_2 and e_3 are as in Fig. 8(c), let $P_1 = C_1 \circ e_3 \circ C_4$, $P_2 = C_3 \circ C_2 \circ e_2$ and $P_3 = C_6 \circ C_5 \circ e_1$. If e_1, e_2 and e_3 are as in Fig.8(d), let $P_1 = C_6 \circ C_1 \circ e_1$, $P_2 = C_2 \circ C_3 \circ e_3$ and $P_3 = C_4 \circ C_5 \circ e_2$. If e_1, e_2 and e_3 are as in Fig.8(e), let $P_1 = e_1 \circ C_1 \circ C_2$, $P_2 = C_4 \circ C_3 \circ e_2$ and $P_3 = C_6 \circ C_5 \circ e_3$. Then $\psi = \{P_1, P_2, P_3\} \bigcup S$ where S is the set of edges of G not covered by P_1, P_2 and P_3 is an OPPD of G such that $|\psi| < q - c + 2$, which is a contradiction. Thus by Claims 1 and 5, G is either an even cycle or a $\theta-$graph of odd size.

Case (ii) $V(G) \neq V(C)$

Let $C = (v_1, v_2, \ldots, v_c, v_1)$ be a longest cycle of length c in G.

Claim 6 Every vertex not on C is a pendant vertex.

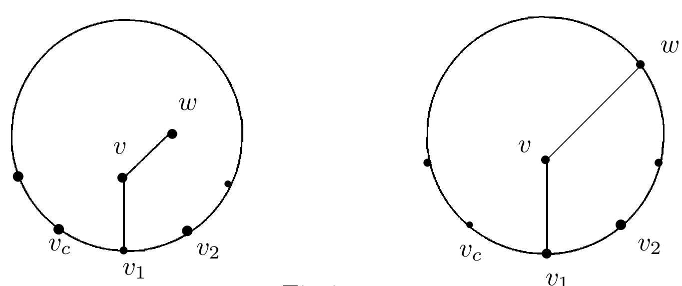


Fig.9

Suppose there exists a vertex v with $degv \geq 2$, not on C adjacent to a vertex of C, say v_1. Let w be a vertex which is adjacent to v. Note that w may be either on C or not on C (Fig 9) . Let $P_1 = (v_1, v_2, \ldots, v_c)$ and $P_2 = (v_c, v_1, v, w)$. Then $\psi = \{P_1, P_2\} \bigcup S$, where S is the set of edges of G not covered by P_1, P_2 is an OPPD of G such that $|\psi| < q - c + 2$ which is a contradiction. Thus every vertex not on C is a pendant vertex.

Claim 7 The cycle C has no chord.

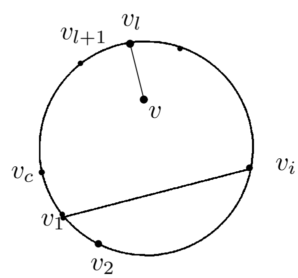


Fig.10

Suppose C has a chord, say v_1v_i (Fig 10). Let v be a pendant vertex not on C, which is adjacent to some vertex, say v_l on C. Suppose v_l is different from v_1 and v_i. If (v_1, v_l)- section is odd, then let $P_1 = (v, v_l, v_{l+1}, \cdots, v_c, v_1, v_i)$ and $P_2 = (v_1, v_2, \cdots, v_i, v_{i+1}, \cdots, v_l)$ and if that section is even, then let $P_1 = (v_l, v_{l+1}, \ldots, v_c, v_1, v_i)$ and $P_2 = (v_1, v_2, \ldots, v_i, v_{i+1}, \ldots, v_l, v)$.

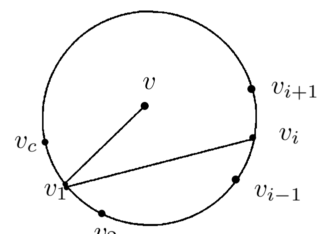


Fig.11

Suppose v_l is either v_1 or v_i. Without loss of generality, let $v_l = v_1$(Fig 11). Let $P_1 = (v, v_1, v_i, v_{i+1})$ and $P_2 = (v_{i+1}, v_{i+2}, \ldots, v_c, v_1, v_2, \ldots, v_{i-1}, v_i)$. Then $\psi = \{P_1, P_2\} \bigcup S$, where S is the set of edges of G not covered by P_1, P_2 is an OPPD of G such that $|\psi| < q - c + 2$ which is a contradiction. Hence the cycle C has no chord. Thus G is a unicyclic graph.

Claim 8 Every vertex on C has degree less than or equal to 3.

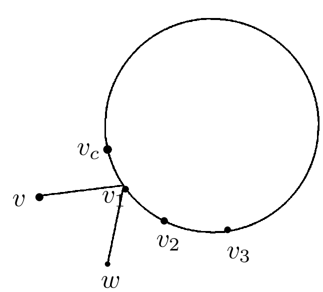


Fig.12

Suppose there is a vertex, say v_1 on C with degree of $v \geq 4$ (Fig 12). From Claims 6 and 7, it follows that there are two pendant vertices, say v and w not on C which are adjacent to v_1. Let $P_1 = (w, v_1, v_2, v_3)$ and $P_2 = (v_3, v_4, \ldots, v_c, v_1, v)$. Since c is even, $\psi = \{P_1, P_2\} \bigcup S$, where S is the set of edges of G not covered by P_1, P_2 is an OPPD of G such that $|\psi| < q - c + 2$ which is a contradiction. Thus every vertex on C has degree less than or equal to 3.

Claim 9 Exactly one vertex on C has degree 3.

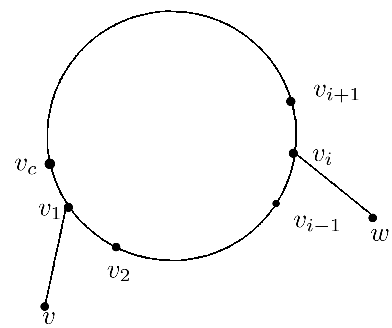


Fig.13

Suppose there are two vertices on C have degree 3, say v_1 and v_i (Fig. 13). By claim 6, there are two pendant vertices v and w not on C, adjacent to v_1 and v_i respectively. If the length of (v_1, v_i)- section not containing v_c is odd, then let $P_1 = (v, v_1, v_2, \ldots, v_{i-1}, v_i, w)$ and $P_2 = (v_i, v_{i+1}, \ldots, v_c, v_1)$ and if that section is even, then let $P_1 = (v, v_1, v_2, \ldots, v_{i-1}, v_i)$ and $P_2 = (w, v_i, v_{i+1}, \ldots, v_c, v_1)$. Since c is even, $\psi = \{P_1, P_2\} \bigcup S$, where S is the set of edges of G not covered by P_1, P_2 is an OPPD of G such that $|\psi| < q - c + 2$ which is a contradiction. Thus exactly one vertex on C has degree 3. Thus G is a kite with tail length 1 of odd size.

The converse is obvious. □

Remark 2.25 In the Theorem 2.24, for the case $V(G) = V(C)$, we have $c = p$ and the condition becomes $\pi_P(G) = q - p + 2$.

We conclude this paper by posing the following problems for further investigation.

(*i*) *For a tree T of even size, prove that $\pi(T) = \pi_P(T)$.*

(*ii*) *If G is a unicyclic graph, find $\pi_P(G)$.*

(*iii*) *For a graph G of even size, prove that $\pi(G) \leq \pi_P(G) \leq \pi(G) + 1$.*

References

[1] B. D. Acharya and E. Sampathkumar, Graphoidal covers and graphoidal covering number of a graph, *Indian J. pure appl. Math.,* **18** (10)(1987), 882 - 890.

[2] S. Arumugam and I. Sahul Hamid, Simple path covers in graphs, *International Journal of Mathematical Combinatorics,* **3** (2008), 94 - 104.

[3] S. Arumugam and I. Sahul Hamid, Simple Acyclic graphoidal covers in a graph, *Australasian Journal of Combinatorics,* **37** (2007), 243 - 255.

[4] S. Arumugam and J. Suresh Suseela, Acyclic graphoidal covers and path partitions in a graph, *Discrete Math.,* **190**(1998), 67 - 77.

[5] G. Chartrand and L. Lesniak, *Graphs and Digraphs,* Fourth Edition, CRC Press, Boca Raton, 2004.

[6] F. Harary,*Graph Theory,* Addison-Wesley, Reading, Mass, 1972.

[7] F. Harary and A. J. Schwenk, Evolution of the path number of a graph, covering and packing in graphs II, *Graph Theory and Computing,* Ed. R. C. Road, Academic Press, New York, (1972), 39 - 45.

[8] K. Nagarajan, A. Nagarajan and S. Somasundram, 2-graphoidal Path Covers *International Journal of Applied Mathematics,* **21**(4) (2008), 615 - 628.

[9] B. Peroche, The path number of some multipartite graphs, *Annals of Discrete Math.,* **9**(1982), 193 - 197.

[10] R. G. Stanton, D. D. Cowan and L. O. James, Some results on path numbers, *Proc. Louisiana Conf. on Combinatorics, Graph Theory and computing,* (1970), 112 - 135.

International J.Math. Combin. Vol.1 (2009), 77-89

Edge Detour Graphs with Edge Detour Number 2

A.P.Santhakumaran and S.Athisayanathan

(Research Department of Mathematics, St.Xavier's College(Autonomous), Palayamkottai-627002, India)

Email: apskumar1953@yahoo.co.in, athisayanathan@yahoo.co.in

Abstract: For two vertices u and v in a graph $G = (V, E)$, the *detour distance* $D(u, v)$ is the length of a longest u–v path in G. A u–v path of length $D(u, v)$ is called a *u–v detour*. For any integer $k \geq 1$, a set $S \subseteq V$ is called a *Smarandache k-edge detour set* if every edge in G lies on at least k detours joining some pairs of vertices of S. The *Smarandache k-edge detour number* $dn_k(G)$ of G is the minimum order of its Smarandache k-edge detour sets and any Smarandache k-edge detour set of order $dn_k(G)$ is a *Smarandache k-edge detour basis* of G. A connected graph G is called a *Smarandache k-edge detour graph* if it has a Smarandache k-edge detour set for an integer k. Smarandache 1-edge detour graphs are refered to as edge detour graphs and in this paper, such graphs G with detour diameter $D \leq 4$ and $dn_1(G) = 2$ are characterized.

Key Words : Detour, Smarandache k-edge detour set, Smarandache k-edge detour number, edge detour set, edge detour graph, edge detour number.

AMS(2000): 05C12

§1. Introduction

By a *graph* $G = (V, E)$, we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic definitions and terminologies, we refer to [1], [6].

For vertices u and v in a connected graph G, the *detour distance* $D(u, v)$ is the length of a longest u–v path in G. A u–v path of length $D(u, v)$ is called a *u–v detour*. It is known that the detour distance is a metric on the vertex set V. Detour distance and detour center of a graph were studied by Chartrand et al. in [2], [5].

A vertex x is said to lie on a u–v detour P if x is a vertex of P including the vertices u and v. A set $S \subseteq V$ is called a *Smarandache k-detour set* if every vertex v in G lies on at least k detours joining some pairs of vertices of S. The *Smarandache k-detour number $dnk(G)$* of G is the minimum order of a Smarandache k-detour set and any Smarandache k-detour set of order $dnk(G)$ is called a *Smarandache k-detour basis* of G. Smarandache 1-detour sets and Smarandache 1-detour number are nothing but the detour sets and the detour number $dn(G)$ of a graph as introduced and studied by Chartrand et al. in [3]. These concepts have interesting applications in Channel Assignment Problem in radio technologies [4], [7].

[1]Received January 12, 2009. Accepted February 16, 2009.

An edge e of G is said to lie on a u–v detour P if e is an edge of P. In general, there are graphs G for which there exist edges which do not lie on a detour joining any pair of vertices of V. For the graph G given in Figure 1.1, the edge v_1v_2 does not lie on a detour joining any pair of vertices of V. This motivated us to introduce the concepts of *weak edge detour set* of a graph [8] and *edge detour graphs* [9].

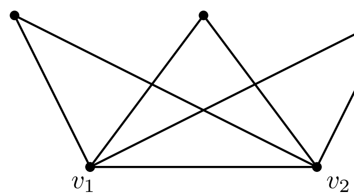
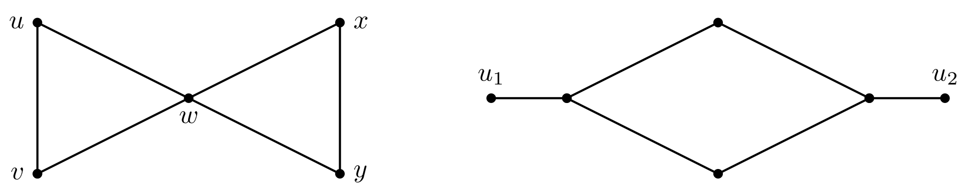


Figure 1.1: G

A set $S \subseteq V$ is called a *weak edge detour set* of G if every edge in G has both its ends in S or it lies on a detour joining a pair of vertices of S. The *weak edge detour number* $dn_w(G)$ of G is the minimum order of its weak edge detour sets and any weak edge detour set of order $dn_w(G)$ is called a *weak edge detour basis* of G. Weak edge detour sets and weak edge detour number of a graph were introduced and studied by Santhakumaran and Athisayanathan in [8].

