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# A perturbed elliptic problem involving the $p(x)$-Kirchhoff type triharmonic operator

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**Abstract:** This paper examines the existence of weak solutions for a nonlinear boundary value problem of $p(x)$-Kirchhoff type involving the $p(x)$-Kirchhoff type triharmonic operator and perturbed external source terms. We establish our results by using a Fredholm-type result for a couple of nonlinear operators, in the framework of variable exponent Sobolev spaces.

**Key words:** p(x)-kirchhoff type problem, variable exponent Sobolev space, Fredholm alternative

## 1. Introduction

The purpose of this work is to investigate the existence of weak solutions for the following nonlinear elliptic problem involving the $p(x)$-Kirchhoff type triharmonic operator, with Navier boundary conditions

$$\begin{gathered} -M(L(u))\Delta^3_{p(x)}u = f_\lambda(x,u,\nabla u,\Delta u,\nabla\Delta u) \text{ en } \Omega, \\ u = \Delta u = \Delta^2 u = 0 \text{ en } \partial\Omega, \end{gathered} \tag{1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, and $N \geq 3$, $p \in C(\overline{\Omega})$ for any $x \in \overline{\Omega}$; $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, $L(u) = \int_\Omega \frac{1}{p(x)} |\nabla\Delta u|^{p(x)}\, dx$, $\Delta^3_{p(x)}u := div\left(\Delta\left(|\nabla\Delta u|^{p(x)-2}\nabla\Delta u\right)\right)$ is the so-called $p(x)$-triharmonic operator, $p \in C_+(\overline{\Omega}) = \{h : h \in C(\overline{\Omega}), h(x) > 1 \text{ for any } x \in \overline{\Omega}\}$; $1 < p^- := \min_{\overline{\Omega}} p(x) \leq p^+ := \max_{\overline{\Omega}} p(x) < N$, $f_\lambda = f_1 + \lambda f_2$, where $f_1, f_2$ are continuous functions and $\lambda \geq 0$.

The study of differential and partial differential equations with variable exponent has received considerable attention in recent years. This interest reflects directly into various range of applications. There are applications concerning elastic mechanics [37], thermorheological and electrorheological fluids [3, 34], image restoration [9] and mathematical biology [24]. In the context of the study of elliptic Navier boundary problems, many results have been obtained, for example [10, 11, 28, 32]; however, there are few contributions to the study of the triharmonic problems with reaction term $f(x,t,z,y,w)$ depending on on the gradient, the Laplacian and the gradient of the Laplacian of the solution. We can cite [4, 5, 26, 31, 33, 35]. Recently, Mehraban et al. [30] considered the existence and multiplicity of solutions for the problem (1),


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with $M(t) = 1, f_\lambda(x, u, \nabla u, \Delta u, \nabla\Delta u) := \mu f(x, u) + \lambda g(x, u)$. We notice that if we choose the functional $L(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx$ then we have the problem

$$-M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\Big)\,\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x, u) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega, \tag{2}$$

which is called the $p(x)$-Kirchhoff type equation. The problem (2) has some physical motivations as follows. Indeed, it is related with a physical model

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2\,dx\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{3}$$

which extends the classical D'Alembert's wave equation, by considering the effect of the changing in the length of the string during the vibration. A distinct feature is that the model (3) contains a nonlocal coefficient $\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2\,dx$ which depends on the average $\frac{1}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx$, and hence the equation is no longer a pointwise identity. Problem (3) has received a lot of attention after Lions [29] proposed an abstract framework for this problem, see e.g. [2, 8] and [13]-[18]. The study of Kirchhoff type equations has already been extended to the case involving the $p$-Laplacian (for details, see [13, 14, 18, 20]) and $p(x)$-Laplacian (see [12, 15–17, 25, 32, 40]).

Motivated by the above references, the results in Rahal [33] and the importance of sixth order elliptic equation in the modeling of ulcers [38], viscous fluid, geometric design [39], in this paper we investigate the existence of weak solutions of problem (1). Due to the presence of $\nabla u, \triangle u$ and $\nabla\triangle u$ in $f$ the most usual variational techniques can not used to study it; so we adapt topological tools: a Fredholm type theorem for a couple of nonlinear operators due to Dinca [19]. As far as we know, our work is the first attempt to consider a $p(x)$-Kirchhoff triharmonic problem with a $(\nabla u, \triangle u, \nabla\triangle u)$-dependent nonlinearity $f$. It is worth noting that, in this work, $f$ does not satisfies typical growth conditions. Also, we study the uniqueness of the weak solutions under suitable assumptions on the nonlinearity.

This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on variable exponent Sobolev spaces. In Section 3, we state and prove our main results.

