

Goldbach's Conjecture — Towards the Inconsistency of Arithmetic

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Abstract. This paper proves an inconsistency in Peano arithmetic (PA). We express a strengthened form of the strong Goldbach conjecture using a specific set. This set varies according to whether the conjecture or its negation is assumed. We show that, on the other hand, the set remains unchanged under these two assumptions. This causes a contradiction.

Notations. Let \mathbb{N} denote the natural numbers starting from 1, let \mathbb{N}_a denote the natural numbers starting from $a > 1$ and let \mathbb{P}_3 denote the prime numbers starting from 3.

Strengthened strong Goldbach conjecture (SSGB): *Every even number greater than 6 is the sum of two different odd primes.*

Theorem. *PA is inconsistent.*

Proof. Assuming that PA is consistent, we will show that the statement FALSE can be deduced.

We define the set $S_g := \{ (pk, mk, qk) \mid k, m \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q; m = (p + q) / 2 \}$.

The term S_g is not a standard part of PA, but it can easily be defined within PA.

SSGB is equivalent to saying that every integer $n \geq 4$ is the arithmetic mean of two different odd primes and so it is equivalent to saying that all integers $n \geq 4$ appear as m in a middle component mk of S_g . So, by the definition of S_g we have

$$\begin{aligned} \text{SSGB} &\Leftrightarrow \forall n \in \mathbb{N}_4 \exists (pk, mk, qk) \in S_g \quad n = m \\ \neg\text{SSGB} &\Leftrightarrow \exists n \in \mathbb{N}_4 \forall (pk, mk, qk) \in S_g \quad n \neq m. \end{aligned}$$

The set S_g has the following two properties.

First, the whole range of \mathbb{N}_3 can be expressed by the triple components of S_g ("covering"). We prove this by dividing it into the following three cases.

- (i) $x \in \mathbb{N}_3$ is prime. Then, $x = pk$ with $p \in \mathbb{P}_3$, $k = 1$.
- (ii) $x \in \mathbb{N}_3$ is composite and not a power of 2. Then, $x = pk$ with $p \in \mathbb{P}_3$, $k \neq 1$.
- (iii) $x \in \mathbb{N}_3$ is a power of 2. Then, $x = (p + q)k / 2$ with $p = 3$, $q = 5$, $k = (\text{a power of } 2)$.

So we have

$$(C) \forall x \in \mathbb{N}_3 \exists (pk, mk, qk) \in S_g \quad x = pk \vee x = mk.$$

A few examples of the covering:

$$x = 19: (19 \cdot 1, 21 \cdot 1, 23 \cdot 1), (19 \cdot 1, 60 \cdot 1, 101 \cdot 1)$$

$$x = 27: (3 \cdot 9, 7 \cdot 9, 11 \cdot 9)$$

$$x = 42: (3 \cdot 14, 5 \cdot 14, 7 \cdot 14), (7 \cdot 6, 9 \cdot 6, 11 \cdot 6)$$

$$x = 4096: (3 \cdot 1024, 4 \cdot 1024, 5 \cdot 1024)$$

$$x = 10000: (5 \cdot 2000, 6 \cdot 2000, 7 \cdot 2000).$$

Second, all pairs (p, q) of distinct odd primes are used in the definition of the set S_g ("maximality"). So we have

$$(M) \forall p, q \in \mathbb{P}_3, p < q \quad \forall k \in \mathbb{N} \quad (pk, mk, qk) \in S_g, \text{ where } m = (p + q) / 2.$$

$\neg(C)$ would immediately imply \neg SSGB since an $n \geq 4$ that is different from all S_g triple components pk and mk is in particular different from all m in S_g . So the property (C) excludes this possibility.

The property (M) excludes the possibility that if there is an $n \geq 4$ different from all m in S_g , then n is the arithmetic mean of a pair of distinct odd primes not used in S_g . So (M) rules out the possibility that the question of whether SSGB holds or not depends on whether (M) holds or not. (The proof would no longer be possible if we left out any pair of distinct odd primes in the formulation of SSGB and S_g .)

Therefore, in both cases SSGB and \neg SSGB, neither $\neg(C)$ nor $\neg(M)$ applies.

The basic idea is now the following.

There are two possibilities for S_g , exactly one of which must occur: Either there is an $n \in \mathbb{N}_4$ in addition to all the numbers m defined in S_g or there is not. The latter is equivalent to SSGB and the former is equivalent to \neg SSGB.

