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Optimal Control of Singular Systems. Continuous Time Case

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***Abstract*—In this paper for the class of singular dynamical systems we present optimal control results. We develop unified framework for feedback optimal and inverse optimal control involving a nonlinear-nonquadratic performance functional. It is shown that the cost functional can be evaluated in closed-form as long as the cost functional considered is related in a specific way to an underlying Lyapunov function that guarantees asymptotic stability of the nonlinear closed-loop singular system. Furthermore, the Lyapunov function is shown to be a solution of a steady-state, Hamilton-Jacobi-Bellman equation.**

I. Introduction

For the class of nonlinear singular dynamical systems we developed optimality results. We consider a feedback optimal control problem over an infinite horizon involving a nonlinear-nonquadratic performance functional. The performance functional involves a continuous-time cost for addressing performance of the continuous-time singular system dynamics. Furthermore, the cost functional can be evaluated in closed-form as long as the nonlinear-nonquadratic cost functional considered is related in a specific way to an underlying Lyapunov function that guarantees asymptotic stability of the nonlinear closed-loop singular system. This Lyapunov function is shown to be a solution of a steady-state, Hamilton-Jacobi-Bellman equation and thus guaranteeing both optimality and stability of the feedback controlled singular dynamical system. The overall framework provides the foundation for extending linear-quadratic feedback control methods to nonlinear singular dynamical systems. We note that the optimal control framework for singular dynamical systems developed herein is quite different from the quasivariational inequality methods for singular and hybrid control developed in the literature (e.g. Barles (1985a-b), Bardi and Dolcetta (1997), and Branicky, Borkar, and Mitter (1998)). Specifically, quasivariational methods do not guarantee asymptotic stability via Lyapunov functions and do not necessarily yield feedback controllers. In contrast, the proposed approach provides feedback controllers guaranteeing closed-loop stability via an underlying Lyapunov function.

An important contribution of the paper is to develop unified framework for the analysis and control synthesis of nonlinear singular dynamical systems.

The contents of the paper are as follows. In Section II we address an optimal control problem with respect to a nonlinear-nonquadratic performance functional for singular dynamical systems. To avoid complexity in solving the Hamilton-Jacobi-Bellman equation, in Section III we specialize the results of Section II to address an inverse optimal control problem for nonlinear affine (in the control) singular systems. Finally, we draw conclusions in Section IV, and define future work in Section V.

Finally, in this paper we use the following standard notation. Let $\mathbb{R}$ denote the set of real numbers, let $\mathcal{N}$ denote the set of nonnegative integers, let $\mathbb{R}^n$ denote the set of $n \times 1$ real column vectors, let $\mathbb{R}^{n\times m}$ denote the set of $n\times m$ real matrices, let $\mathbb{S}^n$ denote the set of $n \times n$ symmetric matrices, and let $\mathbb{N}^n$ (resp., $\mathbb{P}^n$) denote the set of $n \times n$ nonnegative (resp., positive) definite matrices, and let I_n or I denote the $n \times n$ identity matrix. Furthermore, $A \geq 0$ (resp., $A > 0$) denotes the fact that the Hermitian matrix is nonnegative (resp., positive) definite and $A \geq B$ (resp., $A > B$) denotes the fact that $A-B \geq 0$ (resp., $A-B > 0$). In addition, we write $V'(x)$ for the Fréchet derivative of $V(\cdot)$ at x. Finally, let C^0 denote the set of continuous functions and C^{r} denote the set of functions with r continuous derivatives.

II. Optimal Control for Singular Systems. Continuous Time Case

In this section we consider an optimal control problem for nonlinear singular dynamical systems involving notion of optimality with respect to a nonlinear-nonquadratic performance functional. Specifically, we consider the following singular optimal control problem.