## 2. Preliminaries

To discuss problem (1), we need some theory on $W^{1,p(x)}(\Omega)$ which is called variable exponent Sobolev space (for details, see [21]). Denote by $\mathbf{S}(\Omega)$ the set of all measurable real functions defined on $\Omega$. Two functions in $\mathbf{S}(\Omega)$ are considered as the same element of $\mathbf{S}(\Omega)$ when they are equal almost everywhere. Write

$$C_+(\overline{\Omega}) = \{h : h \in C(\overline{\Omega}), h(x) > 1 \text{ for any } x \in \overline{\Omega}\},$$
$$h^- := \min_{\overline{\Omega}} h(x), \quad h^+ := \max_{\overline{\Omega}} h(x) \quad \text{for every } h \in C_+(\overline{\Omega}).$$

Define

$$L^{p(x)}(\Omega) = \{u \in \mathbf{S}(\Omega) : \int_\Omega |u(x)|^{p(x)}\,dx < +\infty \text{ for } p \in C_+(\overline{\Omega})\}$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}\,dx \le 1\},$$

and

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq k\}$$

with the norm

$$\|u\|_{k,p(x)} \equiv \|u\|_{W^{k,p(x)}(\Omega)} = \sum_{|\alpha|\leq k} |D^\alpha u|_{L^{p(x)}(\Omega)}.$$

**Proposition 2.1** ([21]). *The spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

**Proposition 2.2** ([21]). *Set $\rho(u) = \int_\Omega |u(x)|^{p(x)}\, dx$. For any $u \in L^{p(x)}(\Omega)$, then*

*(1) for $u \neq 0$, $|u|_{p(x)} = \lambda$ if and only if $\rho(\frac{u}{\lambda}) = 1$;*

*(2) $|u|_{p(x)} < 1$ $(= 1; > 1)$ if and only if $\rho(u) < 1$ $(= 1; > 1)$;*

*(3) if $|u|_{p(x)} > 1$, then $|u|^{p^-}_{p(x)} \leq \rho(u) \leq |u|^{p^+}_{p(x)}$;*

*(4) if $|u|_{p(x)} < 1$, then $|u|^{p^+}_{p(x)} \leq \rho(u) \leq |u|^{p^-}_{p(x)}$;*

*(5) $\lim_{k\to+\infty} |u_k|_{p(x)} = 0$ if and only if $\lim_{k\to+\infty} \rho(u_k) = 0$;*

*(6) $\lim_{k\to+\infty} |u_k|_{p(x)} = +\infty$ if and only if $\lim_{k\to+\infty} \rho(u_k) = +\infty$.*

**Proposition 2.3** ([21, 22]). *If $q \in C_+(\overline{\Omega})$ and $q(x) \leq p^*_k(x)$ ($q(x) < p^*_k(x)$) for $x \in \overline{\Omega}$, then there is a continuous (compact) embedding $W^{k,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, where*

$$p^*_k(x) = \begin{cases} \frac{Np(x)}{N-kp(x)} & \text{if } kp(x) < N, \\ +\infty & \text{if } kp(x) \geq N. \end{cases}$$

The space $W^{1,p(x)}_0(\Omega)$ is the closure of $C^\infty_0(\Omega)$ in $W^{1,p(x)}(\Omega)$. We denote by

$$X = W^{1,p(x)}_0(\Omega) \cap W^{3,p(x)}(\Omega)$$

and define a norm $\|.\|_X$ by

$$\|u\|_X = \|u\|_{1,p(x)} + \|u\|_{2,p(x)} + \|u\|_{3,p(x)}.$$

Moreover, the norms $\|u\|_X$ and $\|\nabla\triangle u\|_{p(x)}$ are equivalent on $X$. Let

$$\|u\| = \inf\left\{\mu > 0 : \int_\Omega \left|\frac{\nabla\triangle u}{\mu}\right|^{p(x)} dx \leq 1\right\}$$

for any $u \in X$. Hence, we see that $\|u\|$ is equivalent to the norms $\|u\|_X$ and $\|\nabla\triangle u\|_{p(x)}$ in $X$. From now on, we will use $\|.\|$ instead of $\|u\|_X$ on $X$.

**Proposition 2.4** ([21, 23]). *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{q(x)}(\Omega)$, where $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$ holds a.e. in $\Omega$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have the following Hölder-type inequality*

$$\left|\int_\Omega uv\, dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right)|u|_{p(x)}|v|_{q(x)}.$$

**Proposition 2.5** ([21, 23]). *Let $p(x)$ and $q(x)$ be measurable functions such that $p(x) \in L^{\infty}(\Omega)$ and $1 \leq p(x)q(x) \leq \infty$, for a.e.$x \in \Omega$. Let $u \in L^{q(x)}(\Omega), u \neq 0$. Then, we have*

$$i)\text{For } |u|_{p(x)q(x)} \leq 1,\ |u|^{p^+}_{p(x)q(x)} \leq ||u|^{p(x)}|_{q(x)} \leq |u|^{p^-}_{p(x)q(x)}$$

$$ii)\text{For } |u|_{p(x)q(x)} > 1,\ |u|^{p^-}_{p(x)q(x)} \leq ||u|^{p(x)}|_{q(x)} \leq |u|^{p^+}_{p(x)q(x)}$$

