The Polynomial Simplest Equations, the Symmetry Point, the Two Simplest Recurrence Equations, and the Method of Differences.

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Abstract.

This study shows some applications of "Shift, Symmetry and Asymmetry in Polynomial Sequences"[5] study. We show the simplest equations for all polynomials up to 6^{th} degree, the symmetry point coordinates, as well as the two possible simplest recurrence equations for each polynomial. We will show how and why the method of differences works conclusively on polynomials, while in any other non-polynomial function the method of differences never ends in a constant. This is a work that shows how polynomials work. It serves as a reference for many future studies and proofs. As an example, at the end we show an application to solve an open problem. Finally, we make a useful summary to be used on a daily basis.

Keywords.

Polynomial Integer Sequences; Recurrence; Symmetry; Asymmetry; Taylor Series.

Introduction.

Following the prove in "Shift, Symmetry and Asymmetry in Polynomial Sequences"[5], every polynomial sequence of integers always has its simplest equation of all. The simplest equation of all is always the one with the smallest (or largest if the sign is reversed) coefficients. It's the same idea as what happens with fractions.

This way, the elements produced at indexes closest to 0 will always be those with the smallest (or largest if the sign is reversed) values of the polynomial sequence. As a consequence of this property, we can find the absolute offset $f = 0$ for any polynomial sequence.

The equation of the offset sequence $f = 0$ is the one that has the polynomial curve in the XY plane with the symmetry point coordinate closest to 0.

From the finite number of elements of any infinite sequence, we will use the method of differences property to differentiate a polynomial function from a non-polynomial function.

1.1. Notation for polynomials in our studies.

We are adopting the following criteria:

- We reserve the use of parentheses ( ) only in the equation formulas.
- To express the functions and derivatives, we substitute parentheses by square brackets [ ].
- To denote data sequences, we use curly brackets { }.
- To differentiate from finite sequences, infinite data sequences begin and/or end with the three dots...
- Generically, we denote any polynomial function element as being $Y[y]$.
  - The reason to use $Y[y]$ is because when we want to draw the polynomial in the XY plane (like GeoGebra), we make x in the function of the index y, or $x = Y[y]$.
- When we want to distinguish or highlight the d^{th} degree of the polynomial, we note $Yd[y]$ or $x = Yd[y]$.
- When we want to make the p^{th} power operation on a d^{th} degree polynomial:
  - if we do not want to mention the degree d of the polynomial: $(Y[y])^p$.
  - if we want to mention the degree d of the polynomial: $(Yd[y])^p$.
- Notation for derivatives:
  - example for 3^{rd} derivative:

$$Y'''[y] = Y^{[3]}[y] = \frac{d^3Y[y]}{dy^3}$$

$$Y'''d[y] = Y^{[3]}d[y] = \frac{d^3Yd[y]}{dy^3}$$

  - example for n^{th} derivative:

$$Y^{[n]}[y] = \frac{d^n Y[y]}{dy^n}$$

$$Y^{[n]}d[y] = \frac{d^n Yd[y]}{dy^n}$$

1.2. The generic polynomial equation.

The generic equation of d^{th} degree polynomial is:

$$x = Yd[y] = a_d y^d + a_{d-1} y^{d-1} + \cdots + a_4 y^4 + a_3 y^3 + a_2 y^2 + a_1 y + a_0$$

Because our studies use quadratics very intensive, we will stand for the generic polynomial equation as:

$$x = Yd[y] = a_d y^d + a_{d-1} y^{d-1} + \cdots + a_4 y^4 + a_3 y^3 + ay^2 + by + c$$

$$a = a_2$$

$$b = a_1$$

$$c = a_0$$

The nomenclature used to express the number of elements needed to form a polynomial is the same nomenclature used in music to define the number of elements in a band: solo, duet, trio, quartet, etc.

One element, the sole element, defines 0^{th} degree polynomial. We express the solo elements as:

$$x = Y0[y] = c \equiv \{x_1\}$$

Two consecutive elements, a duet, define 1^{st} degree polynomial. We express the duet elements as:

$$x = Y1[y] = by + c \equiv \{x_1, x_2\}^1$$

Three consecutive elements, a trio, define 2^{nd} degree polynomial. We express the trios as:

$$x = Y2[y] = ay^2 + by + c \equiv \{x_1, x_2, x_3\}$$

Four consecutive elements, a quartet, define 3^{rd} degree polynomial. We express the quartets as:

$$x = Y3[y] = a_3 y^3 + ay^2 + by + c \equiv \{x_1, x_2, x_3, x_4\}$$

Five consecutive elements, a quintet, define 4^{th} degree polynomial. We express the quintets as:

$$x = Y4[y] = a_4 y^4 + a_3 y^3 + ay^2 + by + c \equiv \{x_1, x_2, x_3, x_4, x_5\}$$

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[1] If we had used parentheses instead of curly brackets, then there could have been confusion with the point notation in the XY plane.

Six consecutive elements, a sextet, define 5th degree polynomial. We express the sextets as:

$$x = Y5[y] = a_5y^5 + a_4y^4 + a_3y^3 + ay^2 + by + c \equiv \{x_1, x_2, x_3, x_4, x_5, x_6\}$$

This continues for septic, octic, nonic, decic, etc.

1.3. Distribution of the elements along the indexes to produce offset $f = 0$.

Following the study "Shift, Symmetry and Asymmetry in Polynomial Sequences"[5], see in the table below which elements must be used to form the simplest equation at offset $f = 0$ according to the d^{th}-degree of the polynomial.

Index y	Polynomial $Yd[y]$	Elements $x = Yd[y]$	$d = 0$ sole	$d = 1$ duet	$d = 2$ trio	$d = 3$ quartet	$d = 4$ quintet	$d = 5$ sextet	$d = 6$
-3	$Yd[-3]$	e							x_1
-2	$Yd[-2]$	f					x_1	x_1	x_2
-1	$Yd[-1]$	g			x_1	x_1	x_2	x_2	x_3
0	$Yd[0]$	h	x_1	x_1	x_2	x_2	x_3	x_3	x_4
1	$Yd[1]$	i		x_2	x_3	x_3	x_4	x_4	x_5
2	$Yd[2]$	j				x_4	x_5	x_5	x_6
3	$Yd[3]$	k						x_6	x_7

Figure 1. Table of the distribution of the elements of the finite sequences along the indexes that form the dth-degrees of the polynomials at offset $f = 0$.

The letters from "e" to "k" represent the elements $x = Yd[y]$ for indexes range $-3 \leq y \leq 3$.

1.4. Notation for differences between two consecutive elements.

Let us denote the differences between two consecutive elements in any polynomial as:

$$dif_1[y] = dif = Yd[y+1] - Yd[y]$$

Then, the differences between the consecutive differences are:

$$dif_2[y] = difdif = dif_1[y+1] - dif_1[y]$$
$$dif_3[y] = difdifdif = dif_2[y+1] - dif_2[y]$$
$$dif_4[y] = difdifdifdif = dif_3[y+1] - dif_3[y]$$
$$\ldots$$
$$dif_h[y] = difdif \ldots h \ldots dif = dif_{h-1}[y+1] - dif_{h-1}[y]$$

1.5. Notation for index direction in any polynomial sequence.

Any polynomial integer sequence has two directions. This is the reason any polynomial has two recurrence equations.

If the direction is:

$$Yd[y] \equiv \{\dots e, f, g, h, i, j, k \dots\} = \backslash\{\dots k, j, i, h, g, f, e \dots\}\backslash$$

Then, the reverse direction is:

$$\backslash Yd[y]\backslash \equiv \{\dots k, j, i, h, g, f, e \dots\} = \backslash\{\dots e, f, g, h, i, j, k \dots\}\backslash$$

In these studies, if an OEIS sequence is $Axxxxxx$, then we write $\backslash Axxxxxx\backslash$ the representation of that sequence in the reverse direction.

2. The simplest polynomial equation.

In the study "Shift, Symmetry and Asymmetry in Polynomial Sequences"[5] we showed the simplest equation only for quadratics. Now, we will show the simplest equations for polynomials up to degree 6th.

2.1. The simplest equation for 0th degree polynomial (constant).

The general polynomial equation of 0th degree, is:

$$x = Y0[y] = c$$

We must find the value of only one coefficient c. Then, to find all the coefficients we must choose the only one element from the $x = Y0[y]$.

Following the study "Shift, Symmetry and Asymmetry in Polynomial Sequences"[5], we can get the simplest equation of all when the central index is $y = 0$. Then,

$$x_1 = Y0[0] = h = c$$

Using Cramer's rule:

$$\Delta = |1| = 1$$

$$\Delta_c = |h| = h$$

$$c = \frac{\Delta_c}{\Delta} = \frac{h}{1} = h$$

The general most simple equation for the polynomial 0th degree is:

$$Y0[y] = h$$

$$x = x_1$$

2.2. The simplest equation for 1st degree polynomial (linear).

The general polynomial equation of 1st degree, is:

$$x = Y1[y] = by + c$$

We must find the value of the 2 coefficients given by b, c. Then, to find all the coefficients we must choose 2 consecutive elements from $x = Y1[y]$.

Following the study "Shift, Symmetry and Asymmetry in Polynomial Sequences"[5], we can get the simplest equation of all when the central index is $y = 0$. Then,

$$x_1 = Y1[0] = h = c$$
$$x_2 = Y1[1] = i = b + c$$

Using Cramer's rule:

$$\Delta = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1$$

$$\Delta_b = \begin{vmatrix} h & 1 \\ i & 1 \end{vmatrix} = h - i$$
$$\Delta_c = \begin{vmatrix} 0 & h \\ 1 & i \end{vmatrix} = -h$$
$$b = \frac{\Delta_b}{\Delta} = \frac{h - i}{-1} = i - h$$
$$c = \frac{\Delta_c}{\Delta} = \frac{-h}{-1} = h$$

The general most simple equation for the polynomial 1st degree is:

$$Y1[y] = (i - h)y + h$$

$$x = (x_2 - x_1)y + x_1$$

2.3. The simplest equation for 2nd degree polynomial (quadratic).

The general polynomial equation of 2nd degree, is:

$$x = Y2[y] = ay^2 + by + c$$

We must find the value of the 3 coefficients given by a, b, c. Then, to find all the coefficients we must choose 3 consecutive elements from $x = Y2[y]$.

Following the study "Shift, Symmetry and Asymmetry in Polynomial Sequences"[5], we can get the simplest equation of all when the central index is $y = 0$. Then,

$$x_1 = Y2[-1] = g = a(-1)^2 + b(-1) + c$$
$$x_2 = Y2[0] = h = a(0)^2 + b(0) + c$$
$$x_3 = Y2[1] = i = a(1)^2 + b(1) + c$$

Or,

$$x_1 = Y2[-1] = g = a - b + c$$
$$x_2 = Y2[0] = h = c$$
$$x_3 = Y2[1] = i = a + b + c$$

Using Cramer's rule:

$$\Delta = \begin{vmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -2$$

$$\Delta_a = \begin{vmatrix} g & -1 & 1 \\ h & 0 & 1 \\ i & 1 & 1 \end{vmatrix} = -g + 2h - i$$

$$\Delta_b = \begin{vmatrix} 1 & g & 1 \\ 0 & h & 1 \\ 1 & i & 1 \end{vmatrix} = g - i$$

$$\Delta_c = \begin{vmatrix} 1 & -1 & g \\ 0 & 0 & h \\ 1 & 1 & i \end{vmatrix} = -2h$$

Then,

$$a = \frac{\Delta_a}{\Delta} = \frac{-g + 2h - i}{-2} = \frac{g - 2h + i}{2}$$
$$b = \frac{\Delta_b}{\Delta} = \frac{g - i}{-2} = \frac{-g + i}{2}$$
$$c = \frac{\Delta_c}{\Delta} = \frac{-2h}{-2} = h$$

The general most simple equation for the 2nd degree polynomial is:

$$Y2[y] = \frac{g - 2h + i}{2} y^2 + \frac{-g + i}{2} y + h$$

$$x = \frac{x_1 - 2x_2 + x_3}{2} y^2 + \frac{x_3 - x_2}{2} y + x_2$$

2.4. The simplest equation for 3rd degree polynomial (cubic).

The general polynomial equation of 3rd degree, is:

$$x = Y3[y] = a_3 y^3 + a y^2 + b y + c$$

We must find the value of the 4 coefficients given by a_3, a, b, c. Then, to find all the coefficients we must choose 4 consecutive elements from $x = Y3[y]$.

Following the study "Shift, Symmetry and Asymmetry in Polynomial Sequences"[5], we can get the simplest equation of all when the central index is $y = 0$. Then,

$$
\begin{aligned}
x_1 &= Y3[-1] = g = a_3(-1)^3 + a(-1)^2 + b(-1) + c \\
x_2 &= Y3[0] = h = c \\
x_3 &= Y3[1] = i = a_3 + a + b + c \\
x_4 &= Y3[2] = j = a_3(2)^3 + a(2)^2 + b(2) + c
\end{aligned}
$$

Then,

$$
\begin{aligned}
x_1 &= Y3[-1] = g = -a_3 + a - b + c \\
x_2 &= Y3[0] = h = c \\
x_3 &= Y3[1] = i = a_3 + a + b + c \\
x_4 &= Y3[2] = j = 8a_3 + 4a + 2b + c
\end{aligned}
$$

Using Cramer's rule:

$$
\Delta = \begin{vmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \end{vmatrix} = 12
$$

$$
\Delta_{a_3} = \begin{vmatrix} g & 1 & -1 & 1 \\ h & 0 & 0 & 1 \\ i & 1 & 1 & 1 \\ j & 4 & 2 & 1 \end{vmatrix} = -2g + 6h - 6i + 2j
$$

$$
\Delta_a = \begin{vmatrix} -1 & g & -1 & 1 \\ 0 & h & 0 & 1 \\ 1 & i & 1 & 1 \\ 8 & j & 2 & 1 \end{vmatrix} = 6g - 12h + 6i
$$

$$
\Delta_b = \begin{vmatrix} -1 & 1 & g & 1 \\ 0 & 0 & h & 1 \\ 1 & 1 & i & 1 \\ 8 & 4 & j & 1 \end{vmatrix} = -4g - 6h + 12i - 2j
$$

$$
\Delta_c = \begin{vmatrix} -1 & 1 & -1 & g \\ 0 & 0 & 0 & h \\ 1 & 1 & 1 & i \\ 8 & 4 & 2 & j \end{vmatrix} = 12h
$$

Then,

$$
\begin{aligned}
a_3 &= \frac{\Delta_{a_3}}{\Delta} = \frac{-2g + 6h - 6i + 2j}{12} = \frac{-g + 3h - 3i + j}{6} \\
a &= \frac{\Delta_a}{\Delta} = \frac{6g - 12h + 6i}{12} = \frac{g - 2h + i}{2} \\
b &= \frac{\Delta_b}{\Delta} = \frac{-4g - 6h + 12i - 2j}{12} = \frac{-2g - 3h + 6i - j}{6} \\
c &= \frac{\Delta_c}{\Delta} = \frac{12h}{12} = h
\end{aligned}
$$

The general most simple equation for the 3rd degree polynomial is:

$$Y3[y] = \frac{-g+3h-3i+j}{6}y^3 + \frac{g-2h+i}{2}y^2 + \frac{-2g-3h+6i-j}{6}y + h$$

$$x = \frac{-x_1+3x_2-3x_3+x_4}{6}y^3 + \frac{x_1-2x_2+x_3}{2}y^2 + \frac{-2x_1-3x_2+6x_3-x_4}{6}y + x_2$$

2.5. The simplest equation for 4th degree polynomial (quartic).

The general polynomial equation of 4th degree, is:

$$x = Y4[y] = a_4y^4 + a_3y^3 + ay^2 + by + c$$

We must find the value of the 5 coefficients given by a_4, a_3, a, b, c. Then, to find all the coefficients we must choose 5 consecutive elements from $x = Y4[y]$.