Since, due to (M), an $n \in \mathbb{N}_4$ different from all m cannot be the arithmetic mean of a pair of primes not used in S_g and since, due to (C), this n equals a component of some S_g triple that exists by definition, the covering of \mathbb{N}_3 by the S_g triples in the case n exists (\neg SSGB) is equal to that in the case n does not exist (SSGB). This causes a contradiction because in the case SSGB the numbers m defined in S_g take all integer values $x \geq 4$ whereas in the case \neg SSGB they don't.

The following steps are independent of the choice of n if, in the case of \neg SSGB, there is more than one that is different from all m . For example, the minimal such n works.

We split S_g into two complementary subsets in the following way. For any $y \in \mathbb{N}_3$, we write

$S_g = S_{g+(y)} \cup S_{g-(y)}$, with

$S_{g+(y)} := \{ (pk, mk, qk) \in S_g \mid \exists k' \in \mathbb{N} \quad pk = yk' \vee mk = yk' \vee qk = yk' \}$

$S_{g-(y)} := \{ (pk, mk, qk) \in S_g \mid \forall k' \in \mathbb{N} \quad pk \neq yk' \wedge mk \neq yk' \wedge qk \neq yk' \}$.

We define

$S_1 := \{ (pk, mk, qk) \in S_g \mid \text{SSGB} \}$

$S_2 := \{ (pk, mk, qk) \in S_g \mid \neg\text{SSGB} \}$.

Under the assumption \neg SSGB there is an $n \in \mathbb{N}_4$ as described above and under the assumption SSGB there is no such n . Then, since under both assumptions SSGB and \neg SSGB the possibilities \neg (C) and \neg (M) are both excluded,

$$(\forall y \in \mathbb{N}_3 \quad \text{SSGB} \Rightarrow S_1 = S_{g+(y)} \cup S_{g-(y)})$$

(1) \wedge

$$(\neg\text{SSGB} \Rightarrow S_2 = S_{g+(n)} \cup S_{g-(n)}).$$

Since $S_{g^+}(n) \cup S_{g^-}(n)$ is independent of n , we get

$$(\forall y \in \mathbb{N}_3 \quad \text{SSGB} \Rightarrow S_1 = S_{g^+}(y) \cup S_{g^-}(y))$$

(1') \wedge

$$(\forall y \in \mathbb{N}_3 \quad \neg \text{SSGB} \Rightarrow S_2 = S_{g^+}(y) \cup S_{g^-}(y)).$$

Now, we will make use of the following principle.

If two sets of (possibly infinitely many) z -tuples are equal, then the sets of their corresponding i -th components are equal; $1 \leq i \leq z$.

To this end, for each $k \in \mathbb{N}$ we define

$$M_1(k) := \{ mk \mid (pk, mk, qk) \in S_1 \}$$

$$M_2(k) := \{ mk \mid (pk, mk, qk) \in S_2 \}.$$

Then, applying the principle above to the middle component of the triples (pk, mk, qk) , (1') implies

$$(\forall k \in \mathbb{N} \quad \forall y \in \mathbb{N}_3 \quad \text{SSGB} \Rightarrow M_1(k) = \{ mk \mid (pk, mk, qk) \in S_{g^+}(y) \cup S_{g^-}(y) \})$$

(2) \wedge

$$(\forall k \in \mathbb{N} \quad \forall y \in \mathbb{N}_3 \quad \neg \text{SSGB} \Rightarrow M_2(k) = \{ mk \mid (pk, mk, qk) \in S_{g^+}(y) \cup S_{g^-}(y) \}).$$

Setting $M_1 := M_1(1)$ and $M_2 := M_2(1)$, we get

$$(\forall y \in \mathbb{N}_3 \quad \text{SSGB} \Rightarrow M_1 = \{ m \mid (p, m, q) \in S_{g^+}(y) \cup S_{g^-}(y) \})$$

(2') \wedge

$$(\forall y \in \mathbb{N}_3 \quad \neg \text{SSGB} \Rightarrow M_2 = \{ m \mid (p, m, q) \in S_{g^+}(y) \cup S_{g^-}(y) \}).$$

Since for every $y \in \mathbb{N}_3$ $S_{g^+}(y) \cup S_{g^-}(y)$ equals S_g by definition, for every $y \in \mathbb{N}_3$ $\{ m \mid (p, m, q) \in S_{g^+}(y) \cup S_{g^-}(y) \}$ equals the set $M := \{ m \mid (p, m, q) \in S_g \}$. So, from (2') we obtain

$$(3) \quad (SSGB \Rightarrow M_1 = M) \quad \wedge \quad (\neg SSGB \Rightarrow M_2 = M),$$

where M is a bound variable that equals either \mathbb{N}_4 if $SSGB$ is true or some non-empty proper subset Y of \mathbb{N}_4 if $SSGB$ is false.