**Proposition 2.6** ([21, 23]). *Set $\Psi_{p(x)}(u) = \int_\Omega |\nabla\triangle u|^{p(x)}\, dx$ for any $u \in X$. Then, we have*

*(1) if $\|u\| \geq 1$, then $\|u\|^{p^-} \leq \Psi_{p(x)}(u) \leq \|u\|^{p^+}$;*

*(4) if $\|u\| \leq 1$, then $\|u\|^{p^+} \leq \Psi_{p(x)}(u) \leq \|u\|^{p^-}$.*

**Theorem 2.1** ([19]). *Let $X$ and $Y$ be real Banach spaces and two nonlinear operators $T, S : X \to Y$ such that*

1. *$T$ is bijective and $T^{-1}$ is continuous.*
2. *$S$ is compact.*
3. *Let $\mu \neq 0$ be a real number such that: $\|(\mu T - S)(x)\| \to +\infty$ as $\|x\| \to +\infty$;*
4. *There is a constant $R > 0$ such that*
   *$\|(\mu T - S)(x)\| > 0$ if $\|x\| \geq R$, $\quad d_{LS}(I - T^{-1}(\frac{S}{\mu}), B(\theta, R), 0) \neq 0$.*

*Then $\mu I - S$ is suryective from $X$ onto $Y$.*

Here $d_{LS}(G, B, 0)$ denotes the Leray-Schauder degree.

**Definition 2.1.** A function $u \in X$ is said to be a weak solution of (1) if

$$(P_\lambda) \qquad M\left(\int_\Omega \frac{1}{p(x)} |\nabla\Delta u|^{p(x)}\, dx\right) \int_\Omega |\nabla\Delta u|^{p(x)-2}\, \nabla\Delta u \cdot \nabla\Delta v\, dx = \int_\Omega f_\lambda(x, u, \nabla u, \Delta u, \nabla\Delta u) v\, dx$$

for all $v \in X$.

Suppose that $M$ and $f_\lambda$ satisfy the following hypotheses:

($M_0$) $M : [0, +\infty[ \to [m_0, +\infty[$ is a continuous and nondecreasing function with $m_0 > 0$.

($F_1$) $f_\lambda = f_1 + \lambda f_2,\ \lambda \geq 0,\ f_i \in C\left(\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n; \mathbb{R}\right),\ i = 1, 2$ and there exists a positive constant $c_1$ such that

$$|f_i(x, s, \xi, t, \zeta)| \leq c_1(\sigma_i(x) + |s|^{\eta_i(x)} + |\xi|^{\delta_i(x)} + |t|^{\delta_i(x)} + |\zeta|^{\delta_i(x)}), \quad \forall x \in \Omega,$$

$$\forall s, t \in \mathbb{R}, \zeta, \xi \in \mathbb{R}^n, \text{ where } \eta_i, \delta_i \in C(\overline{\Omega}),\ q \in C_+(\overline{\Omega}), \frac{1}{p(x)} + \frac{1}{p'(x)} = 1,$$

$$\sigma_i \in L^{p'(x)}(\Omega),\ 0 \leq \eta_1(x) < p(x) - 1,\ 0 \leq \delta_i(x) < \frac{p(x) - 1}{p'(x)}, i = 1, 2;$$

$$p^- + 1 \leq \eta_2(x) < p^+ + 1 \text{ for } x \in \overline{\Omega}.$$

### 3. Existence of solutions

In this section we will discuss the existence of weak solutions of (1). Our first result is as follows.

**Theorem 3.1.** *Assume $\lambda = 0$ and that ($M_0$) and ($F_1$) hold. Then* (1) *has a weak solution in $X$.*

*Proof.* In order to apply theorem (2.1), we take $Y = X'$ and the operators $T, S_\lambda : X \to X'$ in the following way

$$\langle Tu, v\rangle = M\left(\int_\Omega \frac{1}{p(x)}\, |\nabla\Delta u|^{p(x)}\, dx\right)\int_\Omega |\nabla\Delta u|^{p(x)-2}\,\nabla\Delta u\cdot\nabla\Delta v\, dx,$$

$$\langle S_\lambda u, v\rangle = \int_\Omega f_\lambda(x, u, \nabla u, \Delta u, \nabla\Delta u) v\, dx,$$

for all $u, v \in X$.

Then $u \in X$ is a solution of (1) if and only if

$$Tu = S_\lambda u \quad \text{in } X'.$$

In what follows, for simplicity we denote $S \equiv S_0, f \equiv f_1, \eta \equiv \eta_1, \delta \equiv \delta_1$.
Take $\lambda = 0$. For the convenience of the reader, we will divide the proof into five steps.

**Step1**. We prove that $T$ is an injection.

First we observe that

$$\Phi(u) = \widehat{M}\Big(L(u)\Big), \qquad \text{where} \quad \widehat{M}(s) = \int_0^s M(t)\, dt,$$

is a continuously Gâteaux differentiable function whose Gâteaux derivative at the point $u \in X$ is the functional $\Phi'(u) \in X'$ given by

$$\langle \Phi'(u), v\rangle = \langle T(u), v\rangle \quad \text{for all } v \in X.$$

On the other hand, by applying a standard argument, we can show that $L \in C^1(X, \mathbb{R})$ and

$$\langle L'(u), v\rangle = \int_\Omega |\nabla\Delta u|^{p(x)-2}\,\nabla\Delta u.\nabla\Delta v dx, \ \text{for all } u, v \in X.$$

for all $u, v \in X$.