Following the study "Shift, Symmetry and Asymmetry in Polynomial Sequences"[5], we can get the simplest equation of all when the central index is $y = 0$. Then,

$$\begin{aligned}
x_1 &= Y4[-2] = f = a_4(-2)^4 + a_3(-2)^3 + a(-2)^2 + b(-2) + c \\
x_2 &= Y4[-1] = g = a_4(-1)^4 + a_3(-1)^3 + a(-1)^2 + b(-1) + c \\
x_3 &= Y4[0] = h = c \\
x_4 &= Y4[1] = i = a_4 + a_3 + a + b + c \\
x_5 &= Y4[2] = j = a_4(2)^4 + a_3(2)^3 + a(2)^2 + b(2) + c
\end{aligned}$$

Then,

$$\begin{aligned}
x_1 &= f = 16a_4 - 8a_3 + 4a - 2b + c \\
x_2 &= g = a_4 - a_3 + a - b + c \\
x_3 &= h = c \\
x_4 &= i = a_4 + a_3 + a + b + c \\
x_5 &= j = 16a_4 + 8a_3 + 4a + 2b + c
\end{aligned}$$

Using Cramer's rule:

$$\Delta = \begin{vmatrix} 16 & -8 & 4 & -2 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 16 & 8 & 4 & 2 & 1 \end{vmatrix} = 288$$

$$\Delta_{a_4} = \begin{vmatrix} f & -8 & 4 & -2 & 1 \\ g & -1 & 1 & -1 & 1 \\ h & 0 & 0 & 0 & 1 \\ i & 1 & 1 & 1 & 1 \\ j & 8 & 4 & 2 & 1 \end{vmatrix} = 12f - 48g + 72h - 48i + 12j$$

$$\Delta_{a_3}=\begin{vmatrix}16 & f & 4 & -2 & 1\\ 1 & g & 1 & -1 & 1\\ 0 & h & 0 & 0 & 1\\ 1 & i & 1 & 1 & 1\\ 16 & j & 4 & 2 & 1\end{vmatrix}=-24f+48g-48i+24j$$

$$\Delta_{a}=\begin{vmatrix}16 & -8 & f & -2 & 1\\ 1 & -1 & g & -1 & 1\\ 0 & 0 & h & 0 & 1\\ 1 & 1 & i & 1 & 1\\ 16 & 8 & j & 2 & 1\end{vmatrix}=-12f+192g-360h+192i-12j$$

$$\Delta_{b}=\begin{vmatrix}16 & -8 & 4 & f & 1\\ 1 & -1 & 1 & g & 1\\ 0 & 0 & 0 & h & 1\\ 1 & 1 & 1 & i & 1\\ 16 & 8 & 4 & j & 1\end{vmatrix}=24f-192g+192i-24j$$

$$\Delta_{c}=\begin{vmatrix}16 & -8 & 4 & -2 & f\\ 1 & -1 & 1 & -1 & g\\ 0 & 0 & 0 & 0 & h\\ 1 & 1 & 1 & 1 & i\\ 16 & 8 & 4 & 2 & j\end{vmatrix}=288h$$

Then,

$$a_4=\frac{\Delta_{a_4}}{\Delta}=\frac{12f-48g+72h-48i+12j}{288}=\frac{f-4g+6h-4i+j}{24}$$

$$a_3=\frac{\Delta_{a_3}}{\Delta}=\frac{-24f+48g-48i+24j}{288}=\frac{-f+2g-2i+j}{12}$$

$$a=\frac{\Delta_{a}}{\Delta}=\frac{-12f+192g-360h+192i-12j}{288}=\frac{-f+16g-30h+16i-j}{24}$$

$$b=\frac{\Delta_{b}}{\Delta}=\frac{24f-192g+192i-24j}{288}=\frac{f-8g+8i-j}{12}$$

$$c=\frac{\Delta_{c}}{\Delta}=\frac{288h}{288}=h$$

The general most simple equation for the 4th degree polynomial is:

$$Y4[y]=\frac{f-4g+6h-4i+j}{24}y^4+\frac{-f+2g-2i+j}{12}y^3+\frac{-f+16g-30h+16i-j}{24}y^2+\frac{f-8g+8i-j}{12}y+h$$

$$x=\frac{x_1-4x_2+6x_3-4x_4+x_5}{24}y^4+\frac{-x_1+2x_2-2x_4+x_5}{12}y^3+\frac{-x_1+16x_2-30x_3+16x_4-x_5}{24}y^2+\frac{x_1-8x_2+8x_4-x_5}{12}y+x_3$$

2.6. The simplest equation for 5th degree polynomial (quintic).

The general polynomial equation of 5th degree, is:

$$x = Y5[y] = a_5 y^5 + a_4 y^4 + a_3 y^3 + a y^2 + b y + c$$

We must find the value of the 6 coefficients given by a_5, a_4, a_3, a, b, c. Then, to find all the coefficients we must choose 6 consecutive elements from $x = Y5[y]$.

Following the study "Shift, Symmetry and Asymmetry in Polynomial Sequences"[5], we can get the simplest equation of all when the central index is $y = 0$. Then,

$$\begin{aligned}
x_1 &= Y5[-2] = f = a_5(-2)^5 + a_4(-2)^4 + a_3(-2)^3 + a(-2)^2 + b(-2) + c \\
x_2 &= Y5[-1] = g = a_5(-1)^5 + a_4(-1)^4 + a_3(-1)^3 + a(-1)^2 + b(-1) + c \\
x_3 &= Y5[0] = h = c \\
x_4 &= Y5[1] = i = a_5 + a_4 + a_3 + a + b + c \\
x_5 &= Y5[2] = j = a_5(2)^5 + a_4(2)^4 + a_3(2)^3 + a(2)^2 + b(2) + c \\
x_6 &= Y5[3] = k = a_5(3)^5 + a_4(3)^4 + a_3(3)^3 + a(3)^2 + b(3) + c
\end{aligned}$$

We have a linear system:

$$\begin{aligned}
-32a_5 + 16a_4 - 8a_3 + 4a - 2b + c &= f = x_1 \\
-a_5 + a_4 - a_3 + a - b + c &= g = x_2 \\
c &= h = x_3 \\
a_5 + a_4 + a_3 + a + b + c &= i = x_4 \\
32a_5 + 16a_4 + 8a_3 + 4a + 2b + c &= j = x_5 \\
243a_5 + 81a_4 + 27a_3 + 9a + 3b + c &= k = x_6
\end{aligned}$$

Using Cramer's rule:

$$\Delta = \begin{vmatrix} -32 & 16 & -8 & 4 & -2 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 32 & 16 & 8 & 4 & 2 & 1 \\ 243 & 81 & 27 & 9 & 3 & 1 \end{vmatrix} = -34560$$

$$\Delta_{a_5} = \begin{vmatrix} f & 16 & -8 & 4 & -2 & 1 \\ g & 1 & -1 & 1 & -1 & 1 \\ h & 0 & 0 & 0 & 0 & 1 \\ i & 1 & 1 & 1 & 1 & 1 \\ j & 16 & 8 & 4 & 2 & 1 \\ k & 81 & 27 & 9 & 3 & 1 \end{vmatrix} = 288f - 1440g + 2880h - 2880i + 1440j - 288k$$

$$\Delta_{a_4}=\begin{vmatrix} -32 & f & -8 & 4 & -2 & 1 \\ -1 & g & -1 & 1 & -1 & 1 \\ 0 & h & 0 & 0 & 0 & 1 \\ 1 & i & 1 & 1 & 1 & 1 \\ 32 & j & 8 & 4 & 2 & 1 \\ 243 & k & 27 & 9 & 3 & 1 \end{vmatrix} = -1440f + 5760g - 8640h + 5760\text{i} - 1440j$$

$$\Delta_{a_3}=\begin{vmatrix} -32 & 16 & f & 4 & -2 & 1 \\ -1 & 1 & g & 1 & -1 & 1 \\ 0 & 0 & h & 0 & 0 & 1 \\ 1 & 1 & i & 1 & 1 & 1 \\ 32 & 16 & j & 4 & 2 & 1 \\ 243 & 81 & k & 9 & 3 & 1 \end{vmatrix}$$

$$= 1440f + 1440g - 14400h + 20160\text{i} - 10080j + 1440k$$

$$\Delta_{a}=\begin{vmatrix} -32 & 16 & -8 & f & -2 & 1 \\ -1 & 1 & -1 & g & -1 & 1 \\ 0 & 0 & 0 & h & 0 & 1 \\ 1 & 1 & 1 & i & 1 & 1 \\ 32 & 16 & 8 & j & 2 & 1 \\ 243 & 81 & 27 & k & 3 & 1 \end{vmatrix} = 1440f - 23040g + 43200h - 23040\text{i} + 1440j$$

$$\Delta_{b}=\begin{vmatrix} -32 & 16 & -8 & 4 & f & 1 \\ -1 & 1 & -1 & 1 & g & 1 \\ 0 & 0 & 0 & 0 & h & 1 \\ 1 & 1 & 1 & 1 & i & 1 \\ 32 & 16 & 8 & 4 & j & 1 \\ 243 & 81 & 27 & 9 & k & 1 \end{vmatrix}$$

$$= -1728f + 17280g + 11520h - 34560\text{i} + 8640j - 1152k$$

$$\Delta_{c}=\begin{vmatrix} -32 & 16 & -8 & 4 & -2 & f \\ -1 & 1 & -1 & 1 & -1 & g \\ 0 & 0 & 0 & 0 & 0 & h \\ 1 & 1 & 1 & 1 & 1 & i \\ 32 & 16 & 8 & 4 & 2 & j \\ 243 & 81 & 27 & 9 & 3 & k \end{vmatrix} = -34560h$$

Then,

$$a_5 = \frac{\Delta_{a_5}}{\Delta} = \frac{288f - 1440g + 2880h - 2880i + 1440j - 288k}{-34560}$$

$$= \frac{-f + 5g - 10h + 10i - 5j + k}{120}$$

$$a_4 = \frac{\Delta_{a_4}}{\Delta} = \frac{-1440f + 5760g - 8640h + 5760\text{i} - 1440j}{-34560} = \frac{f - 4g + 6h - 4\text{i} + j}{24}$$

$$a_3 = \frac{\Delta_{a_3}}{\Delta} = \frac{1440f + 1440g - 14400h + 20160\text{i} - 10080j + 1440k}{-34560}$$

$$= \frac{-f - g + 10h - 14\text{i} + 7j - k}{24}$$

$$a = \frac{\Delta_a}{\Delta} = \frac{1440f - 23040g + 43200h - 23040\mathrm{i} + 1440j}{-34560} = \frac{-f + 16g - 30h + 16\mathrm{i} - j}{24}$$

$$b = \frac{\Delta_b}{\Delta} = \frac{-1728f + 17280g + 11520h - 34560\mathrm{i} + 8640j - 1152k}{-34560}$$

$$= \frac{3f - 30g - 20h + 60\mathrm{i} - 15j + 2k}{60}$$

$$c = \frac{\Delta_c}{\Delta} = \frac{-34560h}{-34560} = h$$

The general most simple equation for the 5th degree polynomial is:

$$Y5[y] = \frac{-f + 5g - 10h + 10i - 5j + k}{120}y^5 + \frac{f - 4g + 6h - 4i + j}{24}y^4 + \frac{-f - g + 10h - 14i + 7j - k}{24}y^3 + \frac{-f + 16g - 30h + 16i - j}{24}y^2 + \frac{3f - 30g - 20h + 60i - 15j + 2k}{60}y + h$$

$$x = \frac{-x_1 + 5x_2 - 10x_3 + 10x_4 - 5x_5 + x_6}{120}y^5 + \frac{x_1 - 4x_2 + 6x_3 - 4x_4 + x_5}{24}y^4 + \frac{-x_1 - x_2 + 10x_3 - 14x_4 + 7x_5 - x_6}{24}y^3 + \frac{-x_1 + 16x_2 - 30x_3 + 16x_4 - x_5}{24}y^2 + \frac{3x_1 - 30x_2 - 20x_3 + 60x_4 - 15x_5 + 2x_6}{60}y + x_3$$

2.7. The simplest equation for 6th degree polynomial (sextic).

The general polynomial equation of 6th degree, is:

$$x = Y6[y] = a_6y^6 + a_5y^5 + a_4y^4 + a_3y^3 + ay^2 + by + c$$

We must find the value of the 7 coefficients given by $a_6, a_5, a_4, a_3, a, b, c$. Then, to find all the coefficients we must choose 7 consecutive elements from $x = Y6[y]$.

Following the study "Shift, Symmetry and Asymmetry in Polynomial Sequences"[5], we can get the simplest equation of all when the central index is $y = 0$. Then,

$$x_1 = Y6[-3] = e = a_6(-3)^6 + a_5(-3)^5 + a_4(-3)^4 + a_3(-3)^3 + a(-3)^2 + b(-3) + c$$
$$x_2 = Y6[-2] = f = a_6(-2)^6 + a_5(-2)^5 + a_4(-2)^4 + a_3(-2)^3 + a(-2)^2 + b(-2) + c$$
$$x_3 = Y6[-1] = g = a_6(-1)^6 + a_5(-1)^5 + a_4(-1)^4 + a_3(-1)^3 + a(-1)^2 + b(-1) + c$$
$$x_4 = Y6[0] = h = c$$
$$x_5 = Y6[1] = i = a_6 + a_5 + a_4 + a_3 + a + b + c$$
$$x_6 = Y6[2] = j = a_6(2)^6 + a_5(2)^5 + a_4(2)^4 + a_3(2)^3 + a(2)^2 + b(2) + c$$
$$x_7 = Y6[3] = k = a_6(3)^6 + a_5(3)^5 + a_4(3)^4 + a_3(3)^3 + a(3)^2 + b(3) + c$$

We have a linear system:

$$x_1 = Y6[-3] = e = 729a_6 - 243a_5 + 81a_4 - 27a_3 + 9a - 3b + c$$
$$x_2 = Y6[-2] = f = 64a_6 - 32a_5 + 16a_4 - 8a_3 + 4a - 2b + c$$
$$x_3 = Y6[-1] = g = a_6 - a_5 + a_4 - a_3 + a - b + c$$
$$x_4 = Y6[0] = h = c$$
$$x_5 = Y6[1] = i = a_6 + a_5 + a_4 + a_3 + a + b + c$$
$$x_6 = Y6[2] = j = 64a_6 + 32a_5 + 16a_4 + 8a_3 + 4a + 2b + c$$
$$x_7 = Y6[3] = k = 729a_6 + 243a_5 + 81a_4 + 27a_3 + 9a + 3b + c$$

Using Cramer's rule:

$$\Delta = \begin{vmatrix} 729 & -243 & 81 & -27 & 9 & -3 & 1 \\ 64 & -32 & 16 & -8 & 4 & -2 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 64 & 32 & 16 & 8 & 4 & 2 & 1 \\ 729 & 243 & 81 & 27 & 9 & 3 & 1 \end{vmatrix} = -24883200$$

$$\Delta_{a_6} = \begin{vmatrix} e & -243 & 81 & -27 & 9 & -3 & 1 \\ f & -32 & 16 & -8 & 4 & -2 & 1 \\ g & -1 & 1 & -1 & 1 & -1 & 1 \\ h & 0 & 0 & 0 & 0 & 0 & 1 \\ i & 1 & 1 & 1 & 1 & 1 & 1 \\ j & 32 & 16 & 8 & 4 & 2 & 1 \\ k & 243 & 81 & 27 & 9 & 3 & 1 \end{vmatrix}$$
$$= -34560\mathrm{e} + 207360f - 518400g + 691200h - 518400\mathrm{i} + 207360j - 34560k$$

$$\Delta_{a_5} = \begin{vmatrix} 729 & e & 81 & -27 & 9 & -3 & 1 \\ 64 & f & 16 & -8 & 4 & -2 & 1 \\ 1 & g & 1 & -1 & 1 & -1 & 1 \\ 0 & h & 0 & 0 & 0 & 0 & 1 \\ 1 & i & 1 & 1 & 1 & 1 & 1 \\ 64 & j & 16 & 8 & 4 & 2 & 1 \\ 729 & k & 81 & 27 & 9 & 3 & 1 \end{vmatrix}$$
$$= 103680\mathrm{e} - 414720f + 518400g - 518400\mathrm{i} + 414720j - 103680k$$

$$\Delta_{a_4} = \begin{vmatrix} 729 & -243 & e & -27 & 9 & -3 & 1 \\ 64 & -32 & f & -8 & 4 & -2 & 1 \\ 1 & -1 & g & -1 & 1 & -1 & 1 \\ 0 & 0 & h & 0 & 0 & 0 & 1 \\ 1 & 1 & i & 1 & 1 & 1 & 1 \\ 64 & 32 & j & 8 & 4 & 2 & 1 \\ 729 & 243 & k & 27 & 9 & 3 & 1 \end{vmatrix}$$
$$= 172800\mathrm{e} - 2073600f + 6739200g - 9676800h + 6739200\mathrm{i} - 2073600j + 172800k$$

$$\Delta_{a_3}=\begin{vmatrix}729 & -243 & 81 & e & 9 & -3 & 1\\ 64 & -32 & 16 & f & 4 & -2 & 1\\ 1 & -1 & 1 & g & 1 & -1 & 1\\ 0 & 0 & 0 & h & 0 & 0 & 1\\ 1 & 1 & 1 & i & 1 & 1 & 1\\ 64 & 32 & 16 & j & 4 & 2 & 1\\ 729 & 243 & 81 & k & 9 & 3 & 1\end{vmatrix}$$