A set $S \subseteq V$ is called an *edge detour set* of G if every edge in G lies on a detour joining a pair of vertices of S. The *edge detour number* $dn_1(G)$ of G is the minimum order of its edge detour sets and any edge detour set of order $dn_1(G)$ is an *edge detour basis* of G. A graph G is called an *edge detour graph* if it has an edge detour set. Edge detour graphs were introduced and studied in detail by Santhakumaran and Athisayanathan in [9], [10].

For the graph G given in Figure 1.2(a), the sets $S_1 = \{u, x\}$, $S_2 = \{u, w, x\}$ and $S_3 = \{u, v, x, y\}$ are detour basis, weak edge detour basis and edge detour basis of G respectively and hence $dn(G) = 2$, $dn_w(G) = 3$ and $dn_1(G) = 4$. For the graph G given in Figure 1.2(b), the set $S = \{u_1, u_2\}$ is a detour basis, weak edge detour basis and an edge detour basis so that $dn(G) = dn_w(G) = dn_1(G) = 2$. The graphs G given in Figure 1.2 are edge detour graphs. For the graph G given in Figure 1.1, the set $S = \{v_1, v_2\}$ is a detour basis and also a weak edge detour basis. However, it does not contain an edge detour set and so the graph G in Figure 1.1 is not an edge detour graph. Also, for the graph G given in Figure 1.3, it is clear that no



(a) (b)

Figure 1.2: G

two element subset of V is an edge detour set of G. It is easily seen that $S_1 = \{v_1, v_2, v_4\}$ is an edge detour set of G so that S_1 is an edge detour basis of G and hence $dn_1(G) = 3$. Thus G is an edge detour graph. Also $S_2 = \{v_1, v_2, v_5\}$ is another edge detour basis of G and thus there can be more than one edge detour basis for a graph G.

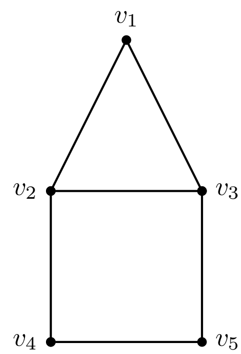


Figure 1.3: G

The following theorems are used in the sequel.

Theorem 1.1([9]) *Every end-vertex of an edge detour graph G belongs to every edge detour set of G. Also if the set S of all end-vertices of G is an edge detour set, then S is the unique edge detour basis for G.*

Theorem 1.2([9]) *If T is a tree with k end-vertices, then $dn_1(T) = k$.*

Theorem 1.3([9]) *If G is the complete graph K_2 or $K_p - e$ $(p \geqslant 3)$ or an even cycle C_n or a non-trivial path P_n or a complete bipartite graph $K_{m,n}$ $(m, n \geqslant 2)$, then G is an edge detour graph and $dn_1(G) = 2$.*

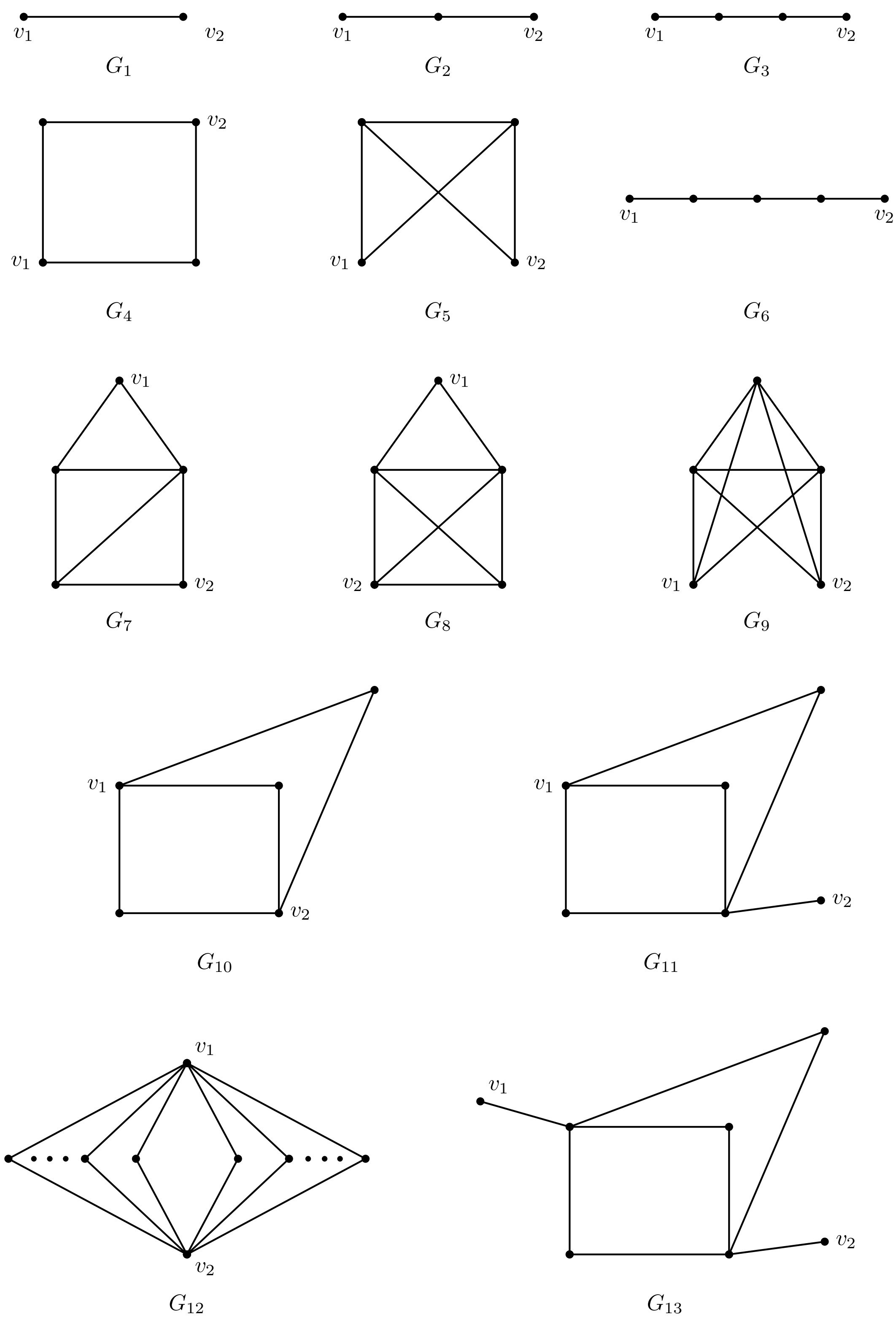
Theorem 1.4([9]) *If G is the complete graph K_p $(p \geqslant 3)$ or an odd cycle C_n, then G is an edge detour graph and $dn_1(G) = 3$.*

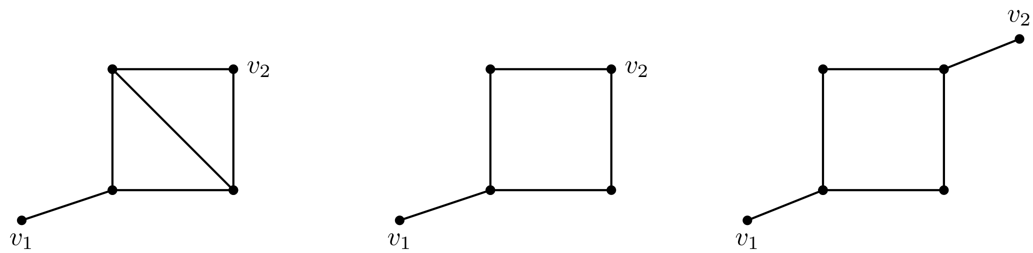
Theorem 1.1([9]) *Let $G = (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_r} \cup kK_1) + v$ be a block graph of order $p \geqslant 5$ such that $r \geqslant 2$, each $n_i \geqslant 2$ and $n_1 + n_2 + \cdots + n_r + k = p - 1$. Then G is an edge detour graph and $dn_1(G) = 2r + k$.*

Throughout this paper G denotes a connected graph with at least two vertices.

§2. Edge detour graphs G with $\boldsymbol{diam}_D G \leq 4$ and $dn_1(G) = 2$

An edge detour set of an edge detour graph G needs at least two vertices so that $dn_1(G) \geq 2$ and the set of all vertices of G is an edge detour set of G so that $dn_1(G) \leq p$. Thus $2 \leq dn_1(G) \leq p$. The bounds in this inequality are sharp. For the complete graph $K_p(p = 2, 3), dn_1(K_p) = p$. The set of two end vertices of a path $P_n(n \geq 2)$ is its unique edge detour set so that $dn_1(P_n) = 2$. Thus the complete graphs $K_p(p = 2, 3)$ have the largest possible edge detour number p and the non-trivial paths have the smallest edge detour number 2. In the following, we characterize graphs G with detour diameter $D \leq 4$ for which $dn_1(G) = 2$. For this purpose we introduce the collection $\mathscr{H}$ of graphs given in Figure 2.1.





G_{14} G_{15} G_{16}

Figure 2.1: Graphs in family $\mathscr{H}$

Theorem 2.1 *Let G be an edge detour graph of order $p \geq 2$ with detour diameter $D \leq 4$. Then $dn_1(G) = 2$ if and only if $G \in \mathscr{H}$ given in Figure 2.1.*

Proof It is straightforward to verify that the set $\{v_1, v_2\}$ as marked in the graphs G_i $(1 \leq i \leq 16)$ given in $\mathscr{H}$ of Figure , is an edge detour set for each G_i. Hence $dn_1(G_i) = 2$ for all the graphs $G_i \in \mathscr{H}$ $(1 \leqslant i \leqslant 16)$ given in Figure 2.1.

For the converse, let G be an edge detour graph of order $p \geqslant 2$, $D \leqslant 4$ and $dn_1(G) = 2$.

If $D = 1$, then it is clear that $G_1 \in \mathscr{H}$ given in Figure 2.1 is the only graph for which $dn_1(G) = 2$.

Suppose $D = 2$. If G is a tree, then it follows from Theorem 1.2 that $G_2 \in \mathscr{H}$ given in Figure 2.1 is the only graph with $dn_1(G) = 2$. If G is not a tree, let $c(G)$ denote the length of a longest cycle in G. Since G is connected and $D = 2$, it is clear that $c(G) = 3$ and G has exactly three vertices so that $G = K_3$ and by Theorem 1.4, $dn_1(G) = 3$. Thus, when $D = 2$, $G_2 \in \mathscr{H}$ given in Figure 2.1 is the only graph that satisfies the requirements of the theorem.

Suppose $D = 3$. If G is a tree, then it follows from Theorem 1.2 that the path $G_3 \in \mathscr{H}$ given in Figure 2.1 is the only graph with $dn_1(G) = 2$. Assume that G is not a tree. Let $c(G)$ denote the length of a longest cycle in G. Since G is connected and $D = 3$, it follows that $p \geq 4$ and $c(G) \leq 4$. We consider two cases.

Case 1 Let $c(G) = 4$. Then, since G is connected and $D = 3$, it is clear that G has exactly four vertices. Hence G_4, $G_5 \in \mathscr{H}$ given in Figure 2.1 and K_4 are the only graphs with these properties. But by Theorem 1.3, $dn_1(G_4) = dn_1(G_5) = 2$ and by Theorem 1.4, $dn_1(K_4) = 3$. Thus in this case G_4, $G_5 \in \mathscr{H}$ given in Figure 2.1 are the only graphs that satisfy the requirements of the theorem.

Case 2: Let $c(G) = 3$. If G contains two or more triangles, then $c(G) = 4$ or $D \geqslant 4$, which is a contradiction. Hence G contains a unique triangle C_3: v_1, v_2, v_3, v_1. Now, if there are two or more vertices of C_3 having degree 3 or more, then $D \geqslant 4$, which is contradiction. Thus exactly one vertex in C_3 has degree 3 or more. Since $D = 3$, it follows that $G = K_{1,p-1} + e$ and so by Theorem 1.5 $dn_1(K_{1,p-1} + e) = p - 1 \geq 3$, which is a contradiction. Thus, in this case, there are no graphs that satisfying the requirements of the theorem.

Suppose $D = 4$. If G is a tree, then it follows from Theorem 1.2 that $G_6 \in \mathscr{H}$ given in Figure 2.1 is the only graph with $dn_1(G) = 2$. Assume that G is not a tree. Let $c(G)$ denote the length of a longest cycle in G. Since $D = 4$, it follows that $p \geq 5$ and $c(G) \leq 5$. We consider three cases.