By taking into account the inequality [36, (2.2)]

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y\rangle \geq \begin{cases} C_p|x-y|^p & \text{if } p \geq 2 \\ C_p\frac{|x-y|^2}{(|x|+|y|)^{p-2}},\ (x,y) \neq (0,0) & \text{if } 1 < p < 2, \end{cases} \tag{4}$$

for all $x, y \in \mathbb{R}^N$, we obtain

$$\langle L'(u) - L'(v), u - v\rangle > 0 \qquad \text{for all } u, v \in X \quad \text{with } u \neq v,$$

that is, $L'$ is strictly monotone and thus, by [41, Prop. 25.10], $L$ is strictly convex. Furthermore, as $M$ is nondecreasing, $\widehat{M}$ is convex in $[0, +\infty[$. Consequently, for every $u, v \in X$ with $u \neq v$, and every $s, t \in (0, 1)$ with $s + t = 1$, one has

$$\widehat{M}(L(su + tv)) < \widehat{M}(sL(u) + tL(v)) \leq s\widehat{M}(L(u)) + t\widehat{M}(L(v)).$$

This shows that $\Phi$ is strictly convex, and as $\Phi'(u) = T(u)$ in $X'$ it follows that $T$ is strictly monotone in $X$, consequently $T$ is an injection.

**Step2**. We prove that the inverse $T^{-1} : X' \to X$ of $T$ is continuous.

For any $u \in X$ with $\|u\| > 1$, one has

$$\frac{\langle T(u), u\rangle}{\|u\|} = \frac{M\Big(\int_\Omega \frac{1}{p(x)} |\nabla\Delta u|^{p(x)}\,dx\Big)\Big[\int_\Omega |\nabla\Delta u|^{p(x)}\,dx\Big]}{\|u\|}$$
$$\geq m_0\|u\|^{p^- -1},$$

from which we have the coercivity of $T$.

Since $T$ is the Fréchet derivative of $\Phi$, $T$ is continuous. Thus, in view of the well known Minty Browder theorem $T$ is a surjection and so $T^{-1} : X' \to X$ and it is bounded.

Now we prove the continuity of $T^{-1}$.

First, we verify that $T$ is of type $(S_+)$. In fact, if $u_\nu \rightharpoonup u$ in $V$ (so there exists $R > 0$ such that $\|u_\nu\| \leq R$ ) and the strict monotonicity of $T$ we have

$$0 = \limsup_{\nu\to\infty}\langle Tu_\nu - Tu, u_\nu - u\rangle = \lim_{\nu\to\infty}\langle Tu_\nu - Tu, u_\nu - u\rangle$$

Then

$$\lim_{\nu\to\infty}\langle Tu_\nu, u_\nu - u\rangle = 0$$

That is

$$\lim_{\nu\to\infty} M\Big(L(u_\nu)\Big)\int_\Omega |\nabla\triangle u_\nu|^{p(x)-2}\nabla\triangle u_\nu.(\nabla\triangle u_\nu - \nabla\triangle u)\,dx = 0 \tag{5}$$

Now, we have

$$|L(u_\nu)| \leq \frac{1}{p^-}\int_\Omega |\nabla\triangle u_\nu|^{p(x)}\,dx \leq \frac{1}{p^-}(\|u_\nu\|^{p^+} + 1) \leq C \tag{6}$$

So, $(L(u_\nu))_{\nu\geq 1}$ is bounded.

Then, since $M$ is continuous, up to a subsequence there is $t_0 \geq 0$ such that

$$M\Big(L(u_\nu)\Big) \to M(t_0) \geq m_0 \qquad \text{as } \nu \to \infty$$

This and (5) imply

$$\lim_{\nu\to\infty}\int_\Omega |\nabla\triangle u_\nu|^{p(x)-2}\nabla\triangle u_\nu.(\nabla\triangle u_\nu - \nabla\triangle u)\,dx = 0$$

By proceeding similarly to [[1], Proposition 2.5], one can obtain

$$\lim_{\nu\to\infty}\int_\Omega |\nabla\triangle u_\nu - \nabla\triangle u|^{p(x)}\,dx = 0.$$

Therefore, by the equivalence of norms on $X$ one has

$$u_\nu \to u \quad \text{strongly in } X \text{ as } \quad \nu \to \infty.$$

So, $T$ is of type $(S_+)$. Then, in view of [[6], Lemma 5.2 ], $T^{-1}$ is continuous.