$$= -518400e + 4147200f - 6739200g + 6739200\mathrm{i} - 4147200j + 518400k$$

$$\Delta_{a}=\begin{vmatrix}729 & -243 & 81 & -27 & e & -3 & 1\\ 64 & -32 & 16 & -8 & f & -2 & 1\\ 1 & -1 & 1 & -1 & g & -1 & 1\\ 0 & 0 & 0 & 0 & h & 0 & 1\\ 1 & 1 & 1 & 1 & i & 1 & 1\\ 64 & 32 & 16 & 8 & j & 2 & 1\\ 729 & 243 & 81 & 27 & k & 3 & 1\end{vmatrix}$$

$$= -138240\mathrm{e} + 1866240f - 18662400g + 33868800h - 18662400\mathrm{i} + 1866240j - 138240k$$

$$\Delta_{b}=\begin{vmatrix}729 & -243 & 81 & -27 & 9 & e & 1\\ 64 & -32 & 16 & -8 & 4 & f & 1\\ 1 & -1 & 1 & -1 & 1 & g & 1\\ 0 & 0 & 0 & 0 & 0 & h & 1\\ 1 & 1 & 1 & 1 & 1 & i & 1\\ 64 & 32 & 16 & 8 & 4 & j & 1\\ 729 & 243 & 81 & 27 & 9 & k & 1\end{vmatrix}$$

$$= 414720\mathrm{e} - 3732480f + 18662400g - 18662400\mathrm{i} + 3732480j - 414720k$$

$$\Delta_{c}=\begin{vmatrix}729 & -243 & 81 & -27 & 9 & -3 & e\\ 64 & -32 & 16 & -8 & 4 & -2 & f\\ 1 & -1 & 1 & -1 & 1 & -1 & g\\ 0 & 0 & 0 & 0 & 0 & 0 & h\\ 1 & 1 & 1 & 1 & 1 & 1 & i\\ 64 & 32 & 16 & 8 & 4 & 2 & j\\ 729 & 243 & 81 & 27 & 9 & 3 & k\end{vmatrix} = -24883200h$$

$$a_6 = \frac{\Delta_{a_6}}{\Delta} = \frac{-34560\mathrm{e} + 207360f - 518400g + 691200h - 518400\mathrm{i} + 207360j - 34560k}{-24883200}$$

$$a_5 = \frac{\Delta_{a_5}}{\Delta} = \frac{103680\mathrm{e} - 414720f + 518400g - 518400\mathrm{i} + 414720j - 103680k}{-24883200}$$

$$a_4 = \frac{\Delta_{a_4}}{\Delta} = \frac{172800\mathrm{e} - 2073600f + 6739200g - 9676800h + 6739200\mathrm{i} - 2073600j + 172800k}{-24883200}$$

$$a_3 = \frac{\Delta_{a_3}}{\Delta} = \frac{-518400e + 4147200f - 6739200g + 6739200\text{i} - 4147200j + 518400k}{-34560}$$

$$a = \frac{\Delta_a}{\Delta}$$

$$= \frac{-138240\text{e} + 1866240f - 18662400g + 33868800h - 18662400\text{i} + 1866240j - 138240k}{-34560}$$

$$b = \frac{\Delta_b}{\Delta} = \frac{414720\text{e} - 3732480f + 18662400g - 18662400\text{i} + 3732480j - 414720k}{-34560}$$

$$c = \frac{\Delta_c}{\Delta} = \frac{-24883200h}{-24883200}$$

Or

$$a_6 = \frac{\Delta_{a_6}}{\Delta} = \frac{e - 6f + 15g - 20h + 15\text{i} - 6j + k}{720}$$

$$a_5 = \frac{\Delta_{a_5}}{\Delta} = \frac{-e + 4f - 5g + 5\text{i} - 4j + k}{240}$$

$$a_4 = \frac{\Delta_{a_4}}{\Delta} = \frac{-e + 12f - 39g + 56h - 39\text{i} + 12j - k}{144}$$

$$a_3 = \frac{\Delta_{a_3}}{\Delta} = \frac{e - 8f + 13g - 13\text{i} + 8j - k}{48}$$

$$a = \frac{\Delta_a}{\Delta} = \frac{2e - 27f + 270g - 490h + 270\text{i} - 27j + 2k}{360} =$$

$$b = \frac{\Delta_b}{\Delta} = \frac{-e + 9f - 45g + 45\text{i} - 9j + k}{60}$$

$$c = \frac{\Delta_c}{\Delta} = h$$

The general most simple equation for the 6th degree polynomial is:

$$Y6[y] = \frac{e - 6f + 15g - 20h + 15i - 6j + k}{720}y^6 + \frac{-e + 4f - 5g + 5i - 4j + k}{240}y^5$$
$$+ \frac{-e + 12f - 39g + 56h - 39i + 12j - k}{144}y^4$$
$$+ \frac{e - 8f + 13g - 13i + 8j - k}{48}y^3$$
$$+ \frac{2e - 27f + 270g - 490h + 270i - 27j + 2k}{360}y^2$$
$$+ \frac{-e + 9f - 45g + 45i - 9j + k}{60}y + h$$

$$x = \frac{x_1 - 6x_2 + 15x_3 - 20x_4 + 15x_5 - 6x_6 + x_7}{720}y^6$$
$$+ \frac{-x_1 + 4x_2 - 5x_3 + 5x_5 - 4x_6 + x_7}{240}y^5$$
$$+ \frac{-x_1 + 12x_2 - 39x_3 + 56x_4 - 39x_5 + 12x_6 - x_7}{144}y^4$$
$$+ \frac{x_1 - 8x_2 + 13x_3 - 13x_5 + 8x_6 - x_7}{48}y^3$$
$$+ \frac{2x_1 - 27x_2 + 27x_3g - 490x_4 + 270x_5 - 27x_6 + 2x_7}{360}y^2$$
$$+ \frac{-x_1 + 9x_2 - 45x_3 + 45x_5 - 9x_6 + x_7}{60}y + x_4$$

3. Symmetry Point of any Polynomial.

Following the “Shift, Symmetry and Asymmetry in Polynomial Sequences”[5] study's reasoning, we will calculate the symmetry points for all polynomials up to the 6th degree.

3.1. Symmetry Point in 0th degree polynomial.

To get the Y-coordinate of the symmetry point of any 0th degree polynomial function, we have to calculate the value of y where the “first-negative-of-the-derivative” results in 0:

$$\frac{d^{-1}Y0[y]}{dy^{-1}} = 0$$

Since we don't know what a "negative-of-the-derivative" is, the Y-coordinate of the symmetry point is undetermined.

$$y_{sp_{Y0[y]}}\left[@\frac{d^{-1}Y0[y]}{dy^{-1}} = 0\right] = undetermined$$

3.2. Symmetry Point in 1st degree polynomial.

To get the Y-coordinate of the symmetry point of any 1st degree polynomial function, we have to calculate the value of y where the “no-derivative” results in 0:

$$\frac{d^0Y1[y]}{dy^0} = 0$$
$$\frac{d^0(by + c)}{dy^0} = 0$$
$$by_{sp} + c = 0$$

$$y_{sp_{Y1[y]}}\left[@\frac{d^0Y1[y]}{dy^0}=0\right]=-\frac{0!\,c}{1!\,b}=-\left(\frac{h}{i-h}\right)$$

$$y_{sp_{Y1[y]}}\left[@\frac{d^0Y1[y]}{dy^0}=0\right]=-\frac{c}{b}=-\left(\frac{x_1}{x_2-x_1}\right)$$

$$x_{sp}[y]=by_{sp}+c$$

$$x_{sp}[y]=b\left(-\frac{c}{b}\right)+c$$

$$x_{sp}[y]=0$$

$$sp_{Y1[y]}(x,y)=\left(0,-\frac{c}{b}\right)$$

3.3. Symmetry Point in 2nd degree polynomial.

We get the Y-coordinate of the symmetry point of any 2nd degree polynomial function at:

$$\frac{d^1Y2[y]}{dy^1}=0$$

So,

$$\frac{d(ay^2+by+c)}{dy}=0$$

$$2!\,ay_{sp}+1!\,b=0$$

$$y_{sp}=-\frac{b}{2a}=-\frac{1}{2}\left(\frac{\frac{-g+i}{2}}{\frac{g-2h+i}{2}}\right)$$

$$y_{sp_{X2[y]}}\left[@\frac{d^1Y2[y]}{dy^1}=0\right]=-\frac{1!\,b}{2!\,a}=-\frac{1}{2}\left(\frac{\frac{-g+i}{2}}{\frac{g-2h+i}{2}}\right)=-\frac{1}{2}\left(\frac{-g+i}{g-2h+i}\right)$$

$$y_{sp_{X2[y]}}\left[@\frac{d^1Y2[y]}{dy^1}=0\right]=-\frac{b}{2a}=-\frac{1}{2}\left(\frac{-x_1+x_3}{x_1-2x_2+x_3}\right)$$

Then,

$$x_{sp}=ay_{sp}^2+by_{sp}+c$$

$$x_{sp}=a\left(-\frac{b}{2a}\right)^2+b\left(-\frac{b}{2a}\right)+c$$

$$x_{sp} = \frac{b^2}{4a} - \frac{b^2}{2a} + c$$
$$x_{sp} = \frac{b^2 - 2b^2 + 4ac}{4a}$$
$$x_{sp} = -\frac{b^2 - 4ac}{4a}$$
$$sp_{Y2[y]}(x, y) = \left(-\frac{b^2 - 4ac}{4a}, -\frac{b}{2a}\right)$$

3.4. Symmetry Point in 3rd degree polynomial.

We get the Y-coordinate of the symmetry point of any 3rd degree polynomial function at:

$$\frac{d^2Y3[y]}{dy^2} = 0$$
$$\frac{d^2(a_3y^3 + ay^2 + by + c)}{dy^2} = 0$$

Then,

$$3!\,y_{sp} + 2!\,a = 0$$
$$y_{sp_{Y3[y]}}\left[@\frac{d^2Y3[y]}{dy^2} = 0\right] = -\frac{2!\,a}{3!\,a_3} = -\frac{1}{3}\left(\frac{\frac{g - 2h + i}{2}}{\frac{-g + 3h - 3i + j}{6}}\right) = -1\left(\frac{g - 2h + i}{-g + 3h - 3i + j}\right)$$
$$y_{sp_{Y3[y]}}\left[@\frac{d^2Y3[y]}{dy^2} = 0\right] = -\frac{a}{3a_3} = -1\left(\frac{x_1 - 2x_2 + x_3}{-x_1 + 3x_2 - 3x_3 + x_4}\right)$$

Then,

$$x_{sp} = a_3y_{sp}^3 + ay_{sp}^2 + by_{sp} + c$$
$$x_{sp} = a_3\left(-\frac{a}{3a_3}\right)^3 + a\left(-\frac{a}{3a_3}\right)^2 + b\left(-\frac{a}{3a_3}\right) + c$$
$$x_{sp} = -\frac{a_3a^3}{27a_3^3} + \frac{a^3}{9a_3^2} - \frac{ab}{3a_3} + c$$
$$x_{sp} = -\frac{a^3}{27a_3^2} + \frac{a^3}{9a_3^2} - \frac{ab}{3a_3} + c$$
$$x_{sp} = -\frac{a^3}{27a_3^2} + \frac{3a^3}{27a_3^2} - \frac{9a_3ab}{27a_3^2} + c$$

$$x_{sp_{Y3[y]}} = \frac{2a^3 - 9a_3ab}{27a_3^2} + c$$

$$sp_{Y3[y]}(x,y) = \left[\frac{2a^3 - 9a_3ab}{27a_3^2} + c, -\frac{a}{3a_3} = -1\left(\frac{g - 2h + i}{-g + 3h - 3i + j}\right)\right]$$

3.5. Symmetry Point in 4th degree polynomial.

We get the Y-coordinate of the symmetry point of any 4th degree polynomial function at:

$$\frac{d^3Y4[y]}{dy^3} = 0$$

$$\frac{d^3(a_4y^4 + a_3y^3 + ay^2 + by + c)}{dy^3} = 0$$

$$4!\,a_4y_{ip} + 3!\,a_3 = 0$$

$$y_{sp_{Y4[y]}}\left[@\frac{d^3Y4[y]}{dy^3} = 0\right] = -\frac{3!\,a_3}{4!\,a_4} = -\frac{a_3}{4a_4} = -\frac{1}{4}\left(\frac{\frac{-f + 2g - 2i + j}{12}}{\frac{f - 4g + 6h - 4i + j}{24}}\right)$$

$$= -\frac{1}{2}\left(\frac{-f + 2g - 2i + j}{f - 4g + 6h - 4i + j}\right)$$

$$y_{sp_{Y4[y]}}\left[@\frac{d^3Y4[y]}{dy^3} = 0\right] = -\frac{a_3}{4a_4} = -\frac{1}{2}\left(\frac{-x_1 + 2x_2 - 2x_4 + x_5}{x_1 - 4x_2 + 6x_3 - 4x_4 + x_5}\right)$$

3.6. Symmetry Point in 5th degree polynomial.

We get the Y-coordinate of the symmetry point of any 5th degree polynomial function at:

$$\frac{d^4Y5[y]}{dy^4} = 0$$

$$\frac{d^4(a_5y^5 + a_4y^4 + a_3y^3 + ay^2 + by + c)}{dy^4} = 0$$

$$5!\,a_5y_{sp} + 4!\,a_4 = 0$$

$$y_{sp_{Y5[y]}}\left[@\frac{d^4Y5[y]}{dy^4} = 0\right] = -\frac{4!\,a_4}{5!\,a_5} = -\frac{1}{5}\left(\frac{\frac{f - 4g + 6h - 4i + j}{24}}{\frac{-f + 5g - 10h + 10i - 5j + k}{120}}\right)$$

$$= -1\left(\frac{f - 4g + 6h - 4i + j}{-f + 5g - 10h + 10i - 5j + k}\right)$$

$$y_{sp_{Y5[y]}}\left[@\frac{d^4Y5[y]}{dy^4}=0\right]=-\frac{a_4}{5a_5}=-1\left(\frac{x_1-4x_2+6x_3-4x_4+x_5}{-x_1+5x_2-10x_3+10x_4-5x_5+x_6}\right)$$

3.7. Symmetry Point in 6th degree polynomial.

We get the Y-coordinate of the symmetry point of any 6th degree polynomial function at:

$$\frac{d^5Y6[y]}{dy^5}=0$$

$$\frac{d^5(a_6y^6+a_5y^5+a_4y^4+a_3y^3+ay^2+by+c)}{dy^5}=0$$

$$6!\,a_6y_{sp}+5!\,a_5=0$$

$$y_{sp_{Y6[y]}}\left[@\frac{d^5Y6[y]}{dy^5}=0\right]=-\frac{5!\,a_5}{6!\,a_6}=-\frac{1}{6}\left(\frac{\frac{-e+4f-5g+5\mathrm{i}-4j+k}{240}}{\frac{e-6f+15g-20h+15\mathrm{i}-6j+k}{720}}\right)$$

$$=-\frac{1}{2}\left(\frac{-e+4f-5g+5\mathrm{i}-4j+k}{e-6f+15g-20h+15\mathrm{i}-6j+k}\right)$$

$$y_{sp_{Y6[y]}}\left[@\frac{d^5Y6[y]}{dy^5}=0\right]=-\frac{a_5}{6a_6}=-\frac{1}{2}\left(\frac{-x_1+4x_2-5x_3+5x_5-4x_6+x_7}{x_1-6x_2+15x_3-20x_4+15x_5-6x_6+x_7}\right)$$

4. Recurrence Equations of any Polynomial.

Because the same recurrence equation can generate infinitely many different polynomial sequences with the same degree, then all recurrences equations should be written in such a way that considers at least any smallest part of the sequence (in any index or any offset).

The minimum number of consecutive elements needed to find the recurrence formula of a polynomial $x=Yd[y]$ of d^{th} degree is $d+1$. This is a direct consequence that each d^{th} degree polynomial has $d+1$ coefficients in its equation.

Because we showed in "Shift, Symmetry and Asymmetry in Polynomial Sequences"[5] that every polynomial has an offset equation $f=0$, here also we will use the smallest possible elements to find the simplest recurrence formulas for polynomials. We'll do this in both directions of the indexes.

We could also obtain the simplest recurrence formulas by directly using the simplest formulas of the polynomial equations obtained above. But so that this study does not take too long, anyone who is curious can consult [9].