Singular Optimal Control Problem. Consider the nonlinear singular controlled system given by

$$E_{\mathrm{c}}\dot{x}(t) = F_{\mathrm{c}}(x(t), u_{\mathrm{c}}(t), t), \qquad u_{\mathrm{c}}(t) \in \mathcal{U}_{\mathrm{c}}, \qquad \text{(II.1)}$$

where $x(t_0) = x_0$, $x(t_f) = x_f$, $t \geq 0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$ is the state vector , $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $u_{\mathrm{c}}(t) \in \mathcal{U}_{\mathrm{c}} \subseteq \mathbb{R}^{m_{\mathrm{c}}}$, $t \in [t_0, t_f]$, is the control input, $x(t_0) = x_0$ is given, $x(t_f) = x_f$ is fixed, $F_{\mathrm{c}} : \mathcal{D}\times\mathcal{U}_{\mathrm{c}}\times\mathbb{R} \to \mathbb{R}^n$ is Lipschitz continuous and satisfies $F_{\mathrm{c}}(0, 0, 0) = 0$. Matrix E_{c} may be singular matrix. Then determine the control input $u_{\mathrm{c}}(t) \in \mathcal{U}_{\mathrm{c}}, t \in [t_0, t_f]$, such that the performance functional

$$J(E_{\mathrm{c}}x_0, u_{\mathrm{c}}(\cdot), t_0) = \int_{t_0}^{t_f} L_{\mathrm{c}}(E_{\mathrm{c}}x(t), u_{\mathrm{c}}(t), t)\mathrm{d}t \quad \text{(II.2)}$$

is minimized, where $L_{\mathrm{c}} : \mathcal{D} \times \mathcal{U}_{\mathrm{c}} \times \mathbb{R} \to \mathbb{R}$ is given, Haddad, Chellaboina, and Kablar (2001b) and Kablar (2005a).

Next, we state a Bellman's principle of optimality for singular systems, Haddad, Chellaboina, and Kablar (2001b)

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and Kablar (2005a), which provides necessary and sufficient conditions, for a given control $u_c(t) \in \mathcal{U}_c, t \geq t_0$, for minimizing the performance functional (II.2).

Lemma. Let $u_c(t) \in \mathcal{U}_c, t \in [t_0, t_f]$, be an optimal control that generates the trajectory $x(t), t \in [t_0, t_f]$, with $x(t_0) = x_0$. Then the trajectory $x(\cdot)$ from (t_0, x_0) to (t_f, x_f) is optimal if and only if for all $t^{'}, t^{''} \in [t_0, t_f]$, the portion of the trajectory $x(\cdot)$ going from $(t', x^{'})$ to $(t^{''}, x(t^{''}))$ optimizes the same cost functional over $[t', t'']$, where $x(t') = x_1$ is a point on the optimal trajectory generated by $u_c(t)$, t$\in [t_0, t')$.

Next, let $u_c^*(t), t \in [t_0, t_f]$, solve the Singular Optimal Control Problem and define the optimal cost $J^*(x_0, t_0) = (x_0, u_c^*(\cdot), t_0)$. Furthermore, define, for $p \in \mathbb{R}^n$, the Hamiltonian $H_c(E_c x, u_c, p, t) = L_c(E_c x, u_c, t) + p^{\mathrm{T}} F_c(x, u_c, t))$.

Theorem II.1. *Let $J^*(E_c x, t)$ denote the minimal cost for the Singular Optimal Control Problem with $x_0 = x$ and $t_0 = t$ and assume that $J^*(\cdot, \cdot)$ is C^1 in x. Then*

$$0 = \frac{\partial J^*(E_c x(t), t)}{\partial t} + \min_{u_c(\cdot) \in \mathcal{U}_c} H_c(E_c x(t), u_c(t), p(t), t), \tag{II.3}$$

where $p(t) = (\frac{\partial J^*(E_c x(t), t)}{\partial t})^{\mathrm{T}}$. Furthermore, if $u_c^*(\cdot)$ solves the Singular Optimal Control Problem, then

$$0 = \frac{\partial J^*(E_c x(t), t)}{\partial t} + H_c(E_c x(t), u_c^*(t), p(t), t), \tag{II.4}$$

Haddad, Chellaboina, and Kablar (2001b) and Kablar (2005a).