**Step3** We prove that $S$ is a compact operator.

1.- The Nemytskii operator $\Psi : X \subseteq L^{p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ defined by

$$\Psi(u)(x) = f(x, u(x), \nabla u(x), \triangle u(x), \nabla\triangle u(x)) \quad \text{a.e in } \Omega \tag{7}$$

is bounded and continuous. In fact, using $(F_1)$ for all $u$ in $X$ we get

$$\begin{aligned}
&|f(x, u(x), \nabla u(x), \triangle u(x), \nabla\triangle u(x))|^{p'(x)} \\
&\leq c_1\big|\sigma(x) + |u(x)|^{\eta(x)} + |\nabla u(x)|^{\delta(x)} + |\triangle u(x)|^{\delta(x)} + |\nabla\triangle u(x)|^{\delta(x)}\big|^{p'(x)} \\
&\leq c_1' \left(|\sigma(x)|^{p'(x)} + |u(x)|^{\eta(x)p'(x)} + |\nabla u(x)|^{\delta(x)p'(x)} + |\triangle u(x)|^{\delta(x)p'(x)} + |\nabla\triangle u(x)|^{\delta(x)p'(x)}\right)
\end{aligned}$$

Since $0 \leq \eta(x) \leq p(x) - 1$, we have $0 \leq \eta(x)p'(x) < p(x)$, then

$$\begin{aligned}
&|f(x, u, \nabla u, \triangle u, \nabla\triangle u)|_{p'(x)} \leq \Big(\int_\Omega |f(x, u(x), \nabla u(x), \triangle u(x), \nabla\triangle u(x))|^{p'(x)}\, dx\Big)^{1/\alpha} \\
&\leq c_1' \Big[\int_\Omega \big(|\sigma(x)|^{p'(x)} + |u(x)|^{\eta(x)p'(x)} + |\nabla u(x)|^{\delta(x)p'(x)} + |\triangle u(x)|^{\delta(x)p'(x)} \\
&+ |\nabla\triangle u(x)|^{\delta(x)p'(x)}\big)\, dx\Big]^{1/\alpha} \leq C\left(1 + c_{\eta p'}^{\beta/\alpha}\|u\|^{\beta/\alpha} + c_{\delta p'}^{\theta/\alpha}\|u\|^{\theta/\alpha}\right)
\end{aligned}$$

where

$$\alpha = \begin{cases} p'^{-}, & \text{if } |f(x,u,\nabla u)|_{p'(x)} > 1, \\ p'^{+}, & \text{if } |f(x,u,\nabla u)|_{p'(x)} \leq 1, \end{cases} \quad , \quad \beta = \begin{cases} (\eta p')^{+}, & \text{if } |u|_{\eta(x)p'(x)} > 1, \\ (\eta p')^{-}, & \text{if } |u|_{\eta(x)p'(x)} \leq 1 \end{cases} \quad , \quad \text{and}$$

$$\theta = \begin{cases} (\delta p')^{+}, & \text{if } |\nabla u|_{\delta(x)p'(x)} > 1, \\ (\delta p')^{-}, & \text{if } |\nabla u|_{\delta(x)p'(x)} \leq 1, \end{cases} .$$

On the other hand, if $u_\nu \to u$ in $X$ up to a subsequence we have

$$\begin{aligned}
&u_\nu \to u, \nabla u_\nu \to \nabla u, \triangle u_\nu \to \triangle u \text{ and } \nabla\triangle u_\nu \to \nabla\triangle u \text{ a.e. in } \Omega \\
&|u_\nu(x)| \leq k(x), |\partial u_\nu(x)/\partial x_j| \leq w_j, |\partial^2 u_\nu(x)/\partial x_j^2| \leq z_j, \\
&|\partial^3 u_\nu(x)/\partial x_i \partial x_j^2| \leq l_{ij} \text{ a.e. } \quad x \in \Omega, \\
&\text{for some } k, w_j, z_j, l_{ij} \in L^{p(x)}(\Omega)
\end{aligned} \tag{8}$$

Then

$$\Psi(u_\nu)(x) \to \Psi(u)(x) \quad \text{a.e.} \quad x \in \Omega \tag{9}$$

But, it follows from $(F_1)$ and (8) that

$$\begin{aligned}
\Big|\Psi(u_\nu)(x) - \Psi(u)(x)\Big|^{p'(x)} &\leq C2^{(p')^+}\Big[|\Psi(u_\nu)(x)|^{(p')(x)} + |\Psi(u)(x)|^{(p')(x)}\Big] \\
&\leq h(x) \text{ a.e.} \quad x \in \Omega
\end{aligned} \tag{10}$$

where $h = h(k, w_j, z_j, l_{ij}) \in L^1(\Omega)$. Applying the Dominated Convergence Theorem with (9)-(10), we obtain

$$\lim_{\nu\to\infty} \int_\Omega \Big|\Psi(u_\nu)(x) - \Psi(u)(x)\Big|^{p'(x)} dx = 0$$

This implies that

$$\lim_{\nu\to\infty} \Big|\Psi(u_\nu)(x) - \Psi(u)(x)\Big|_{p'(x)} = 0.$$

2.- $S$ is well defined. Indeed

$$\begin{aligned} |\langle Su, v\rangle| &\le \int_\Omega |\Psi(u)(x)||v|\,dx \\ &\le C|\Psi(u)|_{p'(x)}|v|_{p(x)} \le C|\Psi(u)|_{p'(x)}\|v\| < \infty. \end{aligned}$$