Case 1 Let $c(G) = 5$. Then, since $D = 4$, it is clear that G has exactly five vertices. Now, it is easily verified that the graphs G_7, G_8, and $G_9 \in \mathscr{H}$ given in Figure 2.1 are the only graphs with $dn_1(G_i) = 2$ ($i = 7, 8, 9$) among all graphs on five vertices having a largest cycle of length 5.

Case 2 Let $c(G) = 4$. Suppose that G contains K_4 as an induced subgraph. Since $p \geqslant 5$, $D = 4$ and $c(G) = 4$, every vertex not on K_4 is pendant and adjacent to exactly one vertex of K_4. Thus the graph reduces to the graph G given in Figure 2.1.

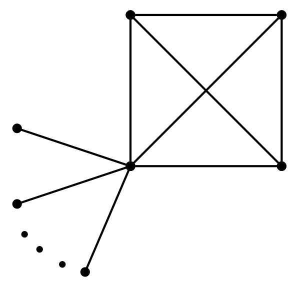
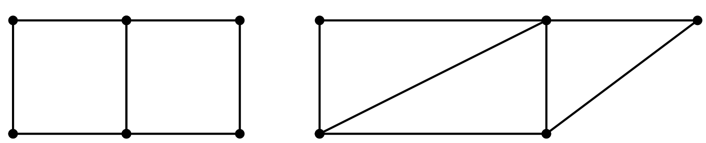


Figure 2.2: G

For this graph G, it follows from Theorem 1.5 that $dn_1(G) = p-1 \geq 4$, which is a contradiction.

Now, suppose that G does not contain K_4 as an induced subgraph. We claim that G contains exactly one 4-cycle C_4. Suppose that G contains two or more 4-cycles. If two 4-cycles in G have no edges in common, then it is clear that $D \geqslant 5$, which is a contradiction. If two 4-cycles in G have exactly one edge in common, then G must contain the graphs given in Figure 2.3 as subgraphs or induced subgraphs. In any case, $D \geqslant 5$ or $c(G) \geqslant 5$, which is a contradiction.



G_1 G_2

Figure 2.3: G

If two 4-cycles in G have exactly two edges in common, then G must contain only the graphs given in Figure 2.4 as subgraphs. It is easily verified that all other subgraphs having two edges in common will have cycles of length $\geqslant 5$ so that $D \geq 5$, which is a contradiction.

Now, if $G = H_1$, then $dn_1(G) = 2$ and it is nothing but the graph $G_{10} \in \mathscr{H}$ given in Figure 2.1. Assume first that G contains H_1 as a proper subgraph. Then there is a vertex x such that $x \notin V(H_1)$ and x is adjacent to at least one vertex of H_1. If x is adjacent to v_1, we get a path x, v_1, v_2, v_3, v_4, v_5 of length 5 so that $D \geqslant 5$, which is a contradiction. Hence x cannot be adjacent to v_1. Similarly x cannot be adjacent to v_3 and v_5. Thus x is adjacent to v_2 or v_4 or both. If x is adjacent only to v_2, then x must be a pendant vertex of G, for otherwise, we get a path of length 5 so that $D \geqslant 5$, which is a contradiction. Thus in this case, the graph G

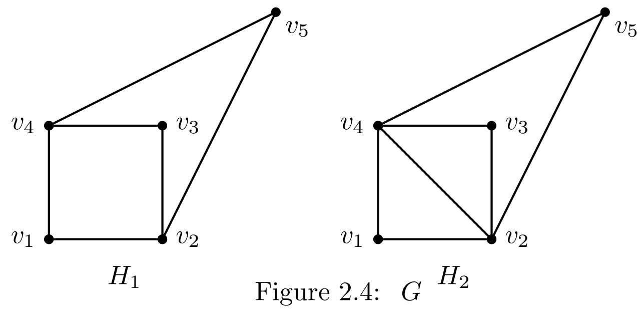


Figure 2.4: G

reduces to the one given in Figure 2.5.

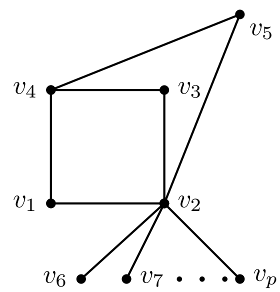


Figure 2.5: G

However, for this graph G, it follows from Theorem 1.1 that the set $\{v_4, v_6, v_7, \ldots, v_p\}$ is an edge detour basis so that $dn_1(G) = p - 4$. Hence $p = 6$ is the only possibility and the graph reduces to $G_{11} \in \mathscr{H}$ given in Figure 2.1 and satisfies the requirements of the theorem. If x is adjacent only to v_4, then we get a graph G isomorphic to the one given in Figure 2.5 and hence we get a graph G isomorphic to $G_{11} \in \mathscr{H}$ given in Figure 2.1 and satisfies the requirements of the theorem. If x is adjacent to both v_2 and v_4, then the graph reduces to the one given in Figure 2.6. This graph G is isomorphic to $G_{12} \in \mathscr{H}$ given in Figure and $\{v_1, v_2\}$ is an edge

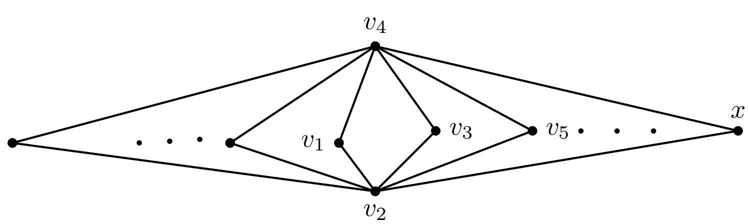


Figure 2.6: G

detour basis for G_{12} so that $dn_1(G) = 2$.

Next, if a vertex x not on H_1 is adjacent only to v_2 and a vertex y not on H_1 is adjacent only to v_4, then x and y must be pendant vertices of G, for otherwise, we get either a path or a cycle of length $\geqslant 5$ so that $D \geqslant 5$, which is a contradiction. Thus in this case, the graph reduces to the one given in Figure 2.7.

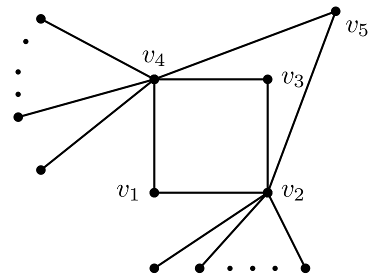


Figure 2.7: G

For this graph G, the set of all end-vertices is an edge detour basis so that by Theorem 1.1, $dn_1(G) = p - 5$. Hence $p = 7$ is the only possibility and the graph reduces to $G_{13} \in \mathscr{H}$ given in Figure 2.1 and satisfies the requirements of the theorem. Thus, in this case, we have G_{10}, G_{11}, G_{12}, $G_{13} \in \mathscr{H}$ given in Figure 2.1 are the only graphs with H_1 as proper subgraph for which $dn_1(G) = 2$.

Next, if $G = H_2$ given in Figure 2.4, then the edge v_2v_4 does not lie on any detour joining a pair of vertices of G so that G is not an edge detour graph. If G contains H_2 as a proper subgraph, then as in the case of H_1, it is easily seen that the graph reduces to any one of the graphs given in Figure 2.8.

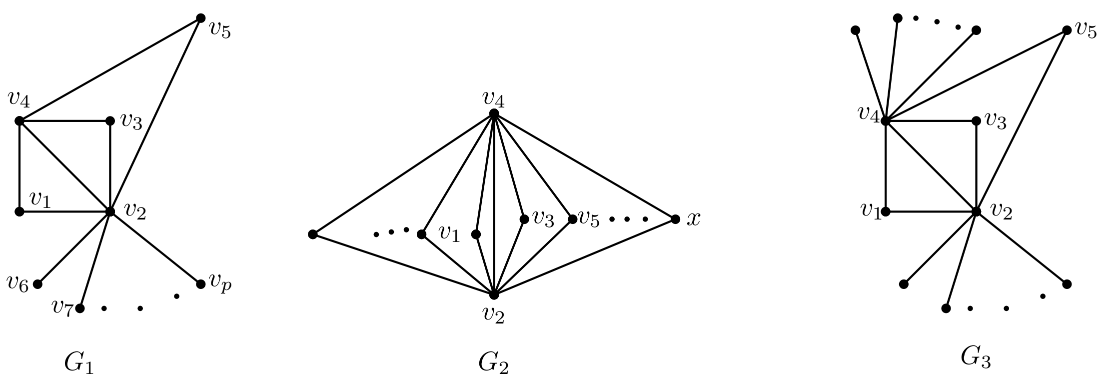


Figure 2.8: G

Since the edge v_2v_4 of G_i $(1 \leqslant i \leqslant 3)$ in Figure 2.8 does not lie on a detour joining any pair of vertices of G_i, these graphs are not edge detour graphs. Thus in this case there are no edge detour graphs G with H_2 as a proper subgraph satisfying the requirements of the theorem. Thus, if G does not contain K_4 as an induced subgraph, we have proved that G has a unique 4-cycle. Now we consider two subcases.

Subcase 1: The unique cycle C_4: v_1, v_2, v_3, v_4, v_1 contains exactly one chord v_2v_4. Since $p \geqslant 5$, $D = 4$ and G is connected, any vertex x not on C_4 is pendant and is adjacent to at least one vertex of C_4. The vertex x cannot be adjacent to both v_1 and v_3, for in this case, we get $c(G) = 5$, which is a contradiction. Suppose that x is adjacent to v_1 or v_3, say v_1. Also, if y is a vertex such that $y \neq x$, v_1, v_2, v_3, v_4, then y cannot be adjacent to v_2 or v_3 or v_4, for in each case $D \geqslant 5$, which is a contradiction. Hence y is a pendant vertex and cannot be adjacent to x or v_2 or v_3 or v_4 so that in this case the graph G reduces to the one given in Figure 2.9.

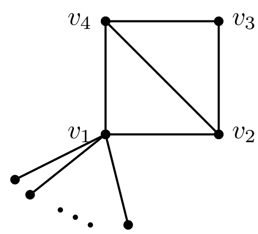


Figure 2.9: G

Since the set of all end vertices together with the vertex v_3 forms an edge detour basis for this graph G, it follows from Theorem 1.1 that $dn_1(G) = p - 3 \geq 2$. Hence $p = 5$ is the only possibility and the graph reduces to $G_{14} \in \mathscr{H}$ given in Figure 2.1 and satisfies the requirements of the theorem. Similarly, if x is adjacent to v_3, then also we get the graph $G_{14} \in \mathscr{H}$ given in Figure 2.1 and satisfies the requirements of the theorem.

Now, if x is adjacent to both v_2 and v_4, we get the graph H given in Figure 2.10 as a subgraph which is isomorphic to the graph H_2 given in Figure 2.4. Then, as in the first part of case 2, we see that there are no edge detour graphs which satisfy the requirements of the theorem.

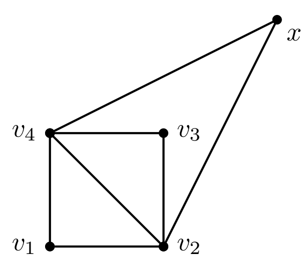


Figure 2.10: H

Thus x is adjacent to exactly one of v_2 or v_4, say v_2. Also, if y is a vertex such that $y \neq x$, v_1, v_2, v_3, v_4, then y cannot be adjacent to x or v_1 or v_3, for in each case $D \geqslant 5$, which is a contradiction. If y is adjacent to v_2 and v_4, then we get the graph H given in Figure 2.11 as a subgraph. Then exactly as in the first part of case 2 it can be seen that there are no graphs satisfying the requirements of the theorem.

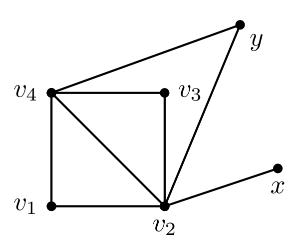
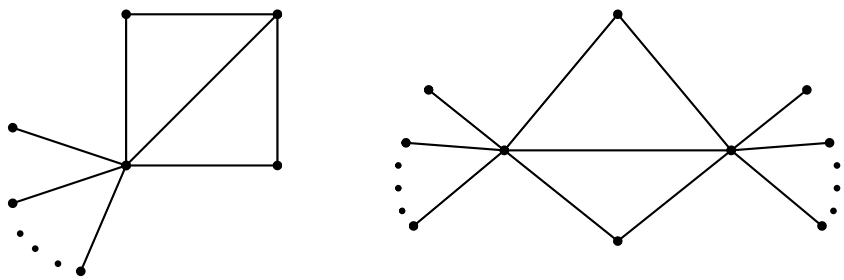


Figure 2.11: H

Thus y must be adjacent to v_2 or v_4 only. Hence we conclude that in either case the graph G

must reduce to the graph G_1 or G_2 as given in Figure 2.12. Similarly, if x is adjacent to v_4, then the graph G reduces to the graph G_1 or G_2 as given in Figure 2.12 and it is clear that $dn_1(G) = p - 2 \geq 3$, which is a contradiction.