Now, we make use of the following rule.

Let $P = P(A)$ be a proposition that depends on a set A . Then, for any set B ,

$$\begin{aligned} & (\text{we have a proof of } P(A) \quad \wedge \quad \text{we have a proof of } (P(A) \Rightarrow P(B))) \\ & \Rightarrow \\ & \text{we have a proof of } P(B). \end{aligned}$$

In the special case that A is a variable that is replaced by B , this becomes

$$\begin{aligned} & (\text{we have a proof of } P(A) \quad \wedge \quad \text{we have a proof of } ((P(A) \wedge A = B) \Rightarrow P(B))) \\ & \Rightarrow \\ & \text{we have a proof of } P(B), \end{aligned}$$

which gives

$$\text{we have a proof of } P(A) \Rightarrow \text{we have a proof of } P(B).$$

Since (3) depends on the variable M and since we have a proof of (3), we can apply the above rule with $P = (3)$. If $M = \mathbb{N}_4$ we use the rule with $A = M$ and $B = \mathbb{N}_4$, and if $M = Y$ we use it with $A = M$ and $B = Y$. Then, since either $M = \mathbb{N}_4$ or $M = Y$, from (3) we obtain

$$(3.1) \quad \text{we have a proof of } (SSGB \Rightarrow M_1 = \mathbb{N}_4 \quad \wedge \quad \neg SSGB \Rightarrow M_2 = \mathbb{N}_4)$$

\vee

$$(3.2) \quad \text{we have a proof of } (SSGB \Rightarrow M_1 = Y \neq \mathbb{N}_4 \quad \wedge \quad \neg SSGB \Rightarrow M_2 = Y \neq \mathbb{N}_4).$$

This implies

(3.1') (we have a proof of ($SSGB \Rightarrow M_1 = \mathbf{N}_4$)
 \wedge
we have a proof of ($\neg SSGB \Rightarrow M_2 = \mathbf{N}_4$))

\vee

(3.2') (we have a proof of ($SSGB \Rightarrow M_1 = Y \neq \mathbf{N}_4$)
 \wedge
we have a proof of ($\neg SSGB \Rightarrow M_2 = Y \neq \mathbf{N}_4$)).

Now, we will establish a contradiction to ((3.1') \vee (3.2')).

Under the assumption $SSGB$ the set $M = \{ m \mid (p, m, q) \in S_g \}$ is equal to \mathbf{N}_4 and under $\neg SSGB$ it is equal to $Y \neq \mathbf{N}_4$. Since $SSGB \Rightarrow M_1 = M$ and $\neg SSGB \Rightarrow M_2 = M$ by definition,

(4.1) we have a proof of ($SSGB \Rightarrow M_1 = \mathbf{N}_4$)

\wedge

(4.2) we have a proof of ($\neg SSGB \Rightarrow M_2 = Y \neq \mathbf{N}_4$).

Then, ((3.1') \vee (3.2')) together with ((4.1) \wedge (4.2)) implies

(5.1) we have a proof of ($\neg SSGB \Rightarrow M_2 = \mathbf{N}_4$)

\vee

(5.2) we have a proof of ($SSGB \Rightarrow M_1 = Y \neq \mathbf{N}_4$).

Because of $((4.1) \wedge (4.2))$ and because

we have a proof of $(\text{SSGB} \Rightarrow M_2 = \{ \} \neq \mathbb{N}_4)$

and

we have a proof of $(\neg \text{SSGB} \Rightarrow M_1 = \{ \} \neq Y)$,

we have a proof that $(M_2 = \mathbb{N}_4)$ is false and we have a proof that $(M_1 = Y \neq \mathbb{N}_4)$ is false.

Therefore, $((5.1) \vee (5.2))$ yields

(6.1) we have a proof of SSGB

\vee

(6.2) we have a proof of $\neg \text{SSGB}$.

Since we have neither a proof of SSGB nor of $\neg \text{SSGB}$, both (6.1) and (6.2) are false.

So, we obtain $(\text{FALSE} \vee \text{FALSE})$ and thus FALSE .

Hence, the assumption that PA is consistent was false.

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