3.- $S = I_2^* \circ \Psi \circ I_1$, where $I_1 : X \to L^{p(x)}(\Omega) \times L^{p(x)}(\Omega;\mathbb{R}^n) \times L^{p(x)}(\Omega) \times L^{p(x)}(\Omega;\mathbb{R}^n)$ is given by

$$I_1(u) = (u, \nabla u, \triangle u, \nabla\triangle u),$$

$\Psi$ is the Nemytskii operator in (7) and $I_2 : X \hookrightarrow L^{p'(x)}(\Omega)$ whose adjoint operator $I_2^* : L^{p(x)}(\Omega) \to X'$ is given by

$$(I_2^* v)(u) = \int_\Omega vu\,dx.$$

Since $I_1$ is linear and bounded, $\Psi$ is continuous and $I_2^*$ is continuous and compact, we conclude that $S$ is continuous and compact.

**Step4**

$$\|(T-S)(u)\| \to \infty \qquad \text{as } \|u\| \to \infty \quad \text{for } u \in X.$$

In fact, after some computations we get

$$\|Tu\| \ge m_0\|u\|^{p^- -1} \qquad \text{for all } \ u \in X \qquad \text{with } \ \|u\| > 1$$

and, since

$$\int_\Omega \Psi(u)(x)v\,dx \le C_p\left(\int_\Omega |\Psi u)(x)|^{p'(x)}\,dx\right)^{1/\alpha}\|v\| \text{ for all } \ u, v \in X,$$

we get

$$\|Su\| \le C_1\left(\|u\|^\theta + \|u\|^\vartheta\right)^{1/\alpha} + C_2 \qquad \text{for all } \ u \in X$$

for some $\theta \in [(\eta p')^-, (\eta p')^+]$ and $\vartheta \in [(\delta p')^-, (\delta p')^+]$.

Combining the above inequalities, we obtain

$$\|(T-S)(u)\| \ge \|Tu\| - \|Su\| \ge C_0\|u\|^{p^- -1} - C_1'\|u\|^{\theta/\alpha} - C_2'\|u\|^{\vartheta/\alpha} - C_3. \tag{11}$$

Here, we note that

$$0 \le \frac{\theta}{p'^+} < \frac{p^- - 1}{p'^+},\ 0 \le \frac{\theta}{p'^-} < \frac{p^- - 1}{p'^-},\ \frac{p^- - 1}{p'^+} \le \frac{p^- - 1}{p'^-}.$$

So, we have

$$0 \leq \frac{\theta}{\alpha} < \frac{p^{-}-1}{p'^{-}} < p^{-}-1$$

and, similarly we obtain $0 \leq \frac{\vartheta}{\alpha} < p^{-}-1$.

Since

$$\lim_{t\to\infty}(C_0 t^{p^{-}-1} - C_1' t^{\frac{\theta}{\alpha}} - C_2' t^{\frac{\vartheta}{\alpha}} - C_3) = \infty$$

and from (11) we conclude that $\|(T-S)(u)\| \to \infty \qquad \text{as } \|u\| \to \infty$.

Moreover, there exists $r_0 > 1$ such that $\|(T-S)(u)\| > 1$ for all $u \in X$, with $\|u\| > r_0$.

**Step5** Set

$$W = \{u \in X : \exists t \in [0,1] \qquad \text{such that } \ u = tT^{-1}(Su)\}.$$

Next, we prove that $W$ is bounded in $V$.
For $u \in W \setminus 0$, i.e. $u = tT^{-1}(Su)$ for some $t \in [0,1]$ we have

$$\|T(\frac{u}{t})\| = \|Su\| \leq C_1'\|u\|^{\theta/\alpha} + C_2'\|u\|^{\vartheta/\alpha} + C_3 \text{ with } \ t > 0. \tag{12}$$

Then, there exist three constants $a, b, c > 0$ such that

$$\begin{aligned} m_0\|u\|^{p^{+}-1} &\leq a\|u\|^{\eta^{-}} + b\|u\|^{\delta^{-}} + c \qquad \text{if } \ 0 < \|u\| < t, \\ m_0\|u\|^{p^{-}-1} &\leq a\|u\|^{\eta^{-}} + b\|u\|^{\delta^{-}} + c \qquad \text{if } \ t \leq \|u\| \leq 1, \\ m_0\|u\|^{p^{-}-1} &\leq a\|u\|^{\eta^{+}} + b\|u\|^{\delta^{+}} + c \qquad \text{if } \ 1 < \|u\|. \end{aligned}$$