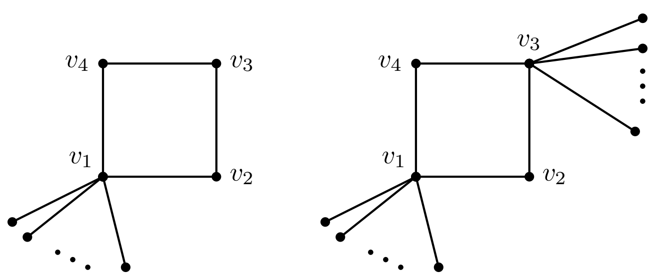


G_1 G_2

Figure 2.12: G

Thus, in this subcase 1, $G_{14} \in \mathscr{H}$ given in Figure 2.1 is the only graph satisfying the requirements of the theorem.

Subcase 2: The unique cycle C_4: v_1, v_2, v_3, v_4, v_1 has no chord. In this case we claim that G contains no triangle. Suppose that G contains a triangle C_3. If C_3 has no vertex in common with C_4 or exactly one vertex in common with C_4, we get a path of length at least 5 so that $D \geqslant 5$. If C_3 has exactly two vertices in common with C_4, we get a cycle of length 5. Thus, in all cases, we have a contradiction and hence it follows that G contains a unique chordless cycle C_4 with no triangles. Since $p \geqslant 5$, $D = 4$, $c(G) = 4$ and G is connected, any vertex x not on C_4 is pendant and is adjacent to exactly one vertex of C_4, say v_1. Also if y is a vertex such that $y \neq x$, v_1, v_2, v_3, v_4, then y cannot be adjacent to v_2 or v_4, for in this case, $D \geqslant 5$, which is a contradiction. Thus y must be adjacent to v_3 only. Hence we conclude that in either case G must reduce to the graphs H_1 or H_2 as given in Figure 2.13.



H_1 H_2

Figure 2.13: G

For these graphs H_1 and H_2 in Figure 2.13, it follows from Theorem 1.1 that $dn_1(H_1) = p - 3$ and $dn_1(H_2) = p - 4$. The only possible vaues are $p = 5$ for H_1 and $p = 6$ for H_2 so that H_1 reduces to $G_{15} \in \mathscr{H}$ and H_2 reduces to $G_{16} \in \mathscr{H}$ as given in Figure 2.1. Thus, in this subcase 2, G_{15}, $G_{16} \in \mathscr{H}$ as given in Figure 2.1 are the only graphs satisfying the requirements of the theorem. Thus, when $D = 4$ and $c(G) = 4$, the graphs satisfying the requirements of the theorem are G_{14}, G_{15}, $G_{16} \in \mathscr{H}$ as in Figure 2.1.

Case 3 Let $c(G) = 3$.

Case 3a G contains exactly one triangle C_3: v_1, v_2, v_3, v_1. Since $p \geqslant 5$, there are vertices not on C_3. If all the vertices of C_3 have degree three or more, then $p \geq 6$ and since $D = 4$, the graph G must reduce to the one given in Figure 2.14. It follows from Theorem 1.1 that $dn_1(G) = p - 3$. Since $p \geq 6$, this is a contradiction. Hence we conclude that at most two vertices of C_3 have degree $\geqslant 3$.

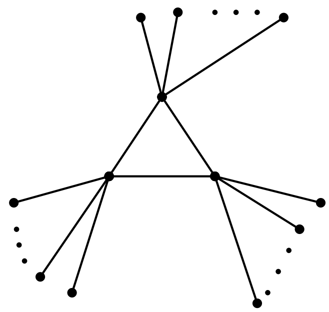


Figure 2.14: G

Subcase 1 Exactly two vertices of C_3 have degree 3 or more. Let *deg* $v_3 = 2$. Now, since $p \geqslant 5$, $D = 4$, $c(G) = 3$ and G is connected, we see that the graph reduces to the graph G given in Figure 2.15. For this graph G, it follows from Theorem 1.1 that $dn_1(G) = p-2$. Since $p \geq 5$, this is a contradiction.

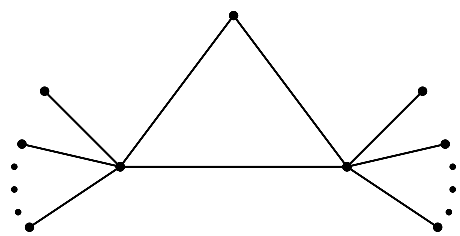


Figure 2.15: G

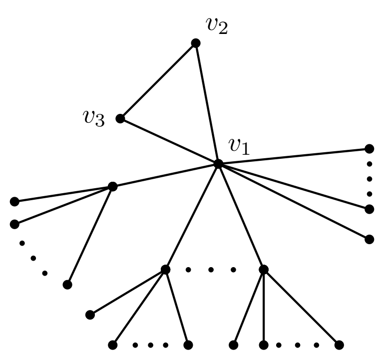


Figure 2.16: G

Subcase 2: Exactly one vertex v_1 of C_3 has degree 3 or more. Since G is connected, $p \geqslant 5$, $D = 4$ and $c(G) = 3$, the graph reduces to the one given in Figure 2.16. We claim that exactly one neighbor of v_1 other than v_2 and v_3 has degree ≥ 2. Otherwise, more than one neighbor of

v_1 other than v_2 and v_3 has degree $\geqslant 2$ so that $p \geq 7$ and set of all end-vertices together with v_2 and v_3 forms an edge detour set of G and so $dn_1(G) \geqslant 4$, which is a contradiction. Thus the graph reduces to the one given in Figure 2.17 and it is clear that $dn_1(G) = p - 2$. Since $p \geq 5$, this is a contradiction.

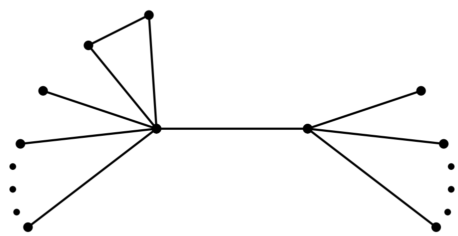
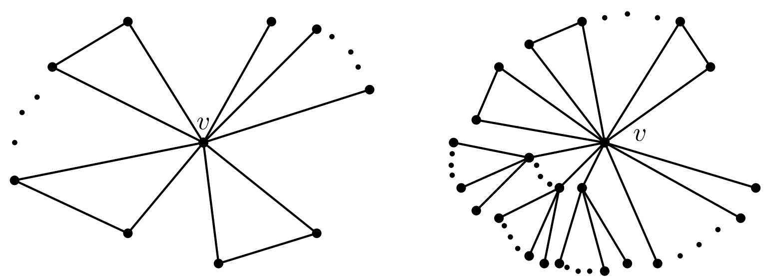


Figure 2.17: G

Case 3b: G contains more than one triangle. Since $D = 4$ and $c(G) = 3$, it is clear that all the triangles must have a vertex v in common. Now, if two triangles have two vertices in common then it is clear that $c(G) \geqslant 4$. Hence all triangles must have exactly one vertex in common. Since $p \geqslant 5$, $D = 4$, $c(G) = 3$ and G is connected, all the vertices of all the triangles are of degree 2 except v. Thus the graph reduces to the graphs given in Figure 2.18.



H_1 H_2

Figure 2.18: G

If $G = H_1$, then by Theorem 1.5, $dn_1(G) = p - 1$. Since $p \geq 5$, this is a contradiction. If $G = H_2$ and more than one neighbor of v not on the triangles has degree $\geqslant 2$, then $p \geq 9$ and the set of all end-vertices together with the all the vertices of all triangles except v forms an edge detour set of G. Hence $dn_1(G) \geqslant 6$, which is a contradiction.

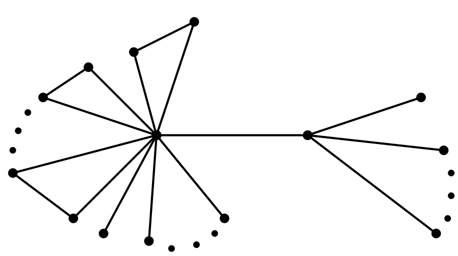


Figure 2.19: G

If $G = H_2$ and exactly one neighbor of v not on the triangles has degree $\geqslant 2$, then the graph reduces to the graph G given in Figure 2.19, and it is easy to verify that $dn_1(G) = p-2$. Since $p \geq 5$, this is a contradiction. Thus we see that when $D = 4$ and $c(G) = 3$, there are no graphs satisfying the requirements of the theorem. This completes the proof of the theorem.□

In view of Theorem 2.1, we leave the following problem as an open question.

Problem 2.2 *Characterize edge detour graphs G with detour diameter $D \geq 5$ for which $dn_1(G) = 2$.*

The following theorem is a characterization for trees.

Theorem 2.3 *For any tree T of order $p \geq 2$, $dn_1(G) = 2$ if and only if T is a path.*

Proof This follows from Theorem 2.1. □

References

[1] F. Buckley and F. Harary, *Distance in Graphs*, Addison-Wesley, Reading MA, (1990).

[2] G. Chartrand, H. Escuadro, and P. Zhang, Detour distance in graphs, *J. Combin. Math. Combin. Comput.* **53** (2005), 75-94.

[3] G. Chartrand, G.L. Johns, and P. Zhang, Detour number of a graph, *Util. Math.* **64** (2003), 97-113.

[4] G. Chartrand, L. Nebesky, and P. Zhang, A survey of Hamilton colorings of graphs, *Preprint*.

[5] G. Chartrand and P. Zang, Distance in graphs-taking the long view, *AKCE J. Graphs. Combin.*, 1, **N**o.1 (2004), 1-13.

[6] G. Chartrand and P. Zang, *Introduction to Graph Theory*, Tata McGraw-Hill, New Delhi (2006).

[7] W. Hale, Frequency assignment; theory and applications, *Proc. IEEE* **68** (1980), 1497-1514.

[8] A. P. Santhakumaran and S. Athisayanathan, Weak edge detour number of a graph, *Ars Combin, to appear*.

[9] A. P. Santhakumaran and S. Athisayanathan, Edge detour graphs, *J. Combin. Math. Combin. Comput., to appear*.

[10] A. P. Santhakumaran and S. Athisayanathan, On edge detour graphs, *communicated*.

International J.Math. Combin. Vol.1 (2009), 90-95

Euclidean Pseudo-Geometry on $\mathbf{R}^n$

Linfan MAO

(Chinese Academy of Mathematics and System Science, Beijing 100080, P.R.China)

E-mail: maolinfan@163.com

Abstract: A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom (1969), i.e., validated and invalided, or only invalided but in multiple distinct ways. Iseri constructed Smarandache 2-manifolds in Euclidean plane $\mathbf{R}^2$ in [1], and later Mao generalized his result to surfaces by *map geometry* in [4]. Then *can we construct Smarandache n-manifolds for $n \geq 3$*? The answer is YES. Not like the technique used in [6], we show how to construct Smarandache geometries in $\mathbf{R}^n$ by an algebraic methods, which was applied in [3] for $\mathbf{R}^2$ first, and then give a systematic way for constructing Smarandache n-manifolds.

Key Words: Smarandache geometries, Euclidean pseudo-geometry, combinatorial system.

AMS(2000): 05E15, 08A02, 15A03, 20E07, 51M15.

§1. Introduction

As it is usually cited in references, a *Smarandache geometry* is defined as follows.

Definition 1.1 *An axiom is said to be Smarandachely denied if the axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways.*

*A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom(*1969*).*

This anti-mathematical or multiple approach on sciences can be used to abstract systems. In the reference [8], we formally generalized it to the conceptions of Smarandachely systems as follows.

Definition 1.2 *A rule in a mathematical system* $(\Sigma; \mathcal{R})$ *is said to be Smarandachely denied if it behaves in at least two different ways within the same set* Σ*, i.e., validated and invalided, or only invalided but in multiple distinct ways.*

A Smarandache system $(\Sigma; \mathcal{R})$ *is a mathematical system which has at least one Smarandachely denied rule in* $\mathcal{R}$*.*

As its a simple or concrete example, a question raised in [4] is to construct a Smarandache geometry on R^n for $n \geq 2$. Certainly, the case of $n = 2$ has be solved by Iseri [1] and Mao [3]-

[1]Received December 25, 2008. Accepted February 16, 2009.

[4]. The main purpose of this paper is to give an algebraic approach for constructing Euclidean pseudo-geometry on $\mathbf{R}^n$ for any integer $n \geq 2$, which also refines the definition of pseudo manifold geometry introduced in [6].