4.1. Recurrence algorithm for 1st degree polynomial in the increasing direction of the indexes.

The general formula of a 1st degree polynomial is:

$$x = Y1[y] = by + c$$

In the upward direction from index 0 we have:

$$x_1 = Y1[0] = c$$
$$x_2 = Y1[1] = b + c$$
$$x_3 = Y1[2] = 2b + c$$

The recurrence algorithm in the increasing direction of the indexes is:

$$x_3 = -x_1 + 2x_2$$

4.2. Recurrence algorithm for 1st degree polynomial in the decreasing direction of the indexes.

In the downward direction from index 0 we have:

$$x_0 = Y1[-1] = -b + c$$
$$x_1 = Y1[0] = c$$
$$x_2 = Y1[1] = b + c$$

The recurrence algorithm in the decreasing direction of the indexes is:

$$x_0 = 2x_1 - x_2$$

4.3. Recurrence algorithm for 2nd degree polynomial in the increasing direction of the indexes.

The general formula of a 2nd degree polynomial is:

$$x = Y2[y] = ay^2 + by + c$$

In the upward direction from index 0 we have:

$$x_1 = Y2[-1] = a - b + c$$
$$x_2 = Y2[0] = c$$
$$x_3 = Y2[1] = a + b + c$$
$$x_4 = Y2[2] = 4a + 2b + c$$

Substituting $Y2[1]$ and $Y2[0]$ in $Y2[2]$,

$$Y2[2] = 4Y2[1] - 2b - 3c$$
$$Y2[2] = 4Y2[1] - 2b - 3Y2[0]$$

If,

$$Y2[-1] = a - b + c$$
$$Y2[1] = a + b + c$$

Then,

$$Y2[-1] - Y2[1] = -2b$$

The value of b:

$$b = \frac{-Y2[-1] + Y2[1]}{2} = \frac{-x_1 + x_3}{2}$$

Substituting $-2b$ in $Y2[2]$,

$$Y2[2] = 4Y2[1] + Y2[-1] - Y2[1] - 3Y2[0]$$

$$Y2[2] = 3Y2[1] + Y2[-1] - 3Y2[0]$$

$$Y2[2] = Y2[-1] - 3Y2[0] + 3Y2[1]$$

The recurrence algorithm in the increasing direction of the indexes is:

$$x_4 = x_1 - 3x_2 + 3x_3$$

4.4. Recurrence algorithm for 2nd degree polynomial in the decreasing direction of the indexes.

In the downward direction from index 0 we have:

$$x_0 = Y2[-2] = 4a - 2b + c$$
$$x_1 = Y2[-1] = a - b + c$$
$$x_2 = Y2[0] = c$$
$$x_3 = Y2[1] = a + b + c$$

Substituting $Y2[1]$ and $Y2[0]$ in $Y2[-2]$,

$$Y2[-2] = 4Y2[1] - 6b - 3c$$
$$Y2[-2] = 4Y2[1] - 6b - 3Y2[0]$$

If,

$$Y2[-1] = a - b + c$$
$$Y2[1] = a + b + c$$

Then,

$$Y2[-1] - Y2[1] = -2b$$
$$3Y2[-1] - 3Y2[1] = -6b$$

Substituting $-6b$ in $Y2[-2]$,

$$Y2[-2] = 4Y2[1] + 3Y2[-1] - 3Y2[1] - 3Y2[0]$$

$$Y2[-2] = Y2[1] + 3Y2[-1] - 3Y2[0]$$

$$Y2[-2] = 3Y2[-1] - 3Y2[0] + Y2[1]$$

The recurrence algorithm in the decreasing direction of the indexes is:

$$x_0 = 3x_1 - 3x_2 + x_3$$

4.5. Recurrence algorithm for 3rd degree polynomial in the increasing direction of the indexes.

The general formula of a 3rd degree polynomial is:

$$x = Y3[y] = a_3y^3 + ay^2 + by + c$$

In the upward direction from index 0 we have:

$$x_1 = Y3[-2] = -8a_3 + 4a - 2b + c$$
$$x_2 = Y3[-1] = -a_3 + a - b + c$$
$$x_3 = Y3[0] = c$$
$$x_4 = Y3[1] = a_3 + a + b + c$$
$$x_5 = Y3[2] = 8a_3 + 4a + 2b + c$$

Putting $Y3[2]$ in function of $Y3[-2]$:

$$Y3[2] = -Y3[-2] + 8a + 2Y3[0]$$

Finding $8a$:

$$Y3[-1] = -a_3 + a - b + c$$
$$Y3[1] = a_3 + a + b + c$$
$$Y3[1] + Y3[-1] = 2a + 2Y3[0]$$
$$2a = Y3[1] + Y3[-1] - 2Y3[0]$$

So, the value of a is:

$$a = \frac{Y3[-1] - 2Y3[0] + Y3[1]}{2} = \frac{x_1 - 2x_2 + x_3}{2}$$

$$8a = 4Y3[-1] - 8Y3[0] + 4Y3[1]$$

Substituting $8a$ in $Y3[2]$:

$$Y3[2] = -Y3[-2] + 4Y3[1] + 4Y3[-1] - 8Y3[0] + 2Y3[0]$$

$$Y3[2] = -Y3[-2] + 4Y3[-1] - 6Y3[0] + 4Y3[1]$$

The recurrence algorithm in the increasing direction of the indexes is:

$$x_5 = -x_1 + 4x_2 - 6x_3 + 4x_4$$

4.6. Recurrence algorithm for 3rd degree polynomial in the decreasing direction of the indexes.

We can put $Y3[-2]$ in function of $Y3[2]$:

$$Y3[-2] = -Y3[2] + 8a + 2Y3[0]$$

Finding $8a$:

$$Y3[-1] = -a_3 + a - b + c$$
$$Y3[1] = a_3 + a + b + c$$
$$Y3[1] + Y3[-1] = 2a + 2Y3[0]$$
$$2a = Y3[1] + Y3[-1] - 2Y3[0]$$

So, the value of a is:

$$a = \frac{Y3[-1] - 2Y3[0] + Y3[1]}{2} = \frac{x_1 - 2x_2 + x_3}{2}$$
$$8a = 4Y3[-1] - 8Y3[0] + 4Y3[1]$$

Substituting $8a$ in $Y3[-2]$:

$$Y3[-2] = -Y3[2] + 4Y3[1] + 4Y3[-1] - 8Y3[0] + 2Y3[0]$$
$$Y3[-2] = 4Y3[-1] - 6Y3[0] + 4Y3[1] - Y3[2]$$

The recurrence algorithm in the decreasing direction of the indexes is:

$$x_0 = 4x_1 - 6x_2 + 4x_3 - x_4$$

4.7. Recurrence algorithm for dth degree polynomial from the Pascal´s Triangle.

The smallest data sequence of a d^{th} degree polynomial sequence is $(d + 1)$ consecutive elements. They are the minimum number of polynomial elements, in any offset, necessary and sufficient to figure out the unique polynomial infinite sequence.

Because all polynomials obey the method of common differences and partial sum, then the same recurrence equation can generate infinitely many different sequences with isomorphic properties. So, all recurrences equations consider at least one smallest size part of the sequence in any offset or index.

The recurrence equation of any polynomial is a direct consequence of the method of common differences.

Because of the polynomial's curve symmetry, we can get the recurrence coefficients directly from the Pascal´s triangle. Each row of Pascal's triangle is palindromic.

Because of the symmetry in Pascal's triangle, we can set up two isomorphic triangular tables. Both tables have a vertical column and a diagonal column of sequences of 1's.

In one table the vertical sequence of 1's is on the left and in the other table the vertical sequence of 1's is on the right, as shown in the figure below:

Row	Pascal's Triangle - vertical 1's to the left											∑
1	1											1
2	1	1										2
3	1	2	1									4
4	1	3	3	1								8
5	1	4	6	4	1							16
6	1	5	10	10	5	1						32
7	1	6	15	20	15	6	1					64
8	1	7	21	35	35	21	7	1				128
9	1	8	28	56	70	56	28	8	1			256
10	1	9	36	84	126	126	84	36	9	1		512
11	1	10	45	120	210	252	210	120	45	10	1	1024

Row	Pascal's Triangle - vertical 1's to the right										
1											1
2										1	1
3									1	2	1
4								1	3	3	1
5							1	4	6	4	1
6						1	5	10	10	5	1
7					1	6	15	20	15	6	1
8				1	7	21	35	35	21	7	1
9			1	8	28	56	70	56	28	8	1
10		1	9	36	84	126	126	84	36	9	1
11	1	10	45	120	210	252	210	120	45	10	1

Figure 2. C001107 The two Pascal's triangle symmetric arrays where each has a vertical column of “1’s”.

Each row of Pascal's triangle has a finite symmetrical sequence in the form of SUB-type or DES-type. The sum of the elements in each row figures out the type of finite sequence.

The sum of Pascal´s triangle rows produces the sequence https://oeis.org/A000079 $2^n = 2^{integer}$.
Rows with sums $2^{2m} = 2^{even}$ https://oeis.org/A000302 are in the form of SUB-type.
Rows with sums $2^{2m-1} = 2^{odd}$ https://oeis.org/A004171 are in the form of DES-type.
https://oeis.org/A000079 = https://oeis.org/A000302 U https://oeis.org/A004171
As we saw in “Shift, Symmetry and Asymmetry in Polynomial Sequences”[5], these very same Pascal triangles are used to obtain the coefficients of the Taylor shift equations.

When we alternately inverse the sign of the elements of the rows of Pascal's triangle, the sum of its rows is reduced to zero:

Row	Pascal's Triangle - vertical 1's to the left and alternating signal											∑
1	-1											-1
2	-1	1										0
3	-1	2	-1									0
4	-1	3	-3	1								0
5	-1	4	-6	4	-1							0
6	-1	5	-10	10	-5	1						0
7	-1	6	-15	20	-15	6	-1					0
8	-1	7	-21	35	-35	21	-7	1				0
9	-1	8	-28	56	-70	56	-28	8	-1			0
10	-1	9	-36	84	-126	126	-84	36	-9	1		0
11	-1	10	-45	120	-210	252	-210	120	-45	10	-1	0

Row	Pascal's Triangle - vertical 1's to the right and alternating signal										
1											-1
2										1	-1
3									-1	2	-1
4								1	-3	3	-1
5							-1	4	-6	4	-1
6						1	-5	10	-10	5	-1
7					-1	6	-15	20	-15	6	-1
8				1	-7	21	-35	35	-21	7	-1
9			-1	8	-28	56	-70	56	-28	8	-1
10		1	-9	36	-84	126	-126	84	-36	9	-1
11	-1	10	-45	120	-210	252	-210	120	-45	10	-1

Figure 3. C001107 Pascal's triangle with alternating signal in each row element. The sum of the elements of each $row > 1$ is 0.

For each polynomial degree, the two recurrence equation coefficients appear when we cut only the vertical column of $\{-1\}$ elements.

Row	Recurrence coefficients towards decreasing index											∑
1	degree											
2	0	1										1
3	1	2	-1									1
4	2	3	-3	1								1
5	3	4	-6	4	-1							1
6	4	5	-10	10	-5	1						1
7	5	6	-15	20	-15	6	-1					1
8	6	7	-21	35	-35	21	-7	1				1
9	7	8	-28	56	-70	56	-28	8	-1			1
10	8	9	-36	84	-126	126	-84	36	-9	1		1
11	9	10	-45	120	-210	252	-210	120	-45	10	-1	1

Row	Recurrence coefficients towards increasing index										
1											degree
2										1	0
3									-1	2	1
4								1	-3	3	2
5							-1	4	-6	4	3
6						1	-5	10	-10	5	4
7					-1	6	-15	20	-15	6	5
8				1	-7	21	-35	35	-21	7	6
9			-1	8	-28	56	-70	56	-28	8	7
10		1	-9	36	-84	126	-126	84	-36	9	8
11	-1	10	-45	120	-210	252	-210	120	-45	10	9

Figure 4. C001107 Each row in the two tables has the two sets of recurrence coefficients, one set for each direction of the index.

Because of the $\{-1\}$ vertical column cut, the sum of the elements in each $row > 1$ is 1.

One table shows the recurrence coefficients in the increasing index direction and the other in the decreasing index direction. Now, the application of the recurrence algorithm no longer depends on the position where consecutive elements will be used. This is because in any range of the polynomial sequence the recurrence algorithm is always the same.

These properties reflect the behavior, or the action, of the method of common differences or the partial sum which is the principle of any polynomial sequence.

So, we can propose two new sequences to the OEIS:

- Coefficients of polynomial recurrence equations with increasing index: C001105 {1, -1, 2, 1, -3, 3, -1, 4, -6, 4, 1, -5, 10, -10, 5, -1, 6, -15, 20, -15, 6, 1, -7, 21, -35, 35, -21, 7, -1, 8, -28, 56, -70, 56, -28, 8, 1, -9, 36, -84, 126, -126, 84, -36, 9, -1, 10, -45, 120, -210, 252, -210, 120, -45, 10, 1, -11, 55, -165, 330, -462, 462, -330, 165, -55, 11, …}. This sequence is based on alternating the sign between the elements of the https://oeis.org/A074909 Running sum of Pascal's triangle (https://oeis.org/A007318), or beheaded Pascal's triangle read by beheaded rows.
- Coefficients of polynomial recurrence equations with decreasing index: C001106 {1, 2, -1, 3, -3, 1, 4, -6, 4, -1, 5, -10, 10, -5, 1, 6, -15, 20, -15, 6, -1, 7, -21, 35, -35, 21, -7, 1, 8, -28, 56, -70, 56, -28, 8, -1, 9, -36, 84, -126, 126, -84, 36, -9, 1, 10, -45, 120, -210, 252, -210, 120, -45, 10, -1, 11, -55, 165, -330, 462, -462, 330, -165, 55, -11, 1, …}. This sequence is based on alternating the sign between the elements from the https://oeis.org/A135278 Triangle read by rows, giving the numbers T(name) = binomial(n+1, m+1); or Pascal's triangle https://oeis.org/A007318 with its left-hand edge removed.

The coefficients of the recurrence equation of the polynomial of degree d are the coefficients of Newton's binomial of power $d+1$ with the exclusion of the binomial $\binom{d+1}{d+1}$ or $\binom{d+1}{0}$.

5. Method of differences in any polynomial.

Let's denote the first difference between consecutive elements in any d^{th} degree polynomial as:

$$dif_1[y] = dif = Yd[y+1] - Yd[y] \neq 0$$

Then, the second difference between the consecutive first differences is:

$$dif_2[y] = difdif = dif_1[y+1] - dif_1[y] \neq 0$$

Then,

$$\begin{gathered} dif_3[y] = difdifdif = dif_2[y+1] - dif_2[y] \neq 0 \\ dif_4[y] = difdifdifdif = dif_3[y+1] - dif_3[y] \neq 0 \\ \dots \\ dif_{n-1}[y] = difdif \dots n-1 \dots dif = dif_{n-2}[y+1] - dif_{n-2}[y] \neq 0 \\ dif_n[y] = difdif \dots n \dots dif = dif_{n-1}[y+1] - dif_{n-1}[y] = 0 \end{gathered}$$

This procedure in polynomials always gets in a result where $dif_n[y] = 0$ for any y.

Then, we know the polynomial has degree $d = n - 1$.