Proof: Follows from the proof of the corresponding theorem of Haddad, Chellaboina, and Kablar (2001b). ■

Next, we provide a converse result to Theorem II.1.

Theorem II.2. *Suppose there exists a C^1 function $V : \mathcal{D} \times \mathbb{R} \to \mathbb{R}$ and an optimal control $u_c^*(\cdot)$ such that $V(E_c x(t_f), t_f) = 0$,*

$$0 = \tfrac{\partial V(E_c x, t)}{\partial t} + H_c(E_c x, u_c^*(t), \tfrac{\partial V^{\mathrm{T}}(E_c x, t)}{\partial x}, t), \tag{II.5}$$

$$H_c(E_c x, u_c^*(t), \tfrac{\partial V^{\mathrm{T}}(E_c x, t)}{\partial x}, t) \leq H_c(E_c x, u_c(t), \tfrac{\partial V^{\mathrm{T}}(E_c x, t)}{\partial x}, t), \quad u_c(\cdot) \in \mathcal{U}_c, \tag{II.6}$$

Then $u_c^*(\cdot)$ solves the Singular Control Problem, that is,

$$J^*(E_c x_0, t_0) = J(E_c x_0, u_c^*(\cdot), t_0) \leq J(E_c x_0, u_c(\cdot), t_0), \quad u_c(\cdot) \in \mathcal{U}_c, \tag{II.7}$$

and

$$J^*(E_c x_0, t_0) = V(E_c x_0, t_0). \tag{II.8}$$

Haddad, Chellaboina, and Kablar (2001b) and Kablar (2005a).

Proof: Follows from the proof of the corresponding theorem of Haddad, Chellaboina, and Kablar (2001b). ■

Next, we use Theorem II.2 to characterize optimal *feedback* controllers for nonlinear singular dynamical systems. To address the optimal nonlinear feedback control problem let $\phi_c : \mathcal{D} \to \mathcal{U}_c$ be such that $\phi_c(0) = 0$. If $u_c(t) = \phi_c(x(t)$, where $x(t),\ t \geq 0$, satisfies (II.1), then $u_c(\cdot)$ is a *feedback control*. Given the feedback control $u_c(t) = \phi_c(x(t)$, the closed-loop singular dynamical system has the form

$$E_c \dot{x}(t) = F_c(x(t), \phi_c(x(t)), \qquad x(t_0) = x_0, \tag{II.9}$$

Now, we present the main theorem for characterizing feedback controllers that guarantee closed-loop stability and minimize a nonlinear-nonquadratic performance functional over the infinite horizon. Furthermore, define the set of asymptotically stabilizing controllers by

$$\mathcal{C}(x_0) = \{ u_c(\cdot) : u_c(\cdot) \text{ is admissible and zero solution } x(t) \equiv 0, \text{ to (II.9) is asimptotically stable} \}. \tag{II.10}$$

Theorem II.3. *Consider the nonlinear controlled singular system (II.9) with performance functional*

$$J(E_c x_0, u_c(\cdot)) = \int_0^\infty L_c(E_c x(t), u_c(t))\mathrm{d}t, \tag{II.11}$$

where $u_c(\cdot)$ is an admissible control. Assume there exists a C^1 function $V : \mathcal{D} \to \mathbb{R}$ and a control law $\phi_c : \mathcal{D} \to \mathcal{U}_c$ such that $V(0) = 0, V(E_c x) \geq 0,\ x \neq 0,\ \phi_c(0) = 0$ and

$$V'(E_c x) F_c(x, F_c(x, \phi_c(x)) \leq 0, \quad x \neq 0, \tag{II.12}$$
$$H_c(E_c x, \phi_c(x)) = 0, \tag{II.13}$$
$$H_c(E_c x, u_c) \geq 0, \quad u_c \in \mathcal{U}_c \tag{II.14}$$