§2. **Euclidean pseudo-geometry**

Let $\mathbf{R}^n$ be an n-dimensional Euclidean space with a normal basis $\overline{\epsilon}_1 = (1, 0, \cdots, 0)$, $\overline{\epsilon}_2 = (0, 1, \cdots, 0)$, $\cdots$, $\overline{\epsilon}_n = (0, 0, \cdots, 1)$. An *orientation* $\overrightarrow{X}$ is a vector $\overrightarrow{OX}$ with $\|\overrightarrow{OX}\| = 1$ in $\mathbf{R}^n$, where $O = (0, 0, \cdots, 0)$. Usually, an orientation $\overrightarrow{X}$ is denoted by its projections of $\overrightarrow{OX}$ on each $\overline{\epsilon}_i$ for $1 \leq i \leq n$, i.e.,

$$\overrightarrow{X} = (\cos(\overrightarrow{OX}, \overline{\epsilon}_1), \cos(\overrightarrow{OX}, \overline{\epsilon}_2), \cdots, \cos(\overrightarrow{OX}, \overline{\epsilon}_n)),$$

where $(\overrightarrow{OX}, \overline{\epsilon}_i)$ denotes the angle between vectors $\overrightarrow{OX}$ and $\overline{\epsilon}_i$, $1 \leq i \leq n$. All possible orientations $\overrightarrow{X}$ in $\mathbf{R}^n$ consist of a set $\mathscr{O}$.

A *pseudo-Euclidean space* is a pair $(\mathbf{R^n}, \omega|_{\overrightarrow{O}})$, where $\omega|_{\overrightarrow{O}} : \mathbf{R}^n \rightarrow \mathscr{O}$ is a continuous function, i.e., a straight line with an orientation $\overrightarrow{O}$ will has an orientation $\overrightarrow{O} + \omega|_{\overrightarrow{O}}(\overline{u})$ after it passing through a point $\overline{u} \in \mathbf{E}$. It is obvious that $(\mathbf{E}, \omega|_{\overrightarrow{O}}) = \mathbf{E}$, namely the Euclidean space itself if and only if $\omega|_{\overrightarrow{O}}(\overline{u}) = \overline{0}$ for $\forall \overline{u} \in \mathbf{E}$.

We have known that a straight line L passing through a point $(x_1^0, x_2^0, \cdots, x_n^0)$ with an orientation $\overrightarrow{O} = (X_1, X_2, \cdots, X_n)$ is defined to be a point set $(x_1, x_2, \cdots, x_n)$ determined by an equation system

$$\begin{cases} x_1 = x_1^0 + tX_1 \\ x_2 = x_2^0 + tX_2 \\ \cdots\cdots\cdots\cdots \\ x_n = x_n^0 + tX_n \end{cases}$$

for $\forall t \in \mathbf{R}$ in analytic geometry on $\mathbf{R}^n$, or equivalently, by the equation system

$$\frac{x_1 - x_1^0}{X_1} = \frac{x_2 - x_2^0}{X_2} = \cdots = \frac{x_n - x_n^0}{X_n}.$$

Therefore, we can also determine its equation system for a straight line L in a pseudo-Euclidean space $(\mathbf{R}^n, \omega)$. By definition, a straight line L passing through a Euclidean point $\overline{x}^0 = (x_1^0, x_2^0, \cdots, x_n^0) \in \mathbf{R}^n$ with an orientation $\overrightarrow{O} = (X_1, X_2, \cdots, X_n)$ in $(\mathbf{R}^n, \omega)$ is a point set $(x_1, x_2, \cdots, x_n)$ determined by an equation system

$$\begin{cases} x_1 = x_1^0 + t(X_1 + \omega_1(\overline{x}^0)) \\ x_2 = x_2^0 + t(X_2 + \omega_2(\overline{x}^0)) \\ \cdots\cdots\cdots\cdots \\ x_n = x_n^0 + t(X_n + \omega_n(\overline{x}^0)) \end{cases}$$

for $\forall t \in \mathbf{R}$, or equivalently,

$$\frac{x_1 - x_1^0}{X_1 + \omega_1(\overline{x}^0)} = \frac{x_2 - x_2^0}{X_2 + \omega_2(\overline{x}^0)} = \cdots = \frac{x_n - x_n^0}{X_n + \omega_n(\overline{x}^0)},$$

where $\omega|_{\overrightarrow{O}}(\overline{x}^0) = (\omega_1(\overline{x}^0), \omega_2(\overline{x}^0), \cdots, \omega_n(\overline{x}^0))$. Notice that this equation system dependent on $\omega|_{\overrightarrow{O}}$, it maybe not a linear equation system.

Similarly, let $\overrightarrow{O}$ be an orientation. A point $\overline{u} \in \mathbf{R}^n$ is said to be *Euclidean* on orientation $\overrightarrow{O}$ if $\omega|_{\overrightarrow{O}}(\overline{u}) = \overline{0}$. Otherwise, let $\omega|_{\overrightarrow{O}}(\overline{u}) = (\omega_1(\overline{u}), \omega_2(\overline{u}), \cdots, \omega_n(\overline{u}))$. The point $\overline{u}$ is *elliptic* or *hyperbolic* determined by the following inductive programming.

STEP 1. If $\omega_1(\overline{u}) < 0$, then $\overline{u}$ is elliptic; otherwise, hyperbolic if $\omega_1(\overline{u}) > 0$;

STEP 2. If $\omega_1(\overline{u}) = \omega_2(\overline{u}) = \cdots = \omega_i(\overline{u} = 0$, but $\omega_{i+1}(\overline{u} < 0$ then $\overline{u}$ is elliptic; otherwise, hyperbolic if $\omega_{i+1}(\overline{u}) > 0$ for an integer $i, 0 \leq i \leq n-1$.

Denote these elliptic, Euclidean and hyperbolic point sets by

$$\overrightarrow{V}_{eu} = \{\ \overline{u} \in \mathbf{R}^n \mid \overline{\mathrm{u}} \text{ an Euclidean point } \},$$
$$\overrightarrow{V}_{el} = \{\ \overline{v} \in \mathbf{R}^n \mid \overline{\mathrm{v}} \text{ an elliptic point } \}.$$
$$\overrightarrow{V}_{hy} = \{\ \overline{v} \in \mathbf{R}^n \mid \overline{\mathrm{w}} \text{ a hyperbolic point } \}.$$

Then we get a partition

$$\mathbf{R}^n = \overrightarrow{V}_{eu} \bigcup \overrightarrow{V}_{el} \bigcup \overrightarrow{V}_{hy}$$

on points in $\mathbf{R}^n$ with $\overrightarrow{V}_{eu} \cap \overrightarrow{V}_{el} = \emptyset$, $\overrightarrow{V}_{eu} \cap \overrightarrow{V}_{hy} = \emptyset$ and $\overrightarrow{V}_{el} \cap \overrightarrow{V}_{hy} = \emptyset$. Points in $\overrightarrow{V}_{el} \cap \overrightarrow{V}_{hy}$ are called *non-Euclidean points*.

Now we introduce a linear order $\prec$ on $\mathscr{O}$ by the dictionary arrangement in the following.

For $(x_1, x_2, \cdots, x_n)$ and $(x_1', x_2', \cdots, x_n') \in \mathscr{O}$, if $x_1 = x_1', x_2 = x_2', \cdots, x_l = x_l'$ and $x_{l+1} < x_{l+1}'$ for any integer $l, 0 \leq l \leq n-1$, then define $(x_1, x_2, \cdots, x_n) \prec (x_1', x_2', \cdots, x_n')$.

By this definition, we know that

$$\omega|_{\overrightarrow{O}}(\overline{u}) \prec \omega|_{\overrightarrow{O}}(\overline{v}) \prec \omega|_{\overrightarrow{O}}(\overline{w})$$

for $\forall \overline{u} \in \overrightarrow{V}_{el}, \overline{v} \in \overrightarrow{V}_{eu}$, $\overline{w} \in \overrightarrow{V}_{hy}$ and a given orientation $\overrightarrow{O}$. This fact enables us to find an interesting result following.

Theorem 2.1 *For any orientation $\overrightarrow{O} \in \mathscr{O}$ in a pseudo-Euclidean space $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$, if $\overrightarrow{V}_{el} \neq \emptyset$ and $\overrightarrow{V}_{hy} \neq \emptyset$, then $\overrightarrow{V}_{eu} \neq \emptyset$.*

Proof By assumption, $\overrightarrow{V}_{el} \neq \emptyset$ and $\overrightarrow{V}_{hy} \neq \emptyset$, we can choose points $\overline{u} \in \overrightarrow{V}_{el}$ and $\overline{w} \in \overrightarrow{V}_{hy}$. Notice that $\omega|_{\overrightarrow{O}} : \mathbf{R}^n \to \mathscr{O}$ is a continuous and $(\mathscr{O}, \prec)$ a linear ordered set. Applying the *generalized intermediate value theorem* on continuous mappings in topology, i.e.,

Let $f : X \to Y$ be a continuous mapping with X a connected space and Y a linear ordered set in the order topology. If $a, b \in X$ and $y \in Y$ lies between $f(a)$ and $f(b)$, then there exists $x \in X$ such that $f(x) = y$.

we know that there is a point $\overline{v} \in \mathbf{R}^n$ such that

$$\omega|_{\overrightarrow{O}}(\overline{v}) = \overline{0},$$

i.e., $\overline{v}$ is a Euclidean point by definition. □

Corollary 2.1 *For any orientation $\overrightarrow{O} \in \mathscr{O}$ in a pseudo-Euclidean space $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$, if $\overrightarrow{V}_{eu} = \emptyset$, then either points in $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$ is elliptic or hyperbolic.*

Certainly, a pseudo-Euclidean space $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$ is a Smarandache geometry sometimes explained in the following.

Theorem 2.2 *A pseudo-Euclidean space $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$ is a Smarandache geometry if $\overrightarrow{V}_{eu}, \overrightarrow{V}_{el} \neq \emptyset$, or $\overrightarrow{V}_{eu}, \overrightarrow{V}_{hy} \neq \emptyset$, or $\overrightarrow{V}_{el}, \overrightarrow{V}_{hy} \neq \emptyset$ for an orientation $\overrightarrow{O}$ in $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$.*

Proof Notice that $\omega|_{\overrightarrow{O}}(\overline{u}) = \overline{0}$ is an axiom in $\mathbf{R}^n$, but a Smarandache denied axiom if $\overrightarrow{V}_{eu}, \overrightarrow{V}_{el} \neq \emptyset$, or $\overrightarrow{V}_{eu}, \overrightarrow{V}_{hy} \neq \emptyset$, or $\overrightarrow{V}_{el}, \overrightarrow{V}_{hy} \neq \emptyset$ for an orientation $\overrightarrow{O}$ in $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$ for $\omega|_{\overrightarrow{O}}(\overline{u}) = \overline{0}$ or $\neq \overline{0}$ in the former two cases and $\omega|_{\overrightarrow{O}}(\overline{u}) \prec \overline{0}$ or $\succ \overline{0}$ both hold in the last one. Whence, we know that $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$ is a Smarandache geometry by definition. □

Notice that there infinite points on a segment of a straight line in $\mathbf{R}^n$. Whence, a necessary for the existence of a straight line is there exist infinite Euclidean points in $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$. We find a necessary and sufficient result for the existence of a curve C in $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$ following.

Theorem 2.3 *A curve $C = (f_1(t), f_2(t), \cdots, f_n(t))$ exists in a pseudo-Euclidean space $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$ for an orientation $\overrightarrow{O}$ if and only if*

$$\frac{df_1(t)}{dt}|_{\overline{u}} = \sqrt{(\frac{1}{\omega_1(\overline{u})})^2 - 1},$$

$$\frac{df_2(t)}{dt}|_{\overline{u}} = \sqrt{(\frac{1}{\omega_2(\overline{u})})^2 - 1},$$

$$\cdots\cdots\cdots\cdots,$$

$$\frac{df_n(t)}{dt}|_{\overline{u}} = \sqrt{(\frac{1}{\omega_n(\overline{u})})^2 - 1}.$$

for $\forall \overline{u} \in C$, where $\omega|_{\overrightarrow{O}} = (\omega_1, \omega_2, \cdots, \omega_n)$.

Proof Let the angle between $\omega|_{\overrightarrow{O}}$ and $\overline{\epsilon}_i$ be θ_i, $1 \leq \theta_i \leq n$.