5.1. Method of differences in 1st-degree polynomials (linear equations).

Given

$$Y1[y] = by + c$$

Then,

$$Y1[y+1] = b(y+1) + c$$
$$Y1[y+1] = by + b + c$$

So,

$$dif_1[y] = (by + b + c) - (by + c)$$

Generically,

$$dif_1[y] = dif = b$$

5.2. Method of differences in 2nd-degree polynomials (quadratic equations).

Given

$$Y2[y] = ay^2 + by + c$$

Then,

$$Y2[y+1] = a(y+1)^2 + b(y+1) + c$$
$$Y2[y+1] = a(y^2 + 2y + 1) + by + b + c$$
$$Y2[y+1] = ay^2 + 2ay + a + by + b + c$$
$$Y2[y+1] = ay^2 + (2a+b)y + a + b + c$$

$$Y2[y+2] = a(y+2)^2 + b(y+2) + c$$
$$Y2[y+2] = a(y^2 + 4y + 4) + by + 2b + c$$
$$Y2[y+2] = ay^2 + 4ay + 4a + by + 2b + c$$
$$Y2[y+2] = ay^2 + (4a+b)y + 4a + 2b + c$$

So,

$$dif_1[y] = (ay^2 + (2a+b)y + a + b + c) - (ay^2 + by + c)$$
$$dif_1[y] = (ay^2 + 2ay + by + a + b + c) - (ay^2 + by + c)$$
$$dif_1[y] = (ay^2 + by + c + 2ay + a + b) - (ay^2 + by + c)$$
$$dif_1[y] = 2ay + a + b$$

$$dif_1[y+1] = (ay^2 + (4a+b)y + 4a + 2b + c) - (ay^2 + (2a+b)y + a + b + c)$$
$$dif_1[y+1] = (ay^2 + 4ay + by + 4a + 2b + c) - (ay^2 + 2ay + by + a + b + c)$$
$$dif_1[y+1] = 2ay + 3a + b$$

Then,

$$dif_2[y] = dif_1[y+1] - dif_1[y]$$

$$dif_2[y] = (2ay + 3a + b) - (2ay + a + b)$$

Generically,

$$dif_2[y] = difdif = 2a$$

5.3. Method of differences in 3rd-degree polynomials (cubic equations).

Given

$$Y3[y] = a_3y^3 + ay^2 + by + c$$

Then,

$$Y3[y+1] = a_3(y+1)^3 + a(y+1)^2 + b(y+1) + c$$
$$Y3[y+1] = a_3(y^3 + 3y^2 + 3y + 1) + a(y^2 + 2y + 1) + by + b + c$$
$$Y3[y+1] = a_3y^3 + 3a_3y^2 + 3a_3y + a_3 + ay^2 + 2ay + a + by + b + c$$
$$Y3[y+1] = a_3y^3 + (3a_3 + a)y^2 + (3a_3 + 2a + b)y + a_3 + a + b + c$$

$$\mathrm{Y}3[y+2] = a_3(y+2)^3 + a(y+2)^2 + b(y+2) + c$$
$$\mathrm{Y}3[y+2] = a_3(y^3 + 6y^2 + 12y + 8) + a(y^2 + 4y + 4) + b(y+2) + c$$
$$\mathrm{Y}3[y+2] = a_3y^3 + 6a_3y^2 + 12a_3y + 8a_3 + ay^2 + 4ay + 4a + by + 2b + c$$
$$Y3[y+2] = a_3y^3 + (6a_3 + a)y^2 + (12a_3 + 4a + b)y + 8a_3 + 4a + 2b + c$$

$$Y3[y+3] = a_3(y+3)^3 + a(y+3)^2 + b(y+3) + c$$
$$Y3[y+3] = a_3(y^3 + 9y^2 + 27y + 27) + a(\mathrm{y}^2 + 6y + 9) + b(y+3) + c$$
$$\mathrm{Y}3[y+3] = a_3y^3 + 9a_3y^2 + 27a_3y + 27a_3 + ay^2 + 6ay + 9a + by + 3b + c$$
$$Y3[y+3] = a_3y^3 + (9a_3 + a)y^2 + (27a_3 + 6a + b)y + 27a_3 + 9a + 3b + c$$

So,

$$dif_1[y] = Y3[y+1] - Y3[y]$$
$$dif_1[y] = a_3y^3 + (3a_3 + a)y^2 + (3a_3 + 2a + b)y + a_3 + a + b + c - (a_3y^3 + ay^2 + by + c)$$
$$dif_1[y] = 3a_3y^2 + (3a_3 + 2a)y + a_3 + a + b$$

$$dif_1[y+1] = Y3[y+2] - Y3[y+3]$$
$$dif_1[y+1] = a_3y^3 + (6a_3 + a)y^2 + (12a_3 + 4a + b)y + 8a_3 + 4a + 2b + c - (a_3y^3 + (3a_3 + a)y^2 + (3a_3 + 2a + b)y + a_3 + a + b + c)$$
$$dif_1[y+1] = 3a_3y^2 + (9a_3 + 2a)y + 7a_3 + 3a + b$$

$$dif_1[y+2] = Y3[y+3] - Y3[y+2]$$

$$dif_1[y+2] = a_3y^3 + (9a_3 + a)y^2 + (27a_3 + 6a + b)y + 27a_3 + 9a + 3b + c$$
$$- (a_3y^3 + (6a_3 + a)y^2 + (12a_3 + 4a + b)y + 8a_3 + 4a + 2b + c)$$
$$dif_1[y+2] = 3a_3y^2 + (15a_3 + 2a)y + 19a_3 + 5a + b$$

Now,

$$dif_2[y] = dif_1[y+1] - dif_1[y]$$
$$dif_2[y] = 3a_3y^2 + (9a_3 + 2a)y + 7a_3 + 3a + b - (3a_3y^2 + (3a_3 + 2a)y + a_3 + a + b)$$
$$dif_2[y] = 6a_3y + 6a_3 + 2a$$

$$dif_2[y+1] = dif_1[y+2] - dif_1[y+1]$$
$$dif_2[y+1] = 3a_3y^2 + (15a_3 + 2a)y + 19a_3 + 5a + b$$
$$- (3a_3y^2 + (9a_3 + 2a)y + 7a_3 + 3a + b)$$
$$dif_2[y+1] = 6a_3y + 12a_3 + 2a$$

Then,

$$dif_3[y] = dif_2[y+1] - dif_2[y]$$
$$dif_3[y] = 6a_3y + 12a_3 + 2a - (6a_3y + 6a_3 + 2a)$$

Generically,

$$dif_3[y] = difdifdif = 6a_3$$

5.4. Method of differences in dth-degree polynomials.

From the generic equation of polynomial d-degree

$$Yd[y] = a_dy^d + a_{d-1}y^{d-1} + \cdots + a_4y^4 + a_3y^3 + ay^2 + by + c$$

We have,

$$dif_d[y] = d!\,a_d$$

6. Understanding the Method of Differences in polynomials.

The addition increases the polynomial degree.

The subtraction decreases the polynomial degree.

This idea comes from The Babbage Engine.

We know that, if $F[y] = y^d$, then,

$$F[y+z] = (y+z)^n$$
$$= \binom{n}{0} y^n + \binom{n}{1} y^{n-1} z + \binom{n}{2} y^{n-2} z^2 + \ldots + \binom{n}{n-2} y^2 z^{n-2} + \binom{n}{n-1} y z^{n-1} + \binom{n}{n} z^n$$

Let's be $z = 1$:

$$F[y+1] = (y+1)^n = \binom{n}{0} y^n + \binom{n}{1} y^{n-1} + \binom{n}{2} y^{n-2} + \ldots + \binom{n}{n-2} y^2 + \binom{n}{n-1} y + \binom{n}{n}$$
$$F[y+1] = (y+1)^n = y^n + \binom{n}{1} y^{n-1} + \binom{n}{2} y^{n-2} + \ldots + \binom{n}{2} y^2 + \binom{n}{1} y + 1$$
$$F[y+1] = F[y] + \binom{n}{1} y^{n-1} + \binom{n}{2} y^{n-2} + \ldots + \binom{n}{2} y^2 + \binom{n}{1} y + 1$$

Being,

$$G[y] = F[y+1] - F[y]$$

Then,

$$G[y] = \binom{n}{1} y^{n-1} + \binom{n}{2} y^{n-2} + \ldots + \binom{n}{2} y^2 + \binom{n}{1} y + 1$$
$$degree\big[G[y]\big] = degree\big[F[y]\big] - 1$$
$$highest\ order\ coefficient[F[y]] = 1$$
$$highest\ order\ coeffcient[G[y]] = \binom{n}{1}$$

Now, from $G[y] = F[y+1] - F[y] = \binom{n}{1} y^{n-1} + \binom{n}{2} y^{n-2} + \ldots + \binom{n}{2} y^2 + \binom{n}{1} y + 1$, we have:

$$G[y+1] = \binom{n}{1} (y+1)^{n-1} + \binom{n}{2} (y+1)^{n-2} + \ldots + \binom{n}{n-2} (y+1)^2 + \binom{n}{n-1} (y+1) + 1$$
$$G[y+1] = \binom{n}{1} \left[y^{n-1} + \binom{n-1}{1} y^{n-2} + \binom{n-1}{2} y^{n-3} + \ldots + \binom{n-1}{2} y^2 + \binom{n-1}{1} y + 1\right] + \binom{n}{2} \left[y^{n-2} + \binom{n-2}{1} y^{n-3} + \binom{n-2}{2} y^{n-4} + \ldots + \binom{n-2}{2} y^2 + \binom{n-2}{1} y + 1\right] + \ldots + \binom{n}{2} [y^2 + 2y + 1] + \binom{n}{1} (y+1) + 1$$
$$G[y+1] = \left\{\binom{n}{1} y^{n-1}\right\} + \binom{n}{1} [\binom{n-1}{1} y^{n-2} + \binom{n-1}{2} y^{n-3} + \ldots + \binom{n-1}{2} y^2 + \binom{n-1}{1} y + 1] + \left\{\binom{n}{2} y^{n-2}\right\} + \binom{n}{2} [\binom{n-2}{1} y^{n-3} + \binom{n-2}{2} y^{n-4} + \ldots + \binom{n-2}{2} y^2 + \binom{n-2}{1} y + 1] + \ldots + \left\{\binom{n}{2} y^2\right\} + \binom{n}{2} [2y+1] + \left\{\binom{n}{1} y\right\} + \binom{n}{1} (1) + 1$$

$$G[y+1]=\left\{\binom{n}{1}y^{n-1}+\binom{n}{2}y^{n-2}+\ldots+\binom{n}{2}y^{2}+\binom{n}{1}y+1\right\}$$
$$+\binom{n}{1}\left[\binom{n-1}{1}y^{n-2}+\binom{n-1}{2}y^{n-3}+\ldots+\binom{n-1}{2}y^{2}+\binom{n-1}{1}y+1\right]$$
$$+\binom{n}{2}\left[\binom{n-2}{1}y^{n-3}+\binom{n-2}{2}y^{n-4}+\ldots+\binom{n-2}{2}y^{2}+\binom{n-2}{1}y+1\right]+\ldots+\binom{n}{2}[2y+1]+\binom{n}{1}$$

$$G[y+1]=G[y]+\binom{n}{1}\left[\binom{n-1}{1}y^{n-2}+\binom{n-1}{2}y^{n-3}+\ldots+\binom{n-1}{2}y^{2}+\binom{n-1}{1}y+1\right]$$
$$+\binom{n}{2}\left[\binom{n-2}{1}y^{n-3}+\binom{n-2}{2}y^{n-4}+\ldots+\binom{n-2}{2}y^{2}+\binom{n-2}{1}y+1\right]+\ldots+\binom{n}{2}[2y+1]+\binom{n}{1}$$

Finally,

$$H[y]=G[y+1]-G[y]$$
$$=\binom{n}{1}\left[\binom{n-1}{1}y^{n-2}+\binom{n-1}{2}y^{n-3}+\ldots+\binom{n-1}{2}y^{2}+\binom{n-1}{1}y+1\right]$$
$$+\binom{n}{2}\left[\binom{n-2}{1}y^{n-3}+\binom{n-2}{2}y^{n-4}+\ldots+\binom{n-2}{2}y^{2}+\binom{n-2}{1}y+1\right]+\ldots+\binom{n}{2}[2y+1]+\binom{n}{1}$$

$$degree\big[H[y]\big]=degree\big[G[y]\big]-1=degree\big[F[y]\big]-2$$

$$highest\ order\ coefficient\big[H[y]\big]=\binom{n}{1}\binom{n-1}{1}$$

7. Understanding the Method of Differences in polynomials.

The addition increases the polynomial degree.

The subtraction decreases the polynomial degree.

This idea comes from The Babbage Engine**Error! Reference source not found.**.

We know that, if $F[y]=y^d$, then,

$$F[y+z]=(y+z)^d$$
$$=\binom{d}{0}y^d+\binom{d}{1}y^{d-1}z+\binom{d}{2}y^{d-2}z^2+\ldots+\binom{d}{d-2}y^2z^{d-2}+\binom{d}{d-1}yz^{d-1}+\binom{d}{d}z^d$$

Let's be $z=1$:

$$F[y+1]=(y+1)^d=\binom{d}{0}y^d+\binom{d}{1}y^{d-1}+\binom{d}{2}y^{d-2}+\ldots+\binom{d}{d-2}y^2+\binom{d}{d-1}y+\binom{d}{d}$$
$$F[y+1]=y^d+\binom{d}{1}y^{d-1}+\binom{d}{2}y^{d-2}+\ldots+\binom{d}{2}y^2+\binom{d}{1}y+1$$

$F[y+1] = F[y] + \binom{d}{1} y^{d-1} + \binom{d}{2} y^{d-2} + \ldots + \binom{d}{2} y^2 + \binom{d}{1} y + 1$

Being,

$$G[y] = F[y+1] - F[y]$$

Then,

$$G[y] = \binom{d}{1} y^{d-1} + \binom{d}{2} y^{d-2} + \ldots + \binom{d}{2} y^2 + \binom{d}{1} y + 1$$

$$degree\big[G[y]\big] = degree\big[F[y]\big] - 1$$

$$highest\ order\ coefficient\big[F[y]\big] = 1$$

$$highest\ order\ coeffcient\big[G[y]\big] = \binom{d}{1}$$

Now, from

$$G[y] = F[y+1] - F[y] = \binom{d}{1} y^{d-1} + \binom{d}{2} y^{d-2} + \ldots + \binom{d}{2} y^2 + \binom{d}{1} y + 1$$

, we have:

$$G[y+1] = \binom{d}{1}(y+1)^{d-1} + \binom{d}{2}(y+1)^{d-2} + \ldots + \binom{d}{d-2}(y+1)^2 + \binom{d}{d-1}(y+1) + 1$$

$$\begin{aligned} G[y+1] = \binom{d}{1}&\left[y^{d-1} + \binom{d-1}{1} y^{d-2} + \binom{d-1}{2} y^{d-3} + \ldots + \binom{d-1}{2} y^2 + \binom{d-1}{1} y + 1\right] \\ &+ \binom{d}{2}\left[y^{d-2} + \binom{d-2}{1} y^{d-3} + \binom{d-2}{2} y^{d-4} + \ldots + \binom{d-2}{2}\ y^2 + \binom{d-2}{1} y \right.\\ &\left.+ 1\right] + \ldots + \binom{d}{2}[y^2 + 2y + 1] + \binom{d}{1}(y+1) + 1 \end{aligned}$$

$$\begin{aligned} G[y+1] = \binom{d}{1}&y^{d-1} \\ &+ \binom{d}{1}\left[\binom{d-1}{1} y^{d-2} + \binom{d-1}{2} y^{d-3} + \ldots + \binom{d-1}{2} y^2 + \binom{d-1}{1} y + 1\right] \\ &+ \binom{d}{2} y^{d-2} \\ &+ \binom{d}{2}\left[\binom{d-2}{1} y^{d-3} + \binom{d-2}{2} y^{d-4} + \ldots + \binom{d-2}{2} y^2 + \binom{d-2}{1} y \right.\\ &\left.+ 1\right] + \ldots + \binom{d}{2} y^2 + \binom{d}{2}[2y + 1] + \binom{d}{1} y + \binom{d}{1}(1) + 1 \end{aligned}$$

$$\begin{aligned} G[y+1] = \binom{d}{1}&y^{d-1} + \binom{d}{2} y^{d-2} + \ldots + \binom{d}{2} y^2 + \binom{d}{1} y + 1 \\ &+ \binom{d}{1}\left[\binom{d-1}{1}\ y^{d-2} + \binom{d-1}{2}\ y^{d-3} + \ldots + \binom{d-1}{2} y^2 + \binom{d-1}{1} y + 1\right] \\ &+ \binom{d}{2}\left[\binom{d-2}{1} y^{d-3} + \binom{d-2}{2} y^{d-4} + \ldots + \binom{d-2}{2} y^2 + \binom{d-2}{1} y \right.\\ &\left.+ 1\right] + \ldots + \binom{d}{2}[2y + 1] + \binom{d}{1} \end{aligned}$$

$$G[y+1] = G[y] + \binom{d}{1}\left[\binom{d-1}{1} y^{d-2} + \binom{d-1}{2} y^{d-3} + \ldots + \binom{d-1}{2} y^2 + \binom{d-1}{1} y + 1\right] + \binom{d}{2}\left[\binom{d-2}{1} y^{d-3} + \binom{d-2}{2} y^{d-4} + \ldots + \binom{d-2}{2} y^2 + \binom{d-2}{1} y + 1\right] + \ldots + \binom{d}{2}[2y+1] + \binom{d}{1}$$

Finally,

$$H[y] = G[y+1] - G[y] = \binom{d}{1}\left[\binom{d-1}{1} y^{d-2} + \binom{d-1}{2} y^{d-3} + \ldots + \binom{d-1}{2} y^2 + \binom{d-1}{1} y + 1\right] + \binom{d}{2}\left[\binom{d-2}{1} y^{d-3} + \binom{d-2}{2} y^{d-4} + \ldots + \binom{d-2}{2} y^2 + \binom{d-2}{1} y + 1\right] + \ldots + \binom{d}{2}[2y+1] + \binom{d}{1}$$

$$degree\big[H[y]\big] = degree\big[G[y]\big] - 1 = degree\big[F[y]\big] - 2$$

$$highest\ order\ coefficient\big[H[y]\big] = \binom{d}{1}\binom{d-1}{1}$$

If we repeat this process d-times we will obtain a polynomial of degree 0, that is, a constant.