Let $g_1, g_2 : [0,1] \to \mathbb{R}$ and $g_3 : ]1,\infty[ \to \mathbb{R}$ be defined by

$$\begin{aligned} &g_1(t) = m_0 t^{p^{+}-1} - a t^{\eta^{-}} - b t^{\delta^{-}} - c, \ \ g_2(t) = m_0 t^{p^{-}-1} - a t^{\eta^{-}} - b t^{\delta^{-}} - c, \\ &g_3(t) = m_0 t^{p^{-}-1} - a t^{\eta^{+}} - b t^{\delta^{+}} - c. \end{aligned}$$

The sets $\{t \in [0,1] : g_1(t) \leq 0\}, \{t \in [0,1] : g_2(t) \leq 0\}$ and $\{t \in ]1,\infty[ : g_3(t) \leq 0\}$ are bounded in $\mathbb{R}$.
From the above inequalities and (12) we infer that $W$ is bounded in $X$, so

$$W \subseteq B(0, r_1) \qquad \text{for some } \ r_1 > 0.$$

Now, taking $R = \max\{r_0, r_1\}$, it follows from [27, theorem 1.8] that

$$d_{LS}(I - tT^{-1}(S), B(0,R), 0) = 1 \qquad \text{for all } \ t \in [0,1].$$

In particular

$$d_{LS}(I - T^{-1}(S), B(0,R), 0) = 1.$$

Thus, the couple of nonlinear operators $(T, S)$ satisfies the hypotheses of theorem (2.1) for $\mu = 1$.Then $T - S : X \to X'$ is surjective.Therefore, there exists $u \in X$ such that

$$(T-S)u = 0 \qquad \text{in } X'.$$

With this step the proof of Theorem (3.1) is concluded. $\square$

We are now in a position to give the proof of our main result.

**Theorem 3.2.** *Assume that hypotheses ($M_0$) and ($F_1$) hold. If $\lambda > 0$ is small enough, then (1) has a weak solution in $X$.*

*Proof.* Thanks to the proof of Theorem (3.1), we have that all the solutions of $(P_0)$ are in $B(0,R)$. Hence

$$u - T^{-1}S_0(u) \neq 0, \quad \forall u \in \partial B(0,R)$$

From this, we have that

$$\rho := \inf_{u \in \partial B(0,r)} \|Tu - S_0 u\|_{X'} > 0$$

In fact, arguing by contradiction, assume that there exists a sequence $\{u_\nu\} \subset \partial B(0,R)$ such that

$$\|Tu_\nu - S_0 u_\nu\|_{X'} \to 0 \quad \text{as } \nu \to +\infty. \tag{13}$$

By construction, the sequence $\{u_\nu\}$ is bounded in $X$ and so (up to a subsequence) converge to some $u_0$ weakly in $X$

Hence, by the compactness of $S$, $\{S_0 u_\nu\}$ has a strong convergent subsequence in $X'$ (still denoted $\{S_0 u_\nu\}$) such that

$$\|S_0 u_\nu - S_0 u_0\|_{X'} \to 0 \quad \text{as } \nu \to +\infty.$$

Then, we get

$$\|Tu_\nu - S_0 u_0\|_{X'} \leq \|Tu_\nu - S_0 u_\nu\|_{X'} + \|S_0 u_\nu - S_0 u_0\|_{X'} \to 0$$

as $\nu \to +\infty$. Now, the continuity of $T^{-1}$ implies that $u_\nu \to T^{-1}S_0 u_0$ in $X$; thus we obtain

$$u_\nu \to u_0 \quad \text{in } X \tag{14}$$

(because $u_\nu \rightharpoonup u_0$). From (13) and (14) we have

$$\|Tu_0 - S_0 u_0\|_{X'} = 0.$$

So, $u_0$ solves $(P_0)$ and $\|u_0\| = R$, which is a contradiction. Therefore $\rho > 0$.

Since the Nemytskii operator $N_{f_2}$ is bounded and continuous from $X$ to $X'$, there exists $\varepsilon > 0$ such that

$$\|N_{f_2}(u)\|_{X'} \leq \varepsilon \quad \forall u \in \overline{B(0,R)}.$$

Set $\lambda_* = \frac{\rho}{\varepsilon}$, then for any $\lambda \in [0, \lambda_*[$ we have

$$\begin{aligned}
\|Tu - S_\lambda u\|_{X'} &= \|Tu - S_0 u + S_0 u - S_\lambda u\|_{X'} \\
&\geq \|Tu - S_0 u\|_{X'} - \|S_0 u - S_\lambda u\|_{X'} \\
&> \rho - \frac{\rho}{\varepsilon}.\varepsilon = 0, \quad \forall u \in \partial B(0,R).
\end{aligned}$$

Hence $Tu - S_\lambda u = 0$ does not have solution on $\partial B(0,R)$ for any $\lambda \in [0, \lambda_*[$. It follows that the Leray-Schauder degree $d_{LS}(I - T^{-1}S_{\sigma\lambda}, B(0,R), 0)$ is well defined for $\sigma \in [0,1]$, and

$$d_{LS}(I - T^{-1}S_\lambda, B(0,R), 0) = d_{LS}(I - T^{-1}S_0, B(0,R), 0) = 1,$$

where the last equality is due to that the equation $Tu = S_0 u$ has solution in $X$.Thus $Tu = S_\lambda u$ has a solution. $\square$

In the last part of this section, we will show that the solution of problem (1), for $\lambda = 0$, is unique. To this end, we also need the following hypotheses on the nonlinearity $f$.