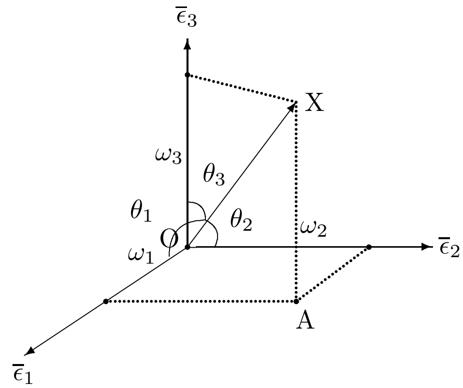


Fig.2.1

Then we know that

$$\cos\theta_i = \omega_i, \quad 1 \leq i \leq n.$$

According to the geometrical implication of differential at a point $\overline{u} \in \mathbf{R}^n$, seeing also Fig.2.1, we know that

$$\frac{df_i(t)}{dt}|_{\overline{u}} = tg\theta_i = \sqrt{(\frac{1}{\omega_i(\overline{u})})^2 - 1}$$

for $1 \leq i \leq n$. Therefore, if a curve $C = (f_1(t), f_2(t), \cdots, f_n(t))$ exists in a pseudo-Euclidean space $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$ for an orientation $\overrightarrow{O}$, then

$$\frac{df_i(t)}{dt}|_{\overline{u}} = \sqrt{(\frac{1}{\omega_2(\overline{u})})^2 - 1}, \quad 1 \leq i \leq n$$

for $\forall \overline{u} \in C$. On the other hand, if

$$\frac{df_i(t)}{dt}|_{\overline{v}} = \sqrt{(\frac{1}{\omega_2(\overline{v})})^2 - 1}, \quad 1 \leq i \leq n$$

hold for points $\overline{v}$ for $\forall t \in \mathbf{R}$, then all points $\overline{v}$, $t \in \mathbf{R}$ consist of a curve $C = (f_1(t), f_2(t), \cdots, f_n(t))$ in $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$ for the orientation $\overrightarrow{O}$. $\square$

Corollary 2.2 *A straight line L exists in $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$ if and only if $\omega|_{\overrightarrow{O}}(\overline{u}) = \overline{0}$ for $\forall \overline{u} \in L$ and $\forall \overrightarrow{O} \in \mathscr{O}$.*

§3. Application to Smarandache n-manifolds

Application of the definition of pseudo-Euclidean space $\mathbf{R}^n$ enables us to formally define a dimensional n pseudo-manifold in [6] following, which makes its structure clear.

Definition 3.1 *An n-dimensional pseudo-manifold $(M^n, \mathcal{A}^\omega)$ is a Hausdorff space such that each points p has an open neighborhood U_p homomorphic to a pseudo-Euclidean space $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$,* *where* $\mathcal{A} = \{(U_p, \varphi_p^\omega) | p \in M^n\}$ *is its atlas with a homomorphism* $\varphi_p^\omega : U_p \to (\mathbf{R}^n, \omega|_{\overrightarrow{O}})$ *and a chart* (U_p, φ_p^ω).

Applications of this definition will rebuilt pseudo-manifold geometries constructed in [6], which will appear in a forthcoming book *Combinatorial Geometry with Applications to Field Theory* of the author in 2009.

References

[1] H.Iseri, *Smarandache manifolds*, American Research Press, Rehoboth, NM,2002.

[2] Linfan Mao, *Automorphism Groups of Maps, Surfaces and Smarandache Geometries*, American Research Press, 2005.

[3] Linfan Mao, *Smarandache Multi-Space Theory*, Hexis, Phoenix,American 2006.

[4] Linfan Mao, A new view of combinatorial maps by Smarandache' notion, e-print: *arXiv: math.GM/0506232*, also in *Selected Papers on Mathematical Combinatorics*(I), World Academic Press, 2006, 49-74.

[5] Linfan Mao, Combinatorial speculation and combinatorial conjecture for mathematics, *International J.Math. Combin.* Vol.1(2007), No.1, 1-19.

[6] Linfan Mao, Pseudo-manifold geometries with applications, *International J.Math. Combin.* Vol.1(2007), No.1, 45-58.

[7] Linfan Mao, An introduction to Smarandache multi-spaces and mathematical combinatorics, *Scientia Magna*, Vol.3, No.1(2007), 54-80.

[8] Linfan Mao, Extending homomorphism theorem to multi-systems, *International J.Math. Combin.* Vol.3(2008), 1-27.

[9] F.Smarandache, *A Unifying Field in Logics. Neutrosopy: Neturosophic Probability, Set, and Logic*, American research Press, Rehoboth, 1999.

[10] F.Smarandache, Mixed noneuclidean geometries, *eprint arXiv: math/0010119*, 10/2000.

International J.Math. Combin. Vol.1 (2009), 96-105

Retraction Effect on Some Geometric Properties of Geometric Figures

M. El-Ghoul and A.T. M-Zidan

(Mathematics Department, Faculty of Science, Tanta University, Tanta, Egypt)

Email: m_elghoul@hotmail.com, ATM_zidan@yahoo.com

Abstract: In this paper, we introduce the effect of some types of retractions on some geometric properties of some geometric figures, which makes the geometric figure that is not manifold to be a manifold. The limit of retractions of some geometric figures is deduced and the types of retractions which fail to change the geometric figure to be a manifold are discussed. Theorems governing these types of retractions are deduced.

Key Words: Manifolds, geometric figures, retraction.

AMS(2000): 51H10, 57N10.

§1. Introduction

The folding of a manifold into another manifold or into itself are presented by EL-Ghoul [4, 7-9], EL-Kholy [12], El-Ahmady [1,2] and in [14], the deformation retract and the topological folding of a manifold are introduced in [1,4,6,7], the retraction of the manifolds are introduced in [5,8,10]. In this paper we have presented the effect of retraction on some geometric properties of some geometric figures, which makes some geometric figures which is not manifolds to be manifolds, also the limit of these retractions is discussed, the types of retractions, which fail to make the non-manifold to be a manifold will be presented, the end of limits of retractions of any geometric figure of dimension n is presented, we introduce a type of retraction ,which makes the non-simple closed curve in R^3 to be a knot, the effect of retraction on some geometric properties of some geometric figures as dimension is discussed, the theorems governing these types of retractions are presented.

§2. Definitions and background

1. Let M and N be two smooth manifolds of dimensions m and n respectively. A map $f : M \to N$ is said to be an isometric folding of M into N if and only if for every piecewise geodesic path $\gamma : I \to M$ the induced path $f \circ \gamma : I \to N$ is piecewise geodesic and of the same length as γ. If f does not preserve the length, it is called topological folding [14].
2. A subset A of a topological space X is called a retract of X, if there exists a continuous map

[1]Received January 10, 2009. Accepted February 18, 2009.

$r : X \to A$ (called retraction) such that $r(a) = a \forall a \in A$ [5,13] .
3. An n-dimensional manifold is a Hausdorff topological space, such that each point has an open neighborhood homeomorphic to an open n-dimension disc [13, 15] .
4. A knot is a subset of 3-space that is homeomorphic to the unit circle in R^3 [16].

§3. The main results

Aiming to our study, we will introduce the following:

our goal is to study the effect of some retractions on the geometric properties of some geometric figures, which are not manifolds as some non simple closed curves and we introduce some types of retractions which makes the geometric figure, which is not manifold to be a manifold and the types of retractions which fail to change the geometric figure to be a manifold.

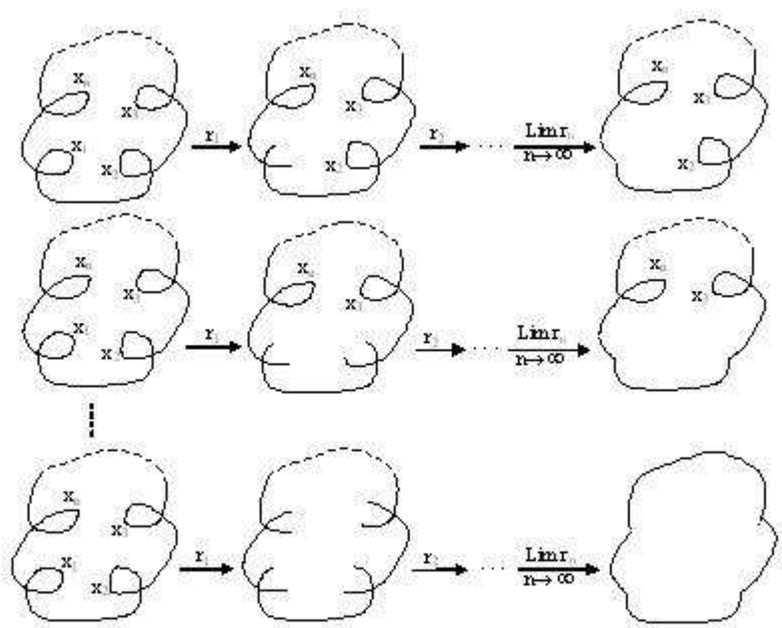


Fig.1

Proposition 3.1 *There is a type of retraction which makes the non-simple closed curve , which is not manifold to be a manifold.*

Proof Let $r : X - \{x_i\} \to X_1$, be a retraction map of X–$\{x_i\}$ into X_1, where X is a non-simple closed curve self-intersection at n-points , since X be a non-simple closed curve self-intersection at n-points, and the neighborhoods of the intersection points different from the neighborhoods of the other points of the curve X, then X is not manifold, let x_i, $i = 1, 2, \cdots, m$ are any points on the loops of the intersection points of X respectively, when the number of the points m is less than the number of the intersection points i.e. $m <$n, then the limit of the retractions of X is not a manifold, when the number of the points m is equal to the number of the intersection points , i.e. $m = n$, then the limit of retractions of X is a simple closed curve, which is a manifold and when m¿n, then the limit of the retractions of X is a 0-manifold, see Fig.1. □

Proposition 3.2 *There is a type of retraction which makes the non-simple closed curve to be a disjoint union of points which is a manifold.*

Proof Let $r : X \setminus \{x_i\} \rightarrow X_2$, be a retraction map of $X \setminus \{x_i\}$ into X_2, where X is a non-simple closed curve self-intersection at n-points, when the number of points x_i, $i = 1, 2, \cdots, m$ is less than the number of intersection points n i.e., $m < n$, then the limit of retractions of X is not a manifold, when the number of the points m is equal the number of the intersection points n then the limit of retractions of X is a disjoint union of points, which is a manifold and when X lies in R^3, we have the same results, see Fig.2. □

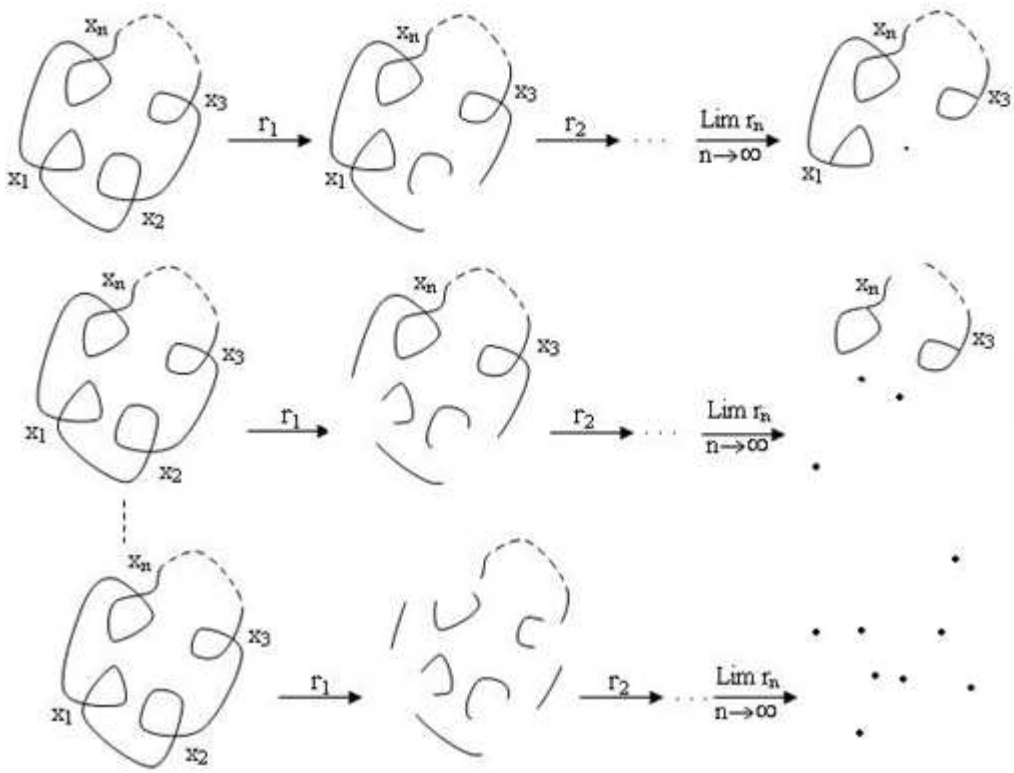


Fig.2

Proposition 3.3 *Let $r : X \setminus \{p_i\} \rightarrow X_3, i = 1, 2, ..., m$ be a retraction map, where X is a non-simple closed curve, p_i are the points on X,which lie between any two consecutive inter-section points of X respectively, then the limit of retractions of X is a manifold.*

Proof Let $r : X \setminus \{p_i\} \rightarrow X_3, i = 1, 2, ..., m$ be a retraction map of $X \setminus \{p_i\}$ into X_3, where X is a non-simple closed curve self-intersection at n-points and p_1, p_2, p_3,$\cdots$, p_m are the points on X, which lie between any two consecutive intersection points of X respectively, when the number of points m is less than the number of intersection points m i.e., $m < n$, then the limit of retractions of X is not a manifold, when the number of the points m is equal to the number of points n i.e. $m = n$, then the limit of retractions of X is a disjoint union of loops which is a manifold see Fig.3. $\square$