where

$$H_c(E_c x, u_c) = L_c(E_c x, u_c) + V'(E_c x) F_c(x, u_c), \tag{II.15}$$

Then, with the feedback control $u_c(\cdot)$, there exists a neighborhood of the origin $\mathcal{D}_0 \subseteq \mathcal{D}$ such that if $x_0 \in \mathcal{D}_0$, the zero solution $x(t) \equiv 0$ of the closed-loop system (II.9) is locally asymptotically stable. Furthermore,

$$\mathcal{J}(E_c x_0, \phi_c(x(\cdot))) = V(E_c x_0), \qquad x_0 \in \mathcal{D}_0. \tag{II.16}$$

In addition, if $x_0 \in \mathcal{D}_0$ then the feedback control $u_c(\cdot) = \phi_c(x(\cdot))$ minimizes $J(E_c x_0, u_c(\cdot)$ in the sense that

$$J(E_c x_0, \phi_c(x(\cdot))) = \min_{u_c(\cdot)} J(E_c x_0, u_c(\cdot)). \tag{II.17}$$

Finally, if $\mathcal{D} = \mathbb{R}^{\ltimes}$, $\mathcal{U}_c = \mathbb{R}^{\gtrdot_c}$, and $V(x) \to \infty$ as $\|x\| \to \infty$, then the zero solution $x(t) \equiv 0$ of the closed-loop system (II.9) is globally asymptotically stable, Haddad, Chellaboina, and Kablar (2001b) and Kablar (2005a).

Proof: Local and global asymptotic stability is a direct consequence of (II.12) by applying Theorem 3.2 of Kablar (2003b) to the closed-loop system (II.9). Conditions (II.16) are a direct consequence of Theorem II.2, with $V(E_c x, t) = V(E_c x)$, $t_0 = 0$, $t_f \to \infty$, and using the fact that $\lim_{t\to\infty} V(E_c x(t)) = 0$. ■

Remark II.1. *Theorem II.3 guarantees optimality with respect to the set of admissible stabilizing controllers $\mathcal{C}(x_0)$. However,* *it is important to note that an explicit characterization of* $\mathcal{C}(x_0)$ *is not required. In addition, the optimal stabilizing* feedback *control law* $u_c = \phi_c(x)$ *is independent of the initial condition* x_0.

Next, we specialize Theorem II.3 to linear singular systems. For the following result let $A_c \in \mathbb{R}^{\ltimes\times\ltimes}$, $B_c \in \mathbb{R}^{\ltimes\times\gtrdot_c}$, $R_{1c} \in \mathbb{R}^{\ltimes\times\ltimes}$, $R_{2c} \in \mathbb{R}^{\gtrdot_c\times\gtrdot_c}$ be given, where R_{1c} and R_{2c} are positive definite.

Corollary II.1. *Consider the linear controlled singular system*

$$E_c\dot{x}(t) = A_c x(t) + B_c u_c(t), \qquad x(0) = x_0, \tag{II.18}$$

with quadratic performance functional

$$J(E_c x_0, u_c(\cdot)) = \int_0^\infty [x^{\mathrm{T}}(t)E_c^{\mathrm{T}}R_{1c}E_c x(t) + u_c^{\mathrm{T}}(t)R_{2c}u_c(t)]\mathrm{d}t \tag{II.19}$$

where $u_c(\cdot)$ *is an admissible control. Furthermore, assume there exists a positive-definite matrix* $P \in \mathbb{R}^{\ltimes\times\ltimes}$ *such that*

$$0 = x^{\mathrm{T}}(A_c^{\mathrm{T}}PE_c + E_c^{\mathrm{T}}PA_c + E_c^{\mathrm{T}}R_{1c}E_c - E_cB_cR_{2c}^{-1}B_c^{\mathrm{T}}PE_c)x, \tag{II.20}$$