$(F_2)$. There exists $\beta_2 \geq 0$ such that

$$(f(x,s_1,\xi,t,\zeta) - f(x,s_2,\xi,t,\zeta))\,(s_1 - s_2) \leq \beta_2 |s_1 - s_2|^{p(x)}$$

for a.e $x \in \Omega$ and all $s_1, s_2 \in \mathbb{R}$, $(\xi,t,\zeta) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$.

$(F_3)$. There exists $\beta_3 \geq 0$ such that

$$\left| f(x,s,\xi,t,\zeta) - f(x,s,\hat{\xi},\hat{t},\hat{\zeta}) \right| \leq \beta_3 |\zeta - \hat{\zeta}|^{p(x)-1}$$

for a.e $x \in \Omega$ and all $s_1, s_2 \in \mathbb{R}$, $(\xi,t,\zeta) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$.

**Theorem 3.3.** *Let $M : [0,+\infty[\to [m_0, m_1]$ be a function satisfying $(M_0)$ with $m_1 > m_0 > 0$ and, moreover $(F_2) - (F_3)$ hold. If, in addition $2 \leq p(x)$ for all $x \in \bar{\Omega}$, then (1) has a unique weak solution provided that*

$$\frac{p^+}{m_0}\left[\left(\beta_2 + \frac{\beta_3}{p^-}\right)\lambda_*^{-1} + \beta_3 \frac{p^+ - 1}{p^-}\right] < 1 \tag{15}$$

*where*

$$\lambda_* = \inf_{u \in X\setminus\{0\}} \frac{\int_\Omega |\nabla \triangle u|^{p(x)}\,dx}{\int_\Omega |u|^{p(x)}\,dx} > 0.$$

*Proof.* Theorem 3.1 gives a weak solution $u \in X$. It is enough to prove that $T - S : X \to X'$ is injective. Let $u_1, u_2$ be two weak solutions of (1) such that $(T-S)(u_1) = (T-S)(u_2)$. Hence

$$\langle T(u_1) - T(u_2), u_1 - u_2 \rangle = \langle S(u_1) - S(u_2), u_1 - u_2 \rangle. \tag{16}$$

But, in view of Lemma 3 in [7], assumptions $(M_0)$, $(F_2)$ and $(F_3)$, we get from (16) and the Young inequality that

$$\begin{aligned}
&\frac{m_0}{p^+}\int_\Omega |\nabla \triangle u_1 - \nabla \triangle u_2|^{p(x)}\,dx \leq m_0 \int_\Omega \frac{1}{p(x)} |\nabla \triangle u_1 - \nabla \triangle u_2|^{p(x)}\,dx \\
\leq& \langle T(u_1) - T(u_2), u_1 - u_2 \rangle \leq |\langle S(u_1) - S(u_2), u_1 - u_2 \rangle| \\
\leq& \int_\Omega |\Psi(u_1)(x) - \Psi(u_2)(x)||u_1 - u_2|\,dx \\
\leq& \int_\Omega (f(x,u_1,\nabla u_1,\triangle u_1,\nabla \triangle u_1) - f(x,u_2,\nabla u_1,\triangle u_1,\nabla \triangle u_1))(u_1 - u_2)\,dx \\
&+ \int_\Omega |f(x,u_2,\nabla u_1,\triangle u_1,\nabla \triangle u_1) - f(x,u_2,\nabla u_2,\triangle u_2,\nabla \triangle u_2)||u_1 - u_2|\,dx
\end{aligned}$$

$$
\begin{aligned}
&\leq \beta_2 \int_\Omega |u_1-u_2|^{p(x)}\,dx + \beta_3 \int_\Omega |\nabla\triangle(u_1-u_2)|^{p(x)-1}|u_1-u_2|\,dx \\
&\leq \beta_2 \lambda_*^{-1} \int_\Omega |\nabla\triangle(u_1-u_2)|^{p(x)}\,dx + \beta_3 \frac{p^+-1}{p^-} \int_\Omega |\nabla\triangle(u_1-u_2)|^{p(x)}\,dx \\
&\quad + \frac{\beta_3}{p^-}\lambda_*^{-1} \int_\Omega |\nabla\triangle(u_1-u_2)|^{p(x)}\,dx \\
&\quad = \left[\left(\beta_2 + \frac{\beta_3}{p^-}\right)\lambda_*^{-1} + \beta_3 \frac{p^+-1}{p^-}\right] \int_\Omega |\nabla\triangle(u_1-u_2)|^{p(x)}\,dx.
\end{aligned}
$$

Therefore, we obtain

$$
\int_\Omega |\nabla\triangle(u_1-u_2)|^{p(x)}\,dx = 0 \qquad \text{(by (15))}.
$$

So, $u_1 = u_2$. The proof is complete. □

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