The value of this constant is $d!$.

Each time we make the difference between two consecutive elements, it will result in a decrease in the degree. If we continue to do differences between consecutive elements recursively until degree zero, it will result in a coefficient equal to a constant $d!$.

$$recursivelly\big[F[y+1] - F[y]\big] = recursivelly[(y+1)^d - y^d] = \frac{d^d}{dy}(y^d) = \mathrm{d}!$$

$d! \equiv$ https://oeis.org/A000142

The method of differences in the elements of a numerical sequence acts as a derivative,

Only in polynomial functions does a finite number of differences result in a constant equal to the factor of the polynomial degree.

7.1. Example with 4th degree polynomial.

$$F[y] = y^4$$

$$G[y] = F[y+1] - F[y]$$
$$G[y] = (y+1)^4 - y^4$$
$$G[y] = 4y^3 + 6y^2 + 4y + 1$$

$$G[y+1] = 4(y+1)^3 + 6(y+1)^2 + 4(y+1) + 1$$
$$G[y+1] = 4y^3 + 12y^2 + 12y + 4 + 6y^2 + 12y + 6 + 4y + 4 + 1$$

$$G[y+1] - G[y] = H[y] = 12y^2 + 12y + 4 + 12y + 6 + 4$$
$$H[y] = 12y^2 + 24y + 14$$

$$H[y+1] = 12(y+1)^2 + 24(y+1) + 14$$
$$H[y+1] = 12y^2 + 24y + 12 + 24y + 24 + 14$$
$$H[y+1] - H[y] = I[y] = 12y^2 + 24y + 12 + 24y + 24 + 14$$
$$I[y] = 24y + 36$$

$$I[y+1] = 24(y+1) + 36$$
$$I[y+1] = 24y + 24 + 36$$
$$I[y+1] - I[y] = 24 = 4!$$

See how the method of differences acts as a derivative.

7.2. Counter-example in Factorial sequence.

See the method of finite differences applied in factorial sequence https://oeis.org/A000142.

https://oeis.org/A068106							
degree	https://oeis.org/A000142	https://oeis.org/A001563	https://oeis.org/A001564	https://oeis.org/A001565	https://oeis.org/A001688	https://oeis.org/A001689	OEIS
d	d!	dif_1	dif_2	dif_3	dif_4	dif_5	
0	1	0	1	2	9	44	https://oeis.org/A000166
1	1	1	3	11	53	309	https://oeis.org/A000255
2	2	4	14	64	362	2428	https://oeis.org/A055790
3	6	18	78	426	2790	21234	https://oeis.org/A277609
4	24	96	504	3216	24024	205056	https://oeis.org/A277563
5	120	600	3720	27240	229080	2170680	https://oeis.org/A280425
6	720	4320	30960	256320	2399760	25022880	https://oeis.org/A280920
7	5040	35280	287280	2656080	27422640	312273360	
8	40320	322560	2943360	30078720	339696000	4196666880	
9	362880	3265920	33022080	369774720	4536362880	60451816320	
10	3628800	36288000	402796800	4906137600	64988179200		
11	39916800	439084800	5308934400	69894316800			
12	479001600	5748019200	75203251200				
13	6227020800	80951270400					
14	87178291200						

Figure 5. The difference method applied in the sequence https://oeis.org/A000142 factorial numbers.

See that the values of the difference increase in each iteration we make. There will never be a convergence towards a constant, in the way that occurs with polynomials.

8. An application of this theory in practice.

There is a comment in the sequence https://oeis.org/A005563 (square minus 1) numbers saying: *"Erdös conjectured that $n^2 - 1 = k!$ has a solution if and only if n is $5, 11$ or 71 (when k is $4, 5$ or 7)".*

If $k! = (n^2 - 1)$, then $k! = (n+1)(n-1)$.

Because of that, we must have $n = \lceil \sqrt{k!} \rceil$.

Because $n = \lceil \sqrt{k!} \rceil$, then we have:

$$n - 1 = \lceil \sqrt{k!} \rceil - 1 = \lfloor \sqrt{k!} \rfloor < \sqrt{k!} < n = \lceil \sqrt{k!} \rceil < \lceil \sqrt{k!} \rceil + 1 = n + 1$$

Because of this expression, then the equality $k! = (n^2 - 1) = (n+1)(n-1)$ is true if and only if https://oeis.org/A082995 has value 0. See the table:

-	A000142	-	A055228	A055226	C000699	C000699	A082995
k	$k!$	$\sqrt{k!}$	$\lceil \sqrt{k!} \rceil$	$\lceil \sqrt{k!} \rceil - 1$	$\lceil \sqrt{k!} \rceil + 1$	$(\lceil \sqrt{k!} \rceil - 1) * (\lceil \sqrt{k!} \rceil + 1)$	$(\lceil \sqrt{k!} \rceil - 1) * (\lceil \sqrt{k!} \rceil + 1) - k!$
1	1	1	1	0	2	0	-1
2	2	1.41421356	2	1	3	3	1
3	6	2.44948974	3	2	4	8	2
4	24	4.89897949	5	4	6	24	0
5	120	10.9544512	11	10	12	120	0
6	720	26.8328157	27	26	28	728	8
7	5040	70.9929574	71	70	72	5040	0
8	40320	200.798406	201	200	202	40400	80
9	362880	602.395219	603	602	604	363608	728
10	3628800	1904.94094	1905	1904	1906	3629024	224
11	39916800	6317.97436	6318	6317	6319	39917123	323
12	479001600	21886.1052	21887	21886	21888	479040768	39168
13	6227020800	78911.4745	78912	78911	78913	6227103743	82943
14	87178291200	295259.701	295260	295259	295261	87178467599	176399

Because of the method of the differences, the function https://oeis.org/A082995 has a finite number of 0´s.

Because https://oeis.org/A082995 is a function that has a finite number of 0's, then it is impossible to have other solutions than just the solutions of the conjectures of Erdos, Brocard, or the sequence https://oeis.org/A082995.

9. Introduction to the 3 types of the quadratic integer sequences

Given a quadratic $Y[y] = ay^2 + by + c$, we have determined the simplest quadratic equation generated from the any 3 consecutive elements: (x_1, x_2, x_3) as being:

$$Y[y] = x = \left(\frac{x_1 - 2x_2 + x_3}{2}\right) y^2 + \left(\frac{x_3 - x_1}{2}\right) y + x_2$$

When choosing the 3 consecutive elements: to form a quadratic, the only constraint is that $a \neq 0$. So, $x_1 - 2x_2 + x_3 \neq 0$ or $(x_2 - x_1) \neq (x_3 - x_2)$.

Nothing prevents 2 elements from being with the same value.

Now let us check what happens to the main parameters when two of these 3 consecutive elements are equal.

After, we will study the results in a table, and we will classify the quadratics into the 3 fundamental types: ACC, or DES, or SUB.

So, let us see the quadratic's behavior over the 5 main parameters $a, b, y_{ip}, x_{ip}, \Delta$, for all the 6 possible cases:

$$x_1 = x_2$$
$$x_2 = x_3$$
$$x_1 = x_3$$
$$x_1 = -x_2$$
$$x_2 = -x_3$$
$$x_1 = -x_3$$

9.1. For $x_1 = x_2$

$$y_{ip} = -\frac{b}{2a} = \frac{x_1 - x_3}{2x_1 - 4x_2 + 2x_3} = \frac{x_1 - x_3}{2x_1 - 4x_1 + 2x_3} = \frac{x_1 - x_3}{-2x_1 + 2x_3} = -\frac{x_1 - x_3}{2x_1 - 2x_3} = -\frac{1}{2}$$

$$a = \frac{x_1 - 2x_2 + x_3}{2} = \frac{x_1 - 2x_1 + x_3}{2} = \frac{-x_1 + x_3}{2} = b$$

$$b = \frac{x_3 - x_1}{2} = a$$

$$\Delta = \frac{x_1^2 + 16x_2^2 + x_3^2 - 8x_1x_2 - 8x_2x_3 - 2x_1x_3}{4}$$

$$= \frac{x_1^2 + 16x_1^2 + x_3^2 - 8x_1x_1 - 8x_1x_3 - 2x_1x_3}{4} = \frac{9x_1^2 + x_3^2 - 10x_1x_3}{4}$$

$$= \frac{(-x_1 + x_3)^2 + 8x_1^2 - 8x_1x_3}{4} = \frac{(-x_1 + x_3)^2 - (-8x_1^2 + 8x_1x_3)}{4}$$

$$= \frac{(-x_1 + x_3)^2 - 8x_1(-x_1 + x_3)}{4} = (-x_1 + x_3)\frac{(-x_1 + x_3) - 8x_1}{4}$$

$$= (-x_1 + x_3)\frac{x_3 - 9x_1}{4}$$

$$x_{sp} = -\frac{\Delta}{4a} = -\frac{(-x_1 + x_3)\frac{x_3 - 9x_1}{4}}{4\frac{-x_1 + x_3}{2}} = -\frac{\frac{x_3 - 9x_1}{4}}{2} = -\frac{x_3 - 9x_1}{8} = \frac{9x_1 - x_3}{8}$$

$$x_{sp} = \frac{8(x_1 - 2x_2 + x_3)x_2 - (-x_1 + x_3)^2}{8(x_1 - 2x_2 + x_3)} = \frac{8(x_1 - 2x_1 + x_3)x_1 - (-x_1 + x_3)^2}{8(x_1 - 2x_1 + x_3)}$$

$$= \frac{8(-x_1 + x_3)x_1 - (-x_1 + x_3)^2}{8(-x_1 + x_3)} = \frac{-8x_1^2 + 8x_1x_3 - (x_1^2 + x_3^2 - 2x_1x_3)}{8(-x_1 + x_3)}$$

$$= \frac{-8x_1^2 + 8x_1x_3 - x_1^2 - x_3^2 + 2x_1x_3}{8(-x_1 + x_3)} = \frac{-9x_1^2 + 10x_1x_3 - x_3^2}{8(-x_1 + x_3)}$$

$$= \frac{9x_1^2 - 10x_1x_3 + x_3^2}{8(x_1 - x_3)} = \frac{(x_1 - x_3)^2 + 8x_1^2 - 8x_1x_3}{8(x_1 - x_3)}$$

$$= \frac{(x_1 - x_3)^2 + 8x_1(x_1 - x_3)}{8(x_1 - x_3)} = \frac{(x_1 - x_3) + 8x_1}{8} = \frac{9x_1 - x_3}{8}$$

$$x_{sp} = x_2 - \frac{(x_3 - x_1)^2}{8(x_1 - 2x_2 + x_3)} = x_1 - \frac{(x_3 - x_1)^2}{8(x_1 - 2x_1 + x_3)} = x_1 - \frac{(x_3 - x_1)^2}{8(-x_1 + x_3)}$$

$$= \frac{8(-x_1 + x_3)x_1 - (x_3 - x_1)^2}{8(-x_1 + x_3)} = \frac{-8x_1^2 + 8x_1x_3 - (x_3^2 + x_1^2 - 2x_1x_3)}{8(-x_1 + x_3)} =$$

$$= \frac{-8x_1^2 + 8x_1x_3 - x_3^2 - x_1^2 + 2x_1x_3}{8(-x_1 + x_3)} = \frac{-9x_1^2 + 10x_1x_3 - x_3^2}{8(-x_1 + x_3)}$$

$$= \frac{9x_1^2 - 10x_1x_3 + x_3^2}{8(x_1 - x_3)} = \frac{(x_1 - x_3)^2 + 8x_1^2 - 8x_1x_3}{8(x_1 - x_3)}$$

$$= \frac{(x_1 - x_3)^2 + 8x_1(x_1 - x_3)}{8(x_1 - x_3)} = \frac{(x_1 - x_3) + 8x_1}{8} = \frac{9x_1 - x_3}{8}$$

9.2. For $x_2 = x_3$

$$y_{sp} = -\frac{b}{2a} = \frac{x_1 - x_3}{2x_1 - 4x_2 + 2x_3} = \frac{x_1 - x_2}{2x_1 - 2x_2} = \frac{1}{2}$$

$$a = \frac{x_1 - 2x_2 + x_3}{2} = \frac{x_1 - x_2}{2} = -b$$

$$b = \frac{x_3 - x_1}{2} = \frac{x_2 - x_1}{2} = -a$$

$$\Delta = \frac{x_1^2 + 16x_2^2 + x_3^2 - 8x_1x_2 - 8x_2x_3 - 2x_1x_3}{4} = \frac{x_1^2 + 16x_2^2 + x_2^2 - 8x_1x_2 - 8x_2x_2 - 2x_1x_2}{4}$$

$$= \frac{x_1^2 + 9x_2^2 - 10x_1x_2}{4} = \frac{(x_1 - x_2)^2 + 8x_2^2 - 8x_1x_2}{4}$$

$$= \frac{(x_1 - x_2)^2 + 8x_2(x_2 - x_1)}{4} = \frac{(x_1 - x_2)^2 - 8x_2(x_1 - x_2)}{4}$$

$$= (x_1 - x_2)\frac{(x_1 - x_2) - 8x_2}{4} = (x_1 - x_2)\frac{x_1 - 9x_2}{4}$$

$$x_{sp} = -\frac{\Delta}{4a} = -\frac{(x_1 - x_2)\frac{x_1 - 9x_2}{4}}{4\frac{x_1 - x_2}{2}} = -\frac{x_1 - 9x_2}{8} = \frac{-x_1 + 9x_2}{8}$$

$$x_{sp} = \frac{8(x_1 - 2x_2 + x_3)x_2 - (-x_1 + x_3)^2}{8(x_1 - 2x_2 + x_3)} = \frac{8(x_1 - 2x_2 + x_2)x_2 - (-x_1 + x_2)^2}{8(x_1 - 2x_2 + x_2)}$$

$$= \frac{8(x_1 - x_2)x_2 - (-x_1 + x_2)^2}{8(x_1 - x_2)} = \frac{8(x_1 - x_2)x_2 - (-x_1 + x_2)^2}{8(x_1 - x_2)}$$

$$= \frac{8(x_1 - x_2)x_2 - (-x_1 + x_2)(-x_1 + x_2)}{8(x_1 - x_2)}$$

$$= \frac{8(x_1 - x_2)x_2 + (x_1 - x_2)(-x_1 + x_2)}{8(x_1 - x_2)} = \frac{8x_2 + (-x_1 + x_2)}{8} = \frac{-x_1 + 9x_2}{8}$$

$$x_{sp} = x_2 - \frac{(x_3 - x_1)^2}{8(x_1 - 2x_2 + x_3)} = x_2 - \frac{(x_2 - x_1)^2}{8(x_1 - 2x_2 + x_2)} = x_2 - \frac{(x_2 - x_1)^2}{8(x_1 - x_2)} = x_2 + \frac{(x_2 - x_1)^2}{8(x_2 - x_1)}$$

$$= x_2 + \frac{x_2 - x_1}{8} = \frac{-x_1 + 9x_2}{8}$$

9.3. For $x_1 = x_3$

$$y_{sp} = -\frac{b}{2a} = \frac{x_1 - x_3}{2x_1 - 4x_2 + 2x_3} = 0$$

$$a = \frac{x_1 - 2x_2 + x_3}{2} = \frac{x_1 - 2x_2 + x_1}{2} = \frac{2x_1 - 2x_2}{2} = x_1 - x_2$$

$$b = \frac{x_3 - x_1}{2} = \frac{x_3 - x_3}{2} = 0$$

$$\Delta = \frac{x_1^2 + 16x_2^2 + x_3^2 - 8x_1x_2 - 8x_2x_3 - 2x_1x_3}{4} = \frac{x_1^2 + 16x_2^2 + x_1^2 - 8x_1x_2 - 8x_2x_1 - 2x_1x_1}{4}$$

$$= \frac{16x_2^2 - 16x_1x_2}{4} = 4x_2^2 - 4x_1x_2 = 4x_2(x_2 - x_1)$$

$$x_{sp} = -\frac{\Delta}{4a} = -\frac{4x_2(x_2 - x_1)}{4(x_1 - x_2)} = x_2$$

$$x_{sp} = \frac{8(x_1 - 2x_2 + x_3)x_2 + (x_1 - x_3)^2}{2(x_1 - 2x_2 + x_3)} = \frac{8(2x_1 - 2x_2)x_2}{2(2x_1 - 2x_2)} = \frac{8(2x_1 - 2x_2)x_2}{8(2x_1 - 2x_2)} = x_2$$

$$x_{sp} = x_2 - \frac{(x_3 - x_1)^2}{8(x_1 - 2x_2 + x_3)} = x_2 - \frac{(x_1 - x_1)^2}{8(x_1 - 2x_2 + x_1)} = x_2$$