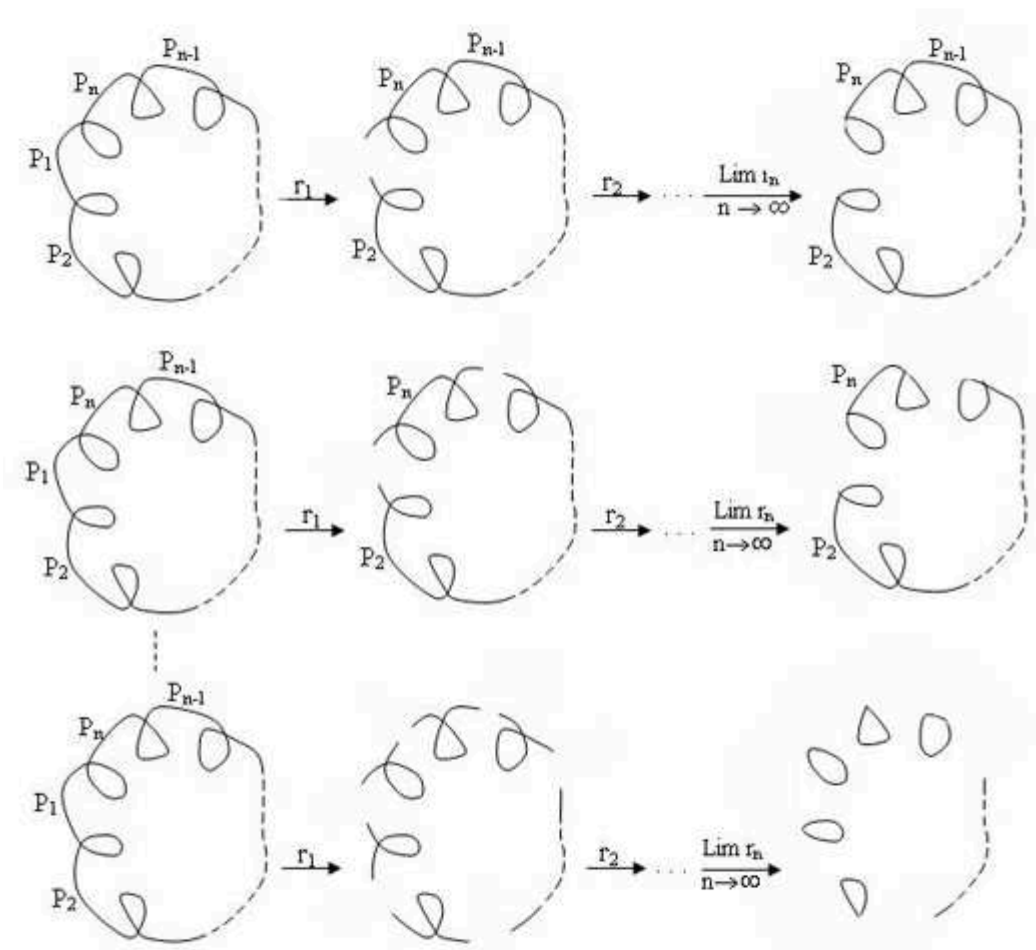


Fig.3

Proposition 3.4 *If $r : X \setminus \{p_i, k_i\} \to X^*$ be a retraction map of X where X is a non-simple closed curve, p_i, k_i are defined as any two points of each loop of the loops of X, then the limit of retraction of X is not a manifold.*

Proof Let $r : X \setminus \{p_i, k_i\} \to X^*$, be a retraction map of $X \setminus \{p_i, k_i\}$ into X^*, where X is a non-simple closed curve self-intersection at n-points of the curve X, let p_i and k_i, $i = 1, 2, \cdots, m$ are the points of each loop of the loops of the curve X i.e., the retraction by removing two points p_i and k_i from each loop respectively ,when the number of points $\{p_i, k_i\}$ is less than n i.e., $m < n$, then the limit of retractions of X is not a manifold, when the number of points $\{p_i, k_i\}$ is equal to the number of points n i.e., $m = n$, then the limit of retractions of X is not a manifold and when X lies in R^3, we have the same results, see Fig.4. $\square$

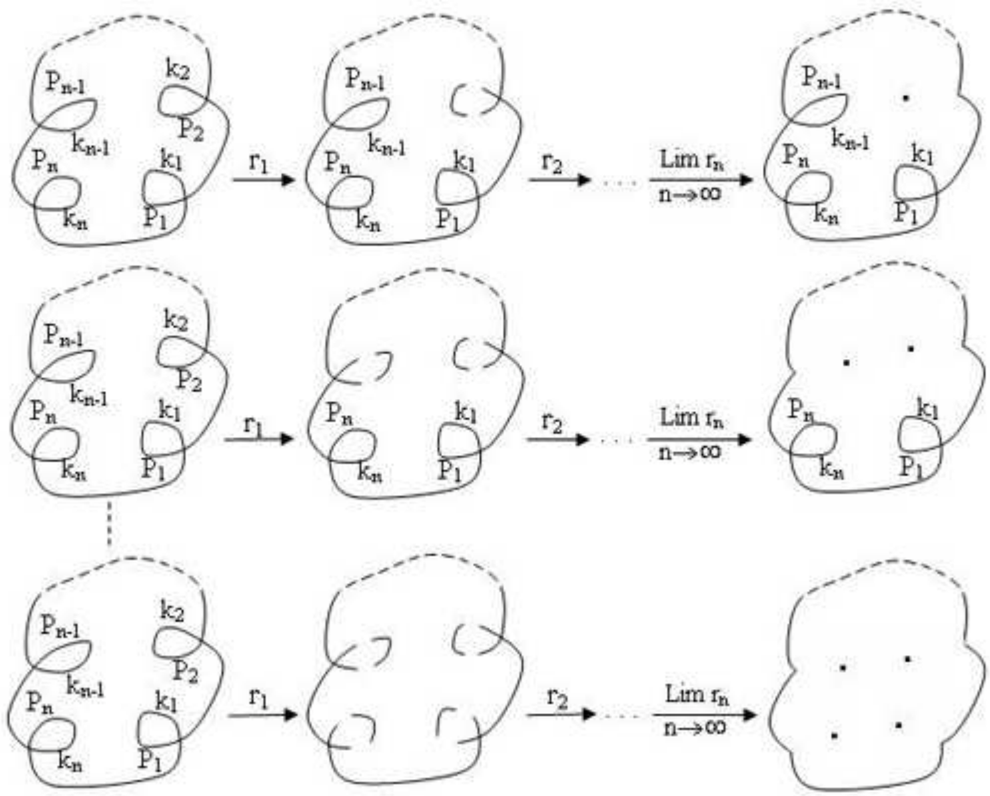


Fig.4

Proposition 3.5 *There is a type of retraction which make the non-simple curve with boundaries which is not a manifold to be a manifold with boundaries.*

Proof Let $r : X \setminus \{x_i\} \to X^r$ be a retraction map of $X \setminus \{x_i\}$ into X^r, where X is a non-simple curve with boundaries b_1 and b_2, which is self-intersection at n-points, let x_i, $i = 1, 2, \cdots, m$ are the points on the loops of the intersection points of X respectively, when the number of the points m are less than n i.e., $m < n$, then the limit of retractions of X is not a manifold, when the number of the points m is equal to the number of the intersection points n, then the limit of retractions of X is a simple curve with boundaries b_1 and b_2, which is a manifold with boundaries and when $m > n$, then the limit of retractions of X is a manifold see Fig.5, when X lies in R^3, we have the same results. $\square$

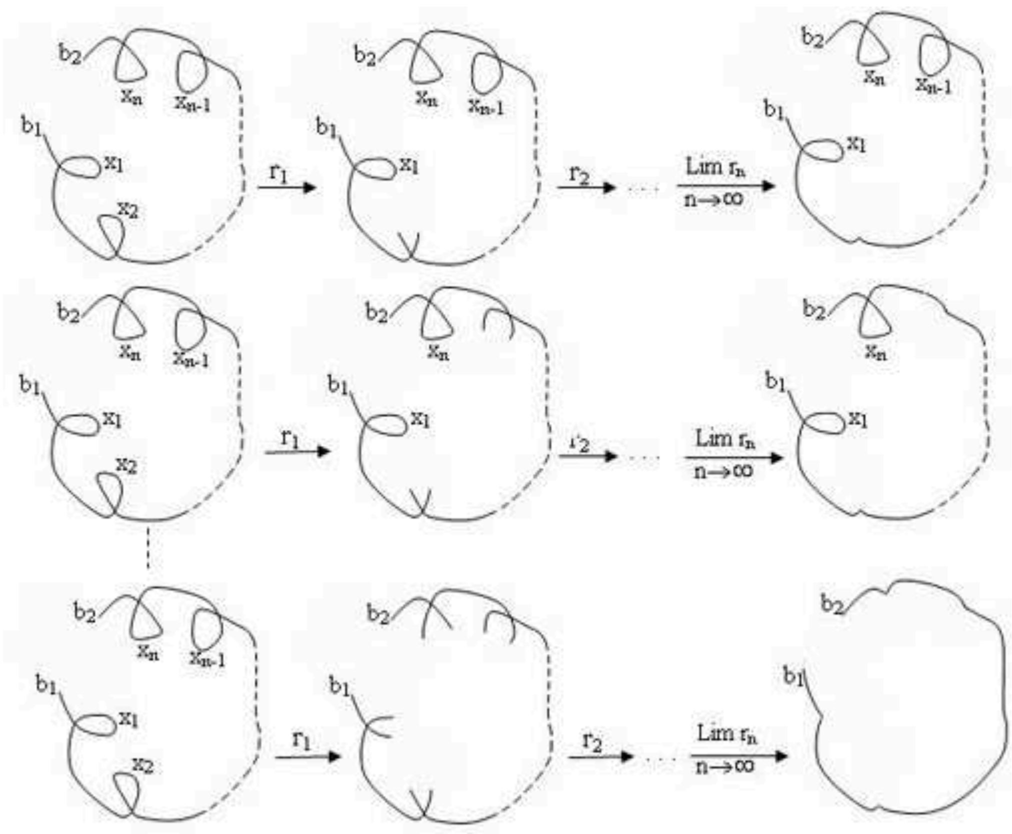


Fig.5

Proposition 3.6 *If $r : X \setminus \{x\} \to X^{/}$ be a retraction map of X, where X is a non-simple curve with boundaries, and x is a point between one point of the boundary and the nearest intersection point, then the limit of retractions of X is not a manifold.*

Proof Let $r : X \setminus \{x\} \to X^{/}$, be a retraction map of $X \setminus \{x\}$ into $X^{/}$, where X is a non-simple curve with boundaries b_1 and b_2 self-intersection at n-points, where x is the point between the boundary b_2 and the nearest intersection point of X, then the limit of retractions of X is not a manifold, when we define the retraction map $r : X \setminus \{x\} \to X^{/}$, where x is the point between the boundary $b_1$and the nearest intersection point of X then the limit of retractions of X is not a manifold, see Fig.6. $\square$

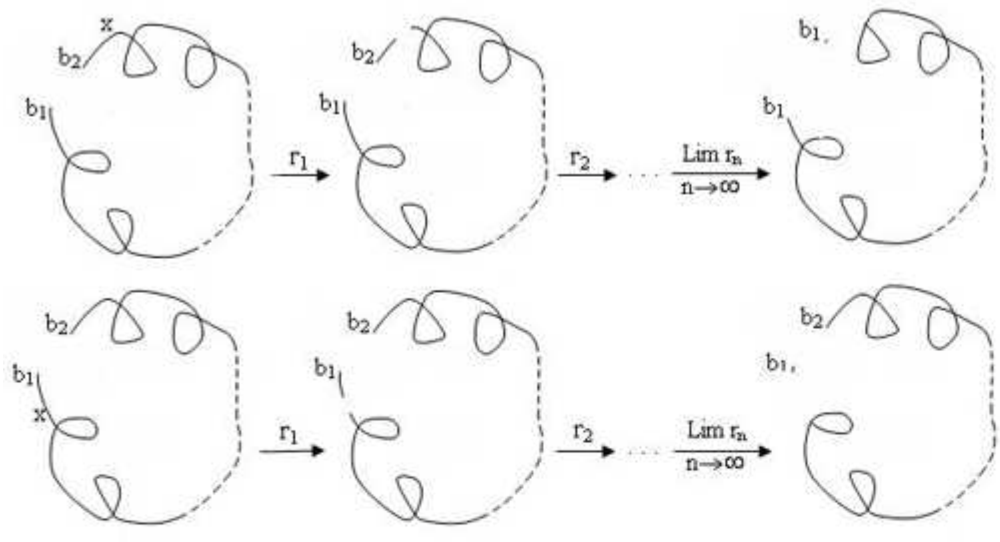


Fig.6

Proposition 3,7 *If $r : X \setminus \{b_i\} \rightarrow X_b, i = 1, 2$ be a retraction map of X, where X is a non simple curve with boundaries b_1 and b_2, then the limit of retractions of X is not a manifold.*