Then, the zero solution $x(t) \equiv 0$ *to (II.18) is globally asymptotically stable with the feedback controller*

$$u_c = \phi_c(x) = -R_{2c}^{-1}B_c^{\mathrm{T}}PE_c x, \tag{II.21}$$

and

$$J(E_c x_0, \phi_c(\cdot)) = x_0^{\mathrm{T}}E_c^{\mathrm{T}}PE_c x_0, \quad x_0 \in \mathbb{R}^{\ltimes}. \tag{II.22}$$

Furthermore,

$$J(E_c x_0, \phi_c(\cdot)) = \min_{(u_c(\cdot))\in\mathcal{C}(x_0)} J(x_0, u_c(\cdot)), \tag{II.23}$$

where $\mathcal{C}(x_0)$ *is the set of asymptotically stabilizing controllers for (II.18) and* $x_0 \in \mathbb{R}^{\ltimes}$, *Haddad, Chellaboina, and Kablar (2001b) and Kablar (2005a).*

Proof: The result is a direct consequence of Theorem II.3 with $F_c(x, u_c) = A_c x + B_c u_c$, $L_c(E_c x, u_c) = x^{\mathrm{T}}E_c^{\mathrm{T}}R_{1c}E_c x + u_c^{\mathrm{T}}R_{2c}u_c$, $V(E_c x) = x^{\mathrm{T}}E_c^{\mathrm{T}}PE_c x$, with argument $E_c x$. $\mathcal{D} = \mathbb{R}^{\ltimes}$, and $\mathcal{U}_c = \mathbb{R}^{\gtrdot_c}$. Specifically, it follows from (II.20) that $H_c(E_c x, \phi_c(x)) = 0, x \notin \mathcal{Z}_x$, and hence $V'(E_c x)F_c(x, \phi_c(x)) < 0$ for all $x \neq 0$. Thus, $H_c(E_c x, u_c) = H_c(E_c x, u_c) - H_c(E_c x, \phi_c(x)) = [u_c - \phi_c(x)]^{\mathrm{T}}R_{2c}[u_c - \phi_c(x)] \geq 0$, $x \notin \mathcal{Z}_x$, so that all conditions of Theorem II.3 are satisfied. Finally, since $V(\cdot)$ is radially unbounded, the zero solution $x(t) \equiv 0$ to (II.18) with $u_c(t) = \phi_c(x(t)) = -R_{2c}^{-1}B_c^{\mathrm{T}}PE_c x(t)$, is globally asymptotically stable. ■

Remark II.2. *The optimal feedback control* $\phi_c(x)$ *in Corollary II.1 is derived using the properties of* $H_c(E_c x, u_c)$ *as defined in Theorem ??. Specifically, since* $H_c(E_c x, u_c) = x^{\mathrm{T}}E_c^{\mathrm{T}}R_{1c}E_c x + u_c^{\mathrm{T}}R_{2c}u_c + x^{\mathrm{T}}(A_c^{\mathrm{T}}PE_c + E_c^{\mathrm{T}}PA_c)x + 2x^{\mathrm{T}}E_c^{\mathrm{T}}PB_cu_c$, *it follows that* $\frac{\partial^2 H_c}{\partial u_c^2} = R_{2c} > 0$. *Now,* $\frac{\partial H_c}{\partial u_c} = 2R_{2c} + 2B_c^{\mathrm{T}}PE_c x = 0$ *give the unique global minimum of* $H_c(E_c x, u_c)$. *Hence, since* $\phi_c(x)$ *minimizes* $H_c(E_c x, u_c)$ *it follows that* $\phi_c(x)$ *satisfies* $\frac{\partial H_c}{\partial u_c} = 0$ *or, equivalently,* $\phi_c(x) = -R_{2c}^{-1}B_c^{\mathrm{T}}PE_c x$.