9.4. For $x_1 = -x_2$

$$y_{sp} = -\frac{b}{2a} = \frac{x_1 - x_3}{2x_1 - 4x_2 + 2x_3} = \frac{x_1 - x_3}{2x_1 + 4x_1 + 2x_3} = \frac{x_1 - x_3}{6x_1 + 2x_3} = \frac{1}{2}\left(\frac{x_1 - x_3}{3x_1 + x_3}\right)$$

$$a = \frac{x_1 - 2x_2 + x_3}{2} = \frac{x_1 + 2x_1 + x_3}{2} = \frac{3x_1 + x_3}{2} = b + 2x_1$$

$$b = \frac{x_3 - x_1}{2} = \frac{3x_1 + x_3 - 4x_1}{2} = a - 2x_1$$

$$\Delta = \frac{x_1^2 + 16x_2^2 + x_3^2 - 8x_1x_2 - 8x_2x_3 - 2x_1x_3}{4} = \frac{x_1^2 + 16x_1^2 + x_3^2 + 8x_1x_1 + 8x_1x_3 - 2x_1x_3}{4}$$

$$= \frac{25x_1^2 + x_3^2 + 6x_1x_3}{4} = \frac{(3x_1 + x_3)^2 + 16x_1^2}{4}$$

$$x_{sp} = -\frac{\Delta}{4a} = -\frac{\frac{(3x_1 + x_3)^2 + 16x_1^2}{4}}{4\frac{3x_1 + x_3}{2}} = -\frac{\frac{(3x_1 + x_3)^2 + 16x_1^2}{4}}{2(3x_1 + x_3)} == -\frac{(3x_1 + x_3)^2 + 16x_1^2}{8(3x_1 + x_3)}$$

$$= -\left(\frac{(3x_1 + x_3)}{8} + \frac{2x_1^2}{(3x_1 + x_3)}\right)$$

$$x_{sp} = \frac{8(x_1 - 2x_2 + x_3)x_2 - (-x_1 + x_3)^2}{8(x_1 - 2x_2 + x_3)} = \frac{-8(x_1 + 2x_1 + x_3)x_1 - (-x_1 + x_3)^2}{8(x_1 + 2x_1 + x_3)}$$

$$= \frac{-8(3x_1 + x_3)x_1 - (-x_1 + x_3)^2}{8(3x_1 + x_3)} = \frac{-24x_1^2 - 8x_1x_3 - (x_1^2 + x_3^2 - 2x_1x_3)}{8(3x_1 + x_3)}$$

$$= \frac{-24x_1^2 - 8x_1x_3 - x_1^2 - x_3^2 + 2x_1x_3}{8(3x_1 + x_3)} = \frac{-25x_1^2 - 6x_1x_3 - x_3^2}{8(3x_1 + x_3)}$$

$$= -\frac{25x_1^2 + 6x_1x_3 + x_3^2}{8(3x_1 + x_3)} = -\frac{(3x_1 + x_3)^2 + 16x_1^2}{8(3x_1 + x_3)}$$

$$= -\left(\frac{(3x_1 + x_3)}{8} + \frac{2x_1^2}{(3x_1 + x_3)}\right)$$

$$x_{sp} = x_2 - \frac{(x_3 - x_1)^2}{8(x_1 - 2x_2 + x_3)} = -x_1 - \frac{(x_3 - x_1)^2}{8(x_1 + 2x_1 + x_3)} = -x_1 - \frac{(x_3 - x_1)^2}{8(3x_1 + x_3)}$$

$$= \frac{-8(3x_1 + x_3)x_1 - (x_3 - x_1)^2}{8(3x_1 + x_3)} = \frac{-24x_1^2 - 8x_1x_3 - (x_1^2 + x_3^2 - 2x_1x_3)}{8(3x_1 + x_3)}$$

$$= \frac{-24x_1^2 - 8x_1x_3 - x_1^2 - x_3^2 + 2x_1x_3}{8(3x_1 + x_3)} = \frac{-25x_1^2 - 6x_1x_3 - x_3^2}{8(3x_1 + x_3)}$$

$$= -\frac{(3x_1 + x_3)^2 + 16x_1^2}{8(3x_1 + x_3)} = -\left(\frac{(3x_1 + x_3)}{8} + \frac{2x_1^2}{(3x_1 + x_3)}\right)$$

9.5. For $x_2 = -x_3$

$$y_{sp} = -\frac{b}{2a} = \frac{x_1 - x_3}{2x_1 - 4x_2 + 2x_3} = \frac{x_1 + x_2}{2x_1 - 4x_2 - 2x_2} = \frac{1}{2}\left(\frac{x_1 + x_2}{x_1 - 3x_2}\right)$$

$$a = \frac{x_1 - 2x_2 + x_3}{2} = \frac{x_1 + 2x_3 + x_3}{2} = \frac{x_1 + 3x_3}{2} = -b + 2x_3$$

$$b = \frac{x_3 - x_1}{2} = \frac{4x_3 - x_1 - 3x_3}{2} = -a + 2x_3$$

$$\Delta = \frac{x_1^2+16x_2^2+x_3^2-8x_1x_2-8x_2x_3-2x_1x_3}{4} = \frac{x_1^2+16x_2^2+x_2^2-8x_1x_2+8x_2x_2+2x_1x_2}{4}$$

$$= \frac{x_1^2+25x_2^2-6x_1x_2}{4} = \frac{(x_1-3x_2)^2+16x_2^2}{4}$$

$$x_{sp} = -\frac{\Delta}{4a} = -\frac{\frac{(x_1-3x_2)^2+16x_2^2}{4}}{4\frac{x_1-3x_2}{2}} = -\frac{\frac{(x_1-3x_2)^2+16x_2^2}{4}}{2(x_1-3x_2)} = -\frac{(x_1-3x_2)^2+16x_2^2}{8(x_1-3x_2)}$$

$$= -\left(\frac{(x_1-3x_2)}{8} + \frac{2x_2^2}{(x_1-3x_2)}\right)$$

$$x_{sp} = \frac{8(x_1-2x_2+x_3)x_2-(-x_1+x_3)^2}{8(x_1-2x_2+x_3)} = \frac{8(x_1-2x_2-x_2)x_2-(-x_1-x_2)^2}{8(x_1-2x_2-x_2)}$$

$$= \frac{8(x_1-3x_2)x_2-(-x_1-x_2)^2}{8(x_1-3x_2)} = \frac{8x_1x_2-24x_2^2-(x_1^2+x_2^2+2x_1x_2)}{8(x_1-3x_2)}$$

$$= \frac{8x_1x_2-24x_2^2-x_1^2-x_2^2-2x_1x_2}{8x_1-24x_2} = \frac{-x_1^2-25x_2^2+6x_1x_2}{8x_1-24x_2}$$

$$= \frac{-x_1^2-25x_2^2+6x_1x_2}{8x_1-24x_2}$$

$$x_{sp} = x_2 - \frac{(x_3-x_1)^2}{8(x_1-2x_2+x_3)} = x_2 - \frac{(-x_2-x_1)^2}{8(x_1-2x_2-x_2)} = x_2 - \frac{(-x_2-x_1)^2}{8(x_1-3x_2)}$$

$$= \frac{8x_2(x_1-3x_2)-(x_1+x_2)^2}{8(x_1-3x_2)}$$

9.6. For $x_1 = -x_3$

$$y_{sp} = -\frac{b}{2a} = \frac{x_1-x_3}{2x_1-4x_2+2x_3} = \frac{x_1+x_1}{2x_1-4x_2-2x_1} = \frac{2x_1}{-4x_2} = -\frac{x_1}{2x_2}$$

$$a = \frac{x_1-2x_2+x_3}{2} = \frac{x_1-2x_2-x_1}{2} = -x_2$$

$$b = \frac{x_3-x_1}{2} = \frac{-2x_1}{2} = -x_1$$

$$\Delta = \frac{x_1^2+16x_2^2+x_3^2-8x_1x_2-8x_2x_3-2x_1x_3}{4} = \frac{x_1^2+16x_2^2+x_1^2-8x_1x_2+8x_2x_1+2x_1x_1}{4}$$

$$= \frac{4x_1^2+16x_2^2}{4} = x_1^2+4x_2^2$$

$$x_{sp} = -\frac{\Delta}{4a} = -\frac{x_1^2+4x_2^2}{4(-x_2)} = \frac{x_1^2+4x_2^2}{4x_2}$$

$$x_{sp} = \frac{8(x_1-2x_2+x_3)x_2-(-x_1+x_3)^2}{8(x_1-2x_2+x_3)} = \frac{8(x_1-2x_2-x_1)x_2-(-x_1-x_1)^2}{8(x_1-2x_2-x_1)}$$

$$= \frac{8(-2x_2)x_2-(-2x_1)^2}{8(-2x_2)} = \frac{-16x_2^2-4x_1^2}{-16x_2} = \frac{4x_2^2+x_1^2}{4x_2}$$

$$x_{sp} = x_2 - \frac{(x_3 - x_1)^2}{8(x_1 - 2x_2 + x_3)} = x_2 - \frac{(-x_1 - x_1)^2}{8(x_1 - 2x_2 - x_1)} = x_2 - \frac{4x_1^2}{-16x_2} = x_2 + \frac{x_1^2}{4x_2} = \frac{4x_2^2 + x_1^2}{4x_2}$$

9.7. Consolidation

Putting all the results together, we get the following table:

	Representation with quadratic coefficients	a	b	$-\frac{b}{2a}$	$-\frac{\Delta}{4a} = -\frac{b^2 - 4ac}{4a}$	$\Delta = b^2 - 4ac$
	Representation with the 3 consecutive elements:	$\frac{x_1 - 2x_2 + x_3}{2}$	$\frac{x_3 - x_1}{2}$	$\frac{x_1 - x_3}{2x_1 - 4x_2 + 2x_3}$	$x_2 - \frac{(-x_1 + x_3)^2}{8(x_1 - 2x_2 + x_3)}$	$\frac{x_1^2 + (4x_2)^2 + x_3^2 - 2x_1(4x_2) - 2(4x_2)x_3 - 2x_1x_3}{4}$
$x_1 = x_2$	DES $\lvert a\rvert = \lvert b\rvert > 0$	$\frac{x_3 - x_1}{2} = b$	$\frac{x_3 - x_1}{2} = a$	$-\frac{1}{2}$	$\frac{9x_1 - x_3}{8}$	$(-x_1 + x_3)\frac{-9x_1 + x_3}{4}$
$x_2 = x_3$	DES $\lvert a\rvert = \lvert b\rvert > 0$	$\frac{x_1 - x_2}{2} = -b$	$\frac{x_3 - x_1}{2} = -a$	$\frac{1}{2}$	$\frac{-x_1 + 9x_2}{8}$	$(x_1 - x_2)\frac{x_1 - 9x_2}{4}$
$x_1 = x_3$	SUB $\lvert a\rvert > \lvert b\rvert = 0$	$x_1 - x_2$	0	0	x_2	$4x_2(-x_1 + x_2)$
$x_1 = -x_2$	ACC $\lvert a\rvert \neq \lvert b\rvert > 0$	$\frac{3x_1 + x_3}{2} = b + 2x_1$	$\frac{x_3 - x_1}{2} = a - 2x_1$	$\frac{1}{2}\left(\frac{x_1 - x_3}{3x_1 + x_3}\right)$	$-\left(\frac{3x_1 + x_3}{8} + \frac{2x_1^2}{3x_1 + x_3}\right)$	$\frac{(3x_1 + x_3)^2 + 16x_1^2}{4}$
$x_2 = -x_3$	ACC $\lvert a\rvert \neq \lvert b\rvert > 0$	$\frac{x_1 + 3x_3}{2} = -b + 2x_3$	$\frac{x_3 - x_1}{2} = -a + 2x_3$	$\frac{1}{2}\left(\frac{x_1 + x_2}{x_1 - 3x_2}\right)$	$-\left(\frac{x_1 - 3x_2}{8} + \frac{2x_2^2}{x_1 - 3x_2}\right)$	$\frac{(x_1 - 3x_2)^2 + 16x_2^2}{4}$
$x_1 = -x_3$	ACC $\lvert a\rvert \neq \lvert b\rvert > 0$	$-x_2$	$-x_1$	$-\frac{x_1}{2x_2}$	$\frac{x_1^2 + 4x_2^2}{4x_2}$	$x_1^2 + 4x_2^2$

	Representation with quadratic coefficients	a	b	$-\frac{b}{2a}$	$-\frac{\Delta}{4a} = -\frac{b^2 - 4ac}{4a}$	$\Delta = b^2 - 4ac$
	Representation with the 3 consecutive elements:	$\frac{x_1 - 2x_2 + x_3}{2}$	$\frac{x_3 - x_1}{2}$	$\frac{x_1 - x_3}{2x_1 - 4x_2 + 2x_3}$	$x_2 - \frac{(-x_1 + x_3)^2}{8(x_1 - 2x_2 + x_3)}$	$\frac{x_1^2 + (4x_2)^2 + x_3^2 - 2x_1(4x_2) - 2(4x_2)x_3 - 2x_1x_3}{4}$

$\boldsymbol{x_1 = x_2}$	DES $\lvert a\rvert = \lvert b\rvert > 0$	$\frac{x_3 - x_1}{2} = b$	$\frac{x_3 - x_1}{2} = a$	$-\frac{1}{2}$	$\frac{9x_1 - x_3}{8}$	$(-x_1 + x_3)\frac{-9x_1 + x_3}{4}$
$\boldsymbol{x_2 = x_3}$	DES $\lvert a\rvert = \lvert b\rvert > 0$	$\frac{x_1 - x_2}{2} = -b$	$\frac{x_3 - x_1}{2} = -a$	$\frac{1}{2}$	$\frac{-x_1 + 9x_2}{8}$	$(x_1 - x_2)\frac{x_1 - 9x_2}{4}$
$\boldsymbol{x_1 = x_3}$	SUB $\lvert a\rvert > \lvert b\rvert = 0$	$x_1 - x_2$	0	0	x_2	$4x_2(-x_1 + x_2)$
$\boldsymbol{x_1 = -x_2}$	ACC $\lvert a\rvert \neq \lvert b\rvert > 0$	$\frac{3x_1 + x_3}{2} = b + 2x_1$	$\frac{x_3 - x_1}{2} = a - 2x_1$	$\frac{1}{2}\left(\frac{x_1 - x_3}{3x_1 + x_3}\right)$	$-\left(\frac{3x_1 + x_3}{8} + \frac{2x_1^2}{3x_1 + x_3}\right)$	$\frac{(3x_1 + x_3)^2 + 16x_1^2}{4}$
$\boldsymbol{x_2 = -x_3}$	ACC $\lvert a\rvert \neq \lvert b\rvert > 0$	$\frac{x_1 + 3x_3}{2} = -b + 2x_3$	$\frac{x_3 - x_1}{2} = -a + 2x_3$	$\frac{1}{2}\left(\frac{x_1 + x_2}{x_1 - 3x_2}\right)$	$-\left(\frac{x_1 - 3x_2}{8} + \frac{2x_2^2}{x_1 - 3x_2}\right)$	$\frac{(x_1 - 3x_2)^2 + 16x_2^2}{4}$
$\boldsymbol{x_1 = -x_3}$	ACC $\lvert a\rvert \neq \lvert b\rvert > 0$	$-x_2$	$-x_1$	$-\frac{x_1}{2x_2}$	$\frac{x_1^2 + 4x_2^2}{4x_2}$	$x_1^2 + 4x_2^2$

Figure 6. C001108 Quadratics classification summary.

9.8. Reasoning on the results

Studying the results, we can conclude that there are only 3 types of quadratics possibilities:

- Type red (SUB type): where $|a| \neq |b| = 0$, and $y_{sp} = integer$, like the example below $x = Y[y] = y^2 + 2 = [3,2,3]$. Generic form: $Y[y] = ay^2 + c$.
- Type blue (DES type): where $|a| = |b| > 0$, and $y_{sp} = \frac{odd}{2}$, like the example below $x = Y[y] = y^2 - y = [2,0,0]$. Generic form: $Y[y] = ay^2 \pm ay + c$.
- Type green (ACC type): where $|a| \neq |b| > 0$, and $y_{sp} \neq integer$, and $y_{sp} \neq \frac{odd}{2}$, like the example $x = Y[y] = 4y^2 - 2y - 1 = [5,-1,1]$. Generic form: $Y[y] = ay^2 \pm by + c$.