Proof Let $r : X \setminus \{b_i\} \rightarrow X_b$ be a retraction map of $X \setminus \{b_i\}$ into X_b, where X is a non-simple curve self-intersection at n-points, b_i, $i = 1, 2$ are the boundaries of $X, i = 1, 2$, then the retraction of X is not a manifold and the limit of retractions of X is not a manifold, see Fig.7. □

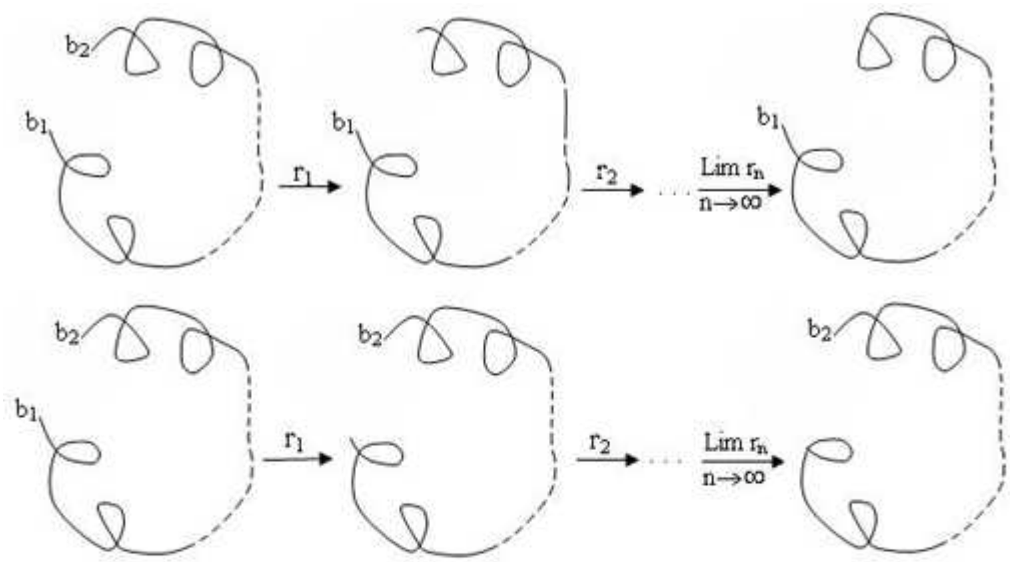


Fig.7

Proposition 3,8 *There is a type of retraction which makes the non-simple closed curve in R^3 to be a knot.*

Proof Let $r : X \setminus \{x_i\} \rightarrow X^*$, be a retraction map of $X \setminus \{x_i\}$ into X^*, where X is a non-simple closed curve in R^3 self-intersection at n-points, X is not a manifold, let x_i, $i = 1, 2, \cdots, m$ are the points on the loops of the intersection points of X respectively, when the number of the points m are less than the number of the intersection points n i.e., $m < n$, then the limit of the retractions of X is not a knot, when the number of the points m is equal to the number of points n of X, i.e., $m = n$, then the limit of retractions of X is a simple closed curve homeomorphic to a circle in R^3 which is a knot, which is also a manifold and when the number of points m¿n, then the limit of the retractions of X is not a knot, but it is a manifold, see Fig.8. $\square$

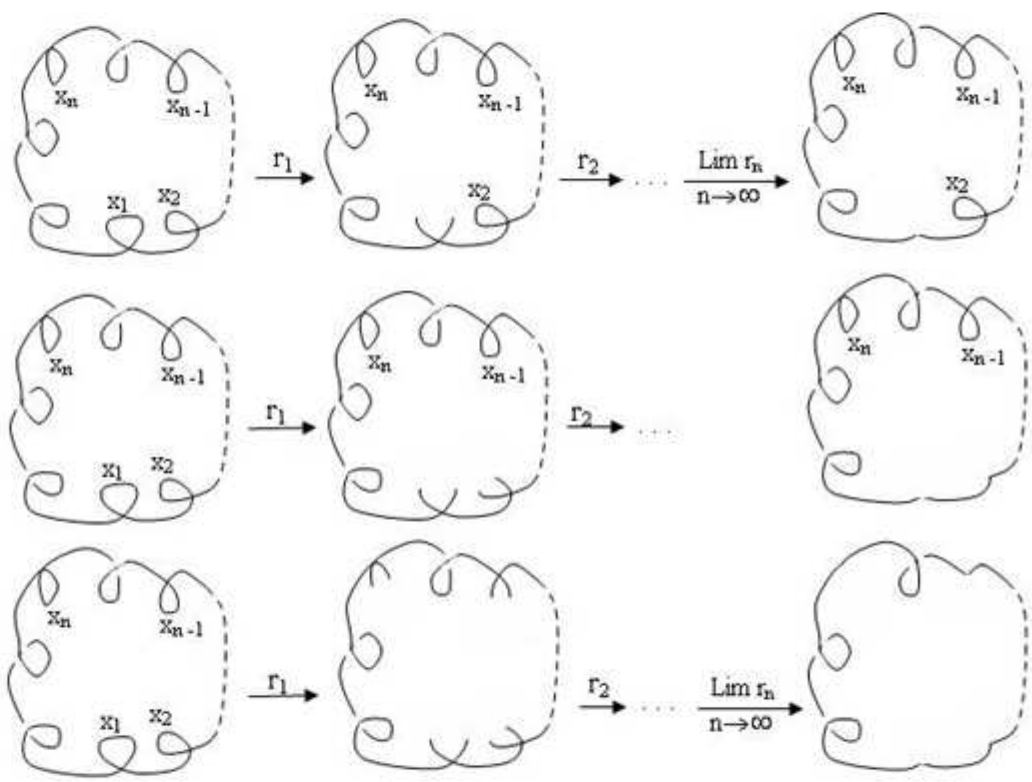


Fig.8

Proposition 3.9 *Let A be a subset of a topological space X and $r : X \rightarrow A$ is a retraction map of X into A, $A = r(X)$, then $dim(X) = dim(r(X)), dim(X) \geq dim(lim\ r(X)), dim(X) \neq dim(lim\ r(X))$ and $dim(r(X)) \geq dim(limr(X))$.*

Proposition 3.10 *The limit of retractions of any geometric figure in R^n, which is not a manifold is not necessary be a manifold, but the end of the limits of retractions of any n-geometric figure is a manifold.*

Proof Let $r : M^n \to M_1^n$ be a retraction map of M^n into M_1^n, M^n is a geometric figure of dimension n, M^n is not a manifold, then the limit of retractions of the geometric figure M^n is M^{n-1} and it may be a manifold or not, there is at least one point , which their neighborhood is not homeomorphic to the other points of M^{n-1}, the limit of retractions of M^{n-1} is M^{n-2} and it may be manifold or not, by using a sequence of retractions of M^n as shown in the following, then we find that the end of limits of retractions of M^n is a 0-manifold.

$$M^n \overset{r_1^1}{\to} M_1^n \overset{r_2^1}{\to} M_2^n \overset{r_2^1}{\to} ...M_{n-1}^n \overset{\lim\limits_{n\to\infty} r_n^1}{\to} M^{n-1},$$

$$M^{n-1} \overset{r_1^2}{\to} M_1^{n-1} \overset{r_2^2}{\to} M_2^{n-1} \overset{r_3^2}{\to} ...M_{n-1}^{n-1} \overset{\lim\limits_{n\to\infty} r_n^2}{\to} M^{n-2},$$

$$M^{n-2} \overset{r_1^3}{\to} M_1^{n-2} \overset{r_2^3}{\to} M_2^{n-2} \overset{r_3^3}{\to} ...M_{n-1}^{n-2} \overset{\lim\limits_{n\to\infty} r_n^3}{\to} M^{n-3},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots ,$$

$$M^1 \overset{r_1^n}{\to} M_1^1 \overset{r_2^n}{\to} M_2^1 \overset{r_3^n}{\to} ...M_{n-1}^1 \overset{\lim\limits_{n\to\infty} r_n^n}{\to} M^0.$$

Then the end of limits of retractions of any n-geometric figure is a 0-manifold. □

References

[1] A.E. El- Ahmady, The deformation retract and topological folding of Buchdahi Space , *Periodica Mathematica Hungarica*, Vol.28 (1994), 19-30.

[2] A.E. El-Ahmady, Fuzzy folding of fuzzy horocycle, *Circolo Mathematico di Palermo Seriell*, TomoL 111(2004), 443-50.

[3] M .P.Docarmo, *Riemannian geometry*, Boston: Birkhauser, 1992.

[4] M .El-Ghoul and A.E. El-Ahmady, The deformation retract and topological folding of the flat space, *J . Inst .Math.. Comp. Sci.*, 5(3)(1992), 349-56.

[5] M .El-Ghoul ,A.E. El-Ahmady and H-Rafat, Folding-retraction of dynamical manifold and the VAK of vacuum fluctuation, *Chaos, Solitons and Fractals*, 20(2004), 209-217.

[6] M .El-Ghoul, The deformation retract of the complex projective space and its topological folding, *J. Mater Sci. V.K*, 30(1995), 44145-44148.

[7] M .El-Ghoul, The deformation retract and topological folding of a manifold.Commun, *Fac. Sic. University of Ankara*, Series A 37(1998), 1-4.

[8] M .El-Ghoul, H. El-Zhony, S.I. Abo-El Fotooh, Fractal retraction and fractal dimension of dynamical manifold, *Chaos, Solitons and Fractals*, 18(2003), 187-192.

[9] M .El-Ghoul, The limit of folding of a manifold and its deformation retract, *J.Egypt, Math. Soc.* 5 (2)(1997), 133-140.

[10] M .El-Ghoul, Fuzzy retraction and folding of fuzzy-orientable compact manifold, *Fuzzy Sets and Systems*, 105(1999), 159-163.

[11] M .El-Ghoul, Unfolding of Riemannian manifolds, *Commun. Fac. Sic. University of Ankara*, Series A, 37(1988) 1-4.

[12] E .EL-Kholy, *Isometric and Topological Folding of Manifold*, Ph .D. Thesis, University of Southampton, UK, 1997.

[13] W.S.Massey, *Algebraic Topology: An Introduction*, Springer-Verlag, New York, etc.(1977).

[14] S.A. Robertson , Isometric folding of Riemannian manifolds. *Proc Roy . Soc Edinburgh*, 77(1977), 275-89.

[15] J.M. Singer, Thorp JA. In: *Lecture notes on elementary topology and geometry*, New York: Springer-Verlag, 1967.

[16] JOHN WILEY& SONS,INC, *Topology of Surfaces, Knots, and Manifolds: A First Undergraduate Course*, New York, 1976.

Nature's mighty law is change.

By Robert Burns, a British poet.

Author Information

Submission: Papers only in electronic form are considered for possible publication. Papers prepared in formats tex, dvi, pdf, or ps may be submitted electronically to one member of the Editorial Board for consideration in the **International Journal of Mathematical Combinatorics** (*ISSN 1937-1055*). An effort is made to publish a paper duly recommended by a referee within a period of 3 months. Articles received are immediately put the referees/members of the Editorial Board for their opinion who generally pass on the same in six week's time or less. In case of clear recommendation for publication, the paper is accommodated in an issue to appear next. Each submitted paper is not returned, hence we advise the authors to keep a copy of their submitted papers for further processing. Each published paper is requested to arrange charges 10-15 USD for per page.

Abstract: Authors are requested to provide an abstract of not more than 250 words, latest Mathematics Subject Classification of the American Mathematical Society, Keywords and phrases. Statements of Lemmas, Propositions and Theorems should be set in italics and references should be arranged in alphabetical order by the surname of the first author in the following style:

Books

[4]K. Kawakubo, The Theory of Transformation Groups, Oxford University Press, New York, 1991.

Research papers

[8]K. K. Azad and Gunjan Agrawal, On the projective cover of an orbit space, *J. Austral. Math. Soc.* 46 (1989), 308-312.
[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

Figures: Figures should be sent as: fair copy on paper, whenever possible scaled to about 200%, or as EPS file. In addition, all figures and tables should be numbered and the appropriate space reserved in the text, with the insertion point clearly indicated.

Copyright: It is assumed that the submitted manuscript has not been published and will not be simultaneously submitted or published elsewhere. By submitting a manuscript, the authors agree that the copyright for their articles is transferred to the publisher, if and when, the paper is accepted for publication. The publisher cannot take the responsibility of any loss of manuscript. Therefore, authors are requested to maintain a copy at their end.

Proofs: One set of galley proofs of a paper will be sent to the author submitting the paper, unless requested otherwise, without the original manuscript, for corrections after the paper is accepted for publication on the basis of the recommendation of referees. Corrections should be restricted to typesetting errors. Authors are advised to check their proofs very carefully before return.

Reprints: One copy of the journal included his or her paper(s) are provided to the authors freely. Additional sets may be ordered at the time of proof correction.

April, 2009

Contents

An International Journal on Mathematical Combinatorics

ISSN 1937-1055