Remark II.3. *For given* R_{1c}, R_{2c}, *(II.20) can be solved using constrained nonlinear programming methods using the structure of* $\mathcal{Z}_x$. *For example, in the case where* $\mathcal{Z}$ *is characterized by the hyperplane* $\mathcal{Z} = \{x \in \mathbb{R}^n : H(E_c x) = 0\}$, *where* $H \in \mathbb{R}^{m\times n}$, *it follows that (II.20) holds when* $x \in [\mathcal{N}(H)]^{\perp} = \mathcal{R}(H)^{\mathrm{T}}$, *where* $\mathcal{N}$ *denotes the null space of* H *and* $\mathcal{R}(H^{\mathrm{T}})$ *denotes the range space of* H^{T}. *Now, reformulating* $\mathcal{Z}$ *as* $\{x \in \mathbb{R}^n : Ex = 0\}$, *where* E *is an elementary matrix composed of zeroes and ones such that the columns of* E *span the nullspace of* H, *and using the fact that* $P > 0$, *(II.20) will hold for* $P > 0$ *with a specific internal matrix structure. This of course reduces the number of free elements in* P *satisfying (II.20). Alternatively, to avoid complexity in solving (II.20) and an inverse optimal control problem can be solved wherein* R_{1c}, R_{2c} *are arbitrary. In this case, (II.20) are implied by*

$$0 = A_c^{\mathrm{T}}PE_c + E_c^{\mathrm{T}}PA_c + E_c^{\mathrm{T}}R_{1c}E_c - E_c^{\mathrm{T}}PB_cR_{2c}B_c^{\mathrm{T}}PE_c, \tag{II.24}$$

Since R_{1c}, R_{2c} *are arbitrary, (II.24) can be cast as an LMI [5] feasibility problem involving*

$$P > 0, \qquad \begin{bmatrix} A_c^{\mathrm{T}}PE_c + E_c^{\mathrm{T}}PA_c & E_c^{\mathrm{T}}PB_c \\ B_c^{\mathrm{T}}PE_c & -R_{2c} \end{bmatrix} < 0, \tag{II.25}$$

III. Inverse Optimal Control for Nonlinear Affine Singular Systems

In this section we specialize Theorem II.3 to affine systems. The controllers obtained are predicated on an *inverse optimal control problem*. In particular, to avoid the complexity in solving steady-state Hamilton-Jacobi-Bellman equation we do not attempt to minimize a *given* cost functional, but rather, we parameterize a family of stabilizing controllers that minimize some *derived* cost functional that provides flexibility in specifying the control law. The performance integrand is shown to explicitly depend on the nonlinear singular system dynamics, the Lyapunov function of the closed-loop system, and the stabilizing feedback control law wherein the coupling is introduced via the Hamilton-Jacobi-Bellman equation. Hence, by varying the parameters in the Lyapunov function and the performance integrand, the proposed framework can be used to characterize a class of globally stabilizing controllers that can meet the closed-loop system response constraints.

Consider the affine (in the control) singular dynamical system

$$E_c\dot{x}(t) = f_c(x(t)) + G_c(x(t))u_c(t), \tag{III.26}$$

where $x(0) = x_0$. Furthermore, we consider performance integrand $L_c(E_c x, u_c)$ of the form

$$L_c(E_c x, u_c) = L_{1c} + u_c^{\mathrm{T}}R_{2c}(x)u_c, \tag{III.27}$$

where $L_{1c} : \mathbb{R}^n \to \mathbb{R}$ and satisfies $L_{1c}(x) \geq 0$, $x \in \mathbb{R}^n$, $R_{2c} : \mathbb{R}^n \to \mathbb{P}^{m_c}$, so that (II.2) becomes

$$J(E_c x_0, u_c(\cdot)) = \int_0^\infty [L_{1c}(E_c x(t)) + u_c^T(t) R_{2c}(x(t)) u_c(t)] dt \tag{III.28}$$