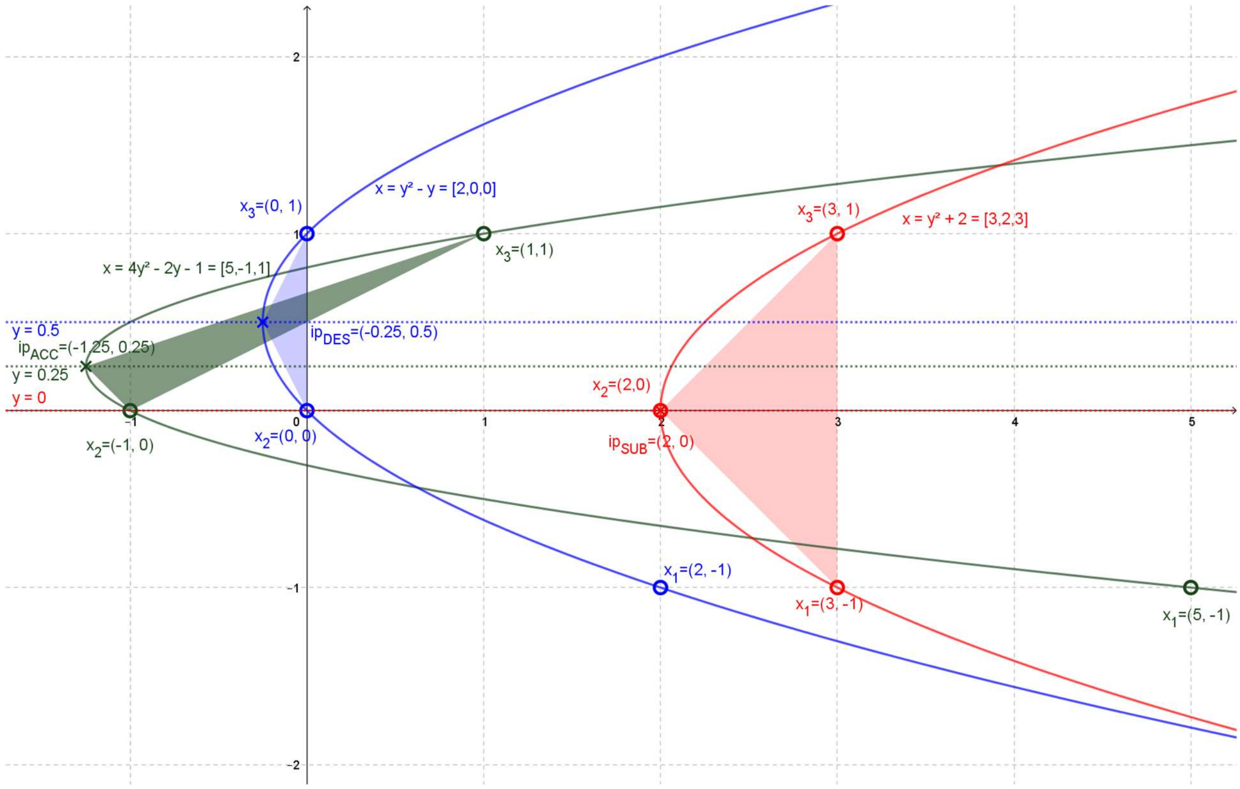

Figure 7. C001108 Example of the 3 types possible of parabolas: ACC type in green, DES type in blue, and SUB type in red. There is no other type of parabola.

Note that we can discard the analysis with the hypothesis of $|b| > |a|$. Whatever $|b| > |a|$, the offset is not zero and we can use the offset equation to get $|a| \geq |b|$ where the offset is zero. Offset does not change the sequence. The sequence is still the same.

Studying these 3 unique possibilities of the types of quadratics we perceive that in each of them have fundamental characteristics that distinguish one from the two others.

There is no other possibility of quadratic classification besides those 3 types above described. All others are isomorphic or equal to one of these 3 cases.

See below:

9.9. Features of the "SUB" (submarine) type quadratics

1. Color red: quadratics where $|a| \neq |b| = 0$ in offset $f = 0$ have the symmetry point is over x_2 and it is equidistant from the x_1 and x_3. And $x_1 = x_3$.
2. For offset $f = 0$, $a \neq b = 0$, $y_{sp} = -\frac{b}{2a} = 0$.
3. For any offset, $b = 2na$, $y_{sp} = -\frac{b}{2a} = integer$.
4. Because of the index y_{sp} always will be an integer, then the value x_{sp} will be always an element of the integer sequence. The symmetry point is over an integer element of the sequence.

5. For all n, the duplicated elements $Y[y_{sp}+n+1] = Y[y_{sp}-n-1]$ will form all the pairs of duplicated quadratic elements of the sequence. Each element of the pair is part of the sequence, and both are equidistant from the symmetry point.
6. Both duplicated elements $Y[y_{sp}+1] = Y[y_{sp}-1]$ will produce the closest pair of duplicated quadratic elements of the symmetry point. The duplicated elements are part of the sequence and are equidistant from the symmetry point.
7. Because $y_{sp} = 0$ for offset $f = 0$, then the symmetry of this even function (quadratic) is in the form of $Y[0+n+1] = Y[0-n-1]$ or $Y[y] = Y[-y]$.
8. The symmetry point is the symmetry element equidistant to all duplicated integers of the sequence. The symmetry point is over an integer element of the sequence and the sequence does not duplicate this only element.
9. Because the symmetry point is over one element, this remembers the submarine in the "Battleship Game".
10. We will call this type of sequences as SUB from the word "submarine".

9.10. Features of the "DES" (destroyer) type quadratics

1. Color blue: quadratics where $|a| = |b| \neq 0$ in offset $f = 0$ have the symmetry point equidistant from the x_2 and x_3. And $x_2 = x_3$.
2. For offset $f = 0$, $a = -b$, $y_{sp} = -\frac{b}{2a} = \frac{1}{2} = 0.5$.
3. For any offset, $b = (2n-1)a$, $y_{sp} = -\frac{b}{2a} = \frac{odd}{2}$.
4. Because of the index y_{sp} always will be an $\frac{odd}{2}$, then the value x_{sp} will not be an element of the integer sequence. The symmetry point is not over any integer element of the sequence.
5. For all n, the duplicated elements $Y[y_{sp}+n+0.5] = Y[y_{sp}-n-0.5]$ will form all the pairs of duplicated quadratic elements of the sequence. Each element of the pair is part of the sequence, and both are equidistant from the sp.
6. Both duplicated elements $Y[y_{sp}+0.5] = Y[y_{sp}-0.5]$ will produce the closest pair of duplicated quadratic elements of the symmetry point. The duplicated elements are part of the sequence and are equidistant from the symmetry point.
7. Because $y_{sp} = 0.5$ for offset $f = 0$, then the symmetry of this even function (quadratic) is in the form of $Y[0.5+n+0.5] = Y[0.5-n-0.5]$, or $Y[y+1] = Y[-y]$.
8. The symmetry point is the symmetry element equidistant to all duplicated integers of the sequence. The symmetry point is not over any integer element of the sequence.
9. Because the symmetry point is equidistant to all pairs of duplicated elements, this remembers the destroyers in the "Battleship Game".
10. We will call this type of sequences as DES from the word "destroyer".

9.11. Features of the "ACC" (aircraft carrier) type quadratics

1. Color green: quadratics where $|a| \neq |b| > 0$ have the symmetry point not equidistant from the any two elements from the $(x_1; x_2; x_3)$ and all the 3 elements are different.
2. For offset $f = 0$, then $|a| > |b| > 0$. So, $0 < |y_{sp}| = \left|-\frac{b}{2a}\right| < 0.5$.

3. For any offset, $y_{sp} = -\frac{b}{2a} \neq integer$ and $y_{sp} = -\frac{b}{2a} \neq \frac{odd}{2}$.
4. Because of the index y_{sp} will never be an integer, then the value x_{sp} will not be an element of the integer sequence. The symmetry point is not over any integer element of the sequence.
5. There are no duplicated elements in the sequence. In the next study we can analyze the pairs of asymmetrical quadratic elements.
6. Always $Y[y] \neq Y[-y]$, and $Y[y+1] \neq Y[-y]$. The symmetry point (sp) not equidistant from the any two integer elements of the sequence.
7. The symmetry point is not equidistant from the any pair of (x_1, x_2, x_3). So, these quadratics are asymmetrical.
8. The symmetry point is not a symmetry element equidistant to any pair of integers of the sequence. There is no repeated integer in the sequence.
9. Because the symmetry point is not equidistant to any pair of elements of the sequence, this remembers the 3 closest points forming the aircraft carrier in the "Battleship Game".
10. We will call this type of sequences as ACC from the word "aircraft carrier".

9.12. C or D even functions

Note that any parabolic curve in the XY plane in the form of $Y[y] = ay^2 + by + c$ will have the "opening mouth", aperture of the parabola, towards the right or left.

- If $a > 0$, then the aperture will be facing right. This remembers the letter "C". So, we will be classifying it in our studies as a C-type parabola or C-type quadratic.
- If $a < 0$, then the aperture will be facing left. This remembers the letter "D". So, we will be classifying it in our studies as a D-type parabola or D-type quadratic.

9.13. S or Z odd functions

Analogously, for odd functions, we can have the shape of the curves resembling the letter S or the letter Z.

- If $a_{d=odd} > 0$, then the shape will remember the letter "S". So, we will be classifying it in our studies as a S-type polynomial or S-type curve.
- If $a_{d=odd} < 0$, then the shape will remember the letter "Z". So, we will be classifying it in our studies as a Z-type polynomial or Z-type curve.

10. General summary.

10.1. The simplest equations up to 6th degree polynomials.

$x = Y0[y] = x_1$

$x = Y1[y] = (x_2 - x_1)y + x_1$

$$x = Y2[y] = \frac{x_1 - 2x_2 + x_3}{2} y^2 + \frac{x_3 - x_2}{2} y + x_2$$

$$x = Y3[y] = \frac{-x_1 + 3x_2 - 3x_3 + x_4}{6} y^3 + \frac{x_1 - 2x_2 + x_3}{2} y^2 + \frac{-2x_1 - 3x_2 + 6x_3 - x_4}{6} y + x_2$$

$$x = Y4[y] = \frac{x_1 - 4x_2 + 6x_3 - 4x_4 + x_5}{24} y^4 + \frac{-x_1 + 2x_2 - 2x_4 + x_5}{12} y^3 + \frac{-x_1 + 16x_2 - 30x_3 + 16x_4 - x_5}{24} y^2 + \frac{x_1 - 8x_2 + 8x_4 - x_5}{12} y + x_3$$

$$x = Y5[y] = \frac{-x_1 + 5x_2 - 10x_3 + 10x_4 - 5x_5 + x_6}{120} y^5 + \frac{x_1 - 4x_2 + 6x_3 - 4x_4 + x_5}{24} y^4 + \frac{-x_1 - x_2 + 10x_3 - 14x_4 + 7x_5 - x_6}{24} y^3 + \frac{-x_1 + 16x_2 - 30x_3 + 16x_4 - x_5}{24} y^2 + \frac{3x_1 - 30x_2 - 20x_3 + 60x_4 - 15x_5 + 2x_6}{60} y + x_3$$

$$x = Y6[y] = \frac{x_1 - 6x_2 + 15x_3 - 20x_4 + 15x_5 - 6x_6 + x_7}{720} y^6 + \frac{-x_1 + 4x_2 - 5x_3 + 5x_5 - 4x_6 + x_7}{240} y^5 + \frac{-x_1 + 12x_2 - 39x_3 + 56x_4 - 39x_5 + 12x_6 - x_7}{144} y^4 + \frac{x_1 - 8x_2 + 13x_3 - 13x_5 + 8x_6 - x_7}{48} y^3 + \frac{2x_1 - 27x_2 + 27x_3 g - 490x_4 + 270x_5 - 27x_6 + 2x_7}{360} y^2 + \frac{-x_1 + 9x_2 - 45x_3 + 45x_5 - 9x_6 + x_7}{60} y + x_4$$

Sequences of the denominators of y^d: $\{1,1,2,6,24,120,720,\ldots\}$ == A000142 Factorial numbers
Sequences of the denominators of y^{d-1}: $\{1,2,2,12,24,240,\ldots\}$ == Axxxxxx

10.2. Symmetry Point Y-coordinate up to 6th degree polynomials.

$$y_{sp_{Y0[y]}}\left[@\frac{d^{-1}Y0[y]}{dy^{-1}} = 0\right] = undetermined$$

$$y_{sp_{Y1[y]}}\left[@\frac{d^{0}Y1[y]}{dy^{0}} = 0\right] = -\frac{c}{b} = -\left(\frac{x_1}{x_2 - x_1}\right)$$

$$y_{sp_{X2[y]}}\left[@\frac{d^{1}Y2[y]}{dy^{1}} = 0\right] = -\frac{b}{2a} = -\frac{1}{2}\left(\frac{-x_1 + x_3}{x_1 - 2x_2 + x_3}\right)$$

$$y_{sp_{Y3[y]}}\left[@\frac{d^2Y3[y]}{dy^2}=0\right]=-\frac{a}{3a_3}=-1\left(\frac{x_1-2x_2+x_3}{-x_1+3x_2-3x_3+x_4}\right)$$

$$y_{sp_{Y4[y]}}\left[@\frac{d^3Y4[y]}{dy^3}=0\right]=-\frac{a_3}{4a_4}=-\frac{1}{2}\left(\frac{-x_1+2x_2-2x_4+x_5}{x_1-4x_2+6x_3-4x_4+x_5}\right)$$

$$y_{sp_{Y5[y]}}\left[@\frac{d^4Y5[y]}{dy^4}=0\right]=-\frac{a_4}{5a_5}=-1\left(\frac{x_1-4x_2+6x_3-4x_4+x_5}{-x_1+5x_2-10x_3+10x_4-5x_5+x_6}\right)$$

$$y_{sp_{Y6[y]}}\left[@\frac{d^5Y6[y]}{dy^5}=0\right]=-\frac{a_5}{6a_6}=-\frac{1}{2}\left(\frac{-x_1+4x_2-5x_3+5x_5-4x_6+x_7}{x_1-6x_2+15x_3-20x_4+15x_5-6x_6+x_7}\right)$$

...

$$y_{sp_{Yd[y]}}\left[@\frac{d^{d-1}Yd[y]}{dy^{d-1}}=0\right]=-\frac{a_{d-1}}{d.a_d}$$

10.3. Recurrence equations with increasing index.

Y0[y]=+1Y0[y-1]
Y1[y]=-1Y1[y-2]+2Y1[y-1]
Y2[y]=+1Y2[y-3]-3Y2[y-2]+3Y2[y-1]
Y3[y]=-1Y3[y-4]+4Y3[y-3]-6Y3[y-2]+4Y3[y-1]
Y4[y]=+1Y4[y-5]-5Y4[y-4]+10Y4[y-3]-10Y4[y-2]+5Y4[y-1]
Y5[y]=-1Y5[y-6]+6Y5[y-5]-15Y5[y-4]+20Y5[y-3]-15Y5[y-2]+6Y5[y-1]
Y6[y]=+1Y6[y-7]-7Y6[y-6]+21Y6[y-5]-35Y6[y-4]+35Y6[y-3]-21Y6[y-2]+7Y6[y-1]

10.4. Recurrence equations with decreasing index.

\Y0 [y]\=1Y0[y+1]

\Y1 [y]\=2Y1 [y+1]-1Y1[y+2]

\Y2[y]\=3Y2[y+1]-3Y2[y+2]+1Y2[y+3]

\Y3 [y]\=4Y3 [y+1]-6Y3 [y+2]+4Y3 [y+3]-1Y3[y+4]

\Y4 [y]\=5Y4 [y+1]-10Y4 [y+2]-10Y4 [y+3]-5Y4 [y+4]+1Y4[y+5]

\Y5 [y]\=6Y5 [y+1]-10Y5 [y+2]+20Y5 [y+3]-15Y5 [y+4]+6Y5[y+5]-1Y5[y+6]

\Y6 [y]\=7Y6 [y+1]-21Y6 [y+2]+35Y6 [y+3]-35Y6 [y+4]+21Y6 [y+5]-7Y6 [y+6]+1Y6 [y+7]

10.5. The method of common differences in polynomials.

$$recursivelly\big[F[y+1]-F[y]\big] = recursivelly[(y+1)^n - y^n] = \frac{d^n}{dy}(y^n) =$$

$n! \equiv$ http://oeis.org/A000142

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Conflicts of Interest.

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