Corollary III.1. *Consider the nonlinear singular controlled system (III.26) with performance functional (III.28). Assume there exists a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$, such that $V(0) = 0, V(E_c x) \geq 0, x \in \mathbb{R}^n, x \neq 0,$*

$$V'(E_c x)[f_c(x) - \tfrac{1}{2} G_c(x) R_2^{-1} G_c^T(x) V'^T(E_c x)] < 0, \quad x \neq 0, \tag{III.29}$$

and

$$V(E_c x) \to \infty \quad \text{as} \quad \|x\| \to \infty. \tag{III.30}$$

Then the zero solution $x(t) \equiv 0$ to the closed-loop system

$$E_c \dot{x}(t) = f_c(x(t)) + G_c(x(t)) \phi_c(x(t)), \tag{III.31}$$

where $x(0) = x_0$, is globally asymptotically stable with the feedback control law

$$\phi_c(x) = \frac{1}{2} R_{2c}^{-1}(x) G_c^T(x) V'^T(E_c x), \tag{III.32}$$

and performance functional (III.28), with

$$L_{1c}(E_c x) = \phi_c^T(x) R_{2c}(x) \phi_c(x) - V'(E_c x) f_c(x), \tag{III.33}$$

is minimized in the sense that

$$J(E_c x_0, \phi_c(x(\cdot))) = \min_{u_c(\cdot) \in \mathcal{C}(x_0)} J(E_c x_0, u_c(\cdot)), \qquad x_0 \in \mathbb{R}^n.$$

Finally,

$$J(E_c x_0, \phi_c(x(\cdot))) = V(E_c x_0), \qquad x_0 \in \mathbb{R}^n. \tag{III.34}$$

Haddad, Chellaboina, and Kablar (2001b) and Kablar (2005a).

Proof: The result is a direct consequence of Theorem II.3 with $\mathcal{D} = \mathbb{R}^n$, $u_c \in \mathbb{R}^{m_c}$, $F_c(x, u_c) = f_c(x) + G_c(x) u_c$, $L_c(E_c x, u_c) = L_{1c} + u_c^T R_{2c}(x) u_c$. Specifically, with (III.27) the Hamiltonian has the form

$$H_c(E_c x, u_c) = L_{1c}(E_c x) + u_c^T R_{2c}(x) u_c + V'(E_c x)(f_c(x) + G_c(x) u_c), \; u_c \in \mathcal{U}_c. \tag{III.35}$$

Now, the feedback control law (III.32) is obtained by setting $\frac{\partial H_c}{\partial u_c} = 0$. With (III.32) it follows that (III.29) imply (II.12), respectively. Next, since $V(\cdot)$ is C^1 and $x = 0$ is a local minimum of $V(\cdot)$, it follows that $V'(0) = 0$, and hence, it follows that $\phi_c(0) = 0$. Next, with $L_{1c}(E_c x)$ given by (III.33), respectively, and $\phi_c(x)$, given by (III.32), (II.13) hold. Finally, since

$$\begin{aligned} H_c(E_c x, u_c) &= H_c(E_c x, u_c) - H_c(E_c x, \phi_c(x)) \\ &= [u_c - \phi_c(x)]^T R_{2c}(x) [u_c - \phi_c(x)], \end{aligned} \tag{III.36}$$

where $R_{2c}(x) > 0$, condition (II.14) hold. The result now follows as a direct consequence of Theorem II.3. ■

Remark III.1. *Note that (III.29) are equivalent to*

$$\dot{V}(E_c x) = V'(E_c x)[f_c(x) + G_c(x) \phi_c(x)] < 0, \qquad x \neq 0, \tag{III.37}$$

with $\phi_c(x)$ given by (III.32). Furthermore, condition (III.37) with $V(0) = 0$ and $V(E_c x) > 0$, $x \in \mathbb{R}^n$, $x \neq 0$, assure that $V(E_c x)$ is a Lyapunov function for the singular closed-loop system (III.31).

IV. Conclusion

In this paper we have developed a unified framework for feedback optimal control over an infinite horizon involving a hybrid nonlinear-nonquadratic performance functional. The overall framework provides the foundation for generalizing optimal linear-quadratic control methods to nonlinear singular dynamical systems.

V. Future Work

Further work will be focused on specializing this results to nonnegative, compartmental and large scale systems. They will be further extended to time-delay systems. Single result on optimality of time-delay singular systems is presented in Kablar (2012).

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