MISCELLANEOUS SERIES IDENTITIES WITH CAUCHY AND HARMONIC NUMBERS, AND THEIR INTERPRETATION AS RAMANUJAN SUMMATION

Marc-Antoine Coppo
Department of Mathematics, University Côte d'Azur, CNRS, Nice, France
`coppo@unice.fr`

Abstract

We provide an overview of several series identities involving the Cauchy numbers and various types of harmonic numbers, all of which are closely related to certain alternating series with zeta values (or harmonic zeta values). We then give, for each of these identities, an interpretation in terms of Ramanujan summation with the hope that this unusual but interesting interpretation of still little-known formulas could inspire further research on the topic.

1. Introduction

A decade ago, we showed how Ramanujan's method of summation of series could be useful to generate a number of identities linking together Cauchy numbers (also known as Bernoulli numbers of the second kind), harmonic numbers, and special values of the Riemann zeta function at positive integers [4]. This powerful method is based on a binomial transformation formula that relates the Cauchy numbers to the Ramanujan summation of series [3, Theorem 18]. More recently, new series identities were obtained by a refinement of the same method, thanks notably to the consideration of a rather natural generalization of the harmonic numbers and new binomial identities (see [8] for details). Some special cases of these results are discussed in Section 3. On the other hand, an alternative efficient method for generating such identities, based on a similar binomial transformation formula, is presented in Section 4. Combining these two methods and reversing the procedure enables us to provide an interesting interpretation in terms of Ramanujan summation of almost all the identities mentioned above, as explained in Section 5.

2. Reminder of Some Basic Definitions

We first recall some basic facts about the Cauchy numbers [1, 2, 4, 10]. We then
introduce various types of harmonic numbers using the notations in [8].

If $s(n,k)$ denotes the (signed) Stirling numbers of the first kind, the non-alternating
Cauchy numbers $\{\lambda_n\}_{n\geq 1}$ can be defined explicitly as follows:

$$\lambda_n := \left| \sum_{k=1}^{n} \frac{s(n,k)}{k+1} \right| \qquad (n \geq 1).$$

Alternatively, they can also be defined recursively by means of the relation

$$\sum_{k=1}^{n-1} \frac{\lambda_k}{k!\,(n-k)} = \frac{1}{n} \qquad (n \geq 2).$$

The first ones are

$$\lambda_1 = \frac{1}{2}\,,\ \lambda_2 = \frac{1}{6}\,,\ \lambda_3 = \frac{1}{4}\,,\ \lambda_4 = \frac{19}{30}\,,\ \lambda_5 = \frac{9}{4}\,,\ \lambda_6 = \frac{863}{84}\,,\ \text{etc.}$$

The non-alternating Cauchy numbers λ_n are closely linked to the Bernoulli numbers
of the second kind b_n (first introduced by Jordan [9]) through the relation

$$\lambda_n = (-1)^{n-1} n!\, b_n = n!\,|b_n| \qquad (n \geq 1).$$

Otherwise, the exponential generating function of the non-alternating Cauchy numbers is given by

$$\sum_{n=1}^{\infty} \lambda_n \frac{x^n}{n!} = 1 + \frac{x}{\ln(1-x)} \qquad (|x| < 1).$$

In particular, the series $\displaystyle\sum_{n\geq 1} \frac{\lambda_n}{n!}$ converges to 1.

The classical harmonic numbers $\{H_n\}_{n\geq 1}$ are defined by

$$H_n = \sum_{j=1}^{n} \frac{1}{j} = \psi(n+1) + \gamma\,,$$

where $\psi = \Gamma'/\Gamma$ denotes the digamma function and $\gamma = -\psi(1)$ is the Euler constant
[5].

For any integer $k \geq 1$, the generalized harmonic numbers $\{H_n^{(k)}\}_{n\geq 1}$ are defined
by $H_n^{(1)} = H_n$, and

$$H_n^{(k)} = \sum_{j=1}^{n} \frac{1}{j^k} = \frac{(-1)^{k-1}}{(k-1)!} \partial^{k-1} \psi(n+1) + \zeta(k) \qquad (k \geq 2).$$

For any integer $k \geq 0$, the Roman harmonic numbers $\{H_{n,k}\}_{n \geq 1}$ are defined by $H_{n,0} = 1$, and

$$H_{n,k} = \sum_{n \geq j_1 \geq \cdots \geq j_k \geq 1} \frac{1}{j_1 \, j_2 \cdots j_k} \qquad (k \geq 1).$$

The Roman harmonic numbers[1] can be expressed as polynomials in the generalized harmonic numbers $H_n, H_n^{(2)}, \cdots, H_n^{(k)}$ [4, Equation (18)], [11, Equation (29)]. More precisely, $H_{n,1} = H_n$, and

$$H_{n,k} = \frac{1}{k!}(H_n)^k + \cdots + \frac{1}{k}H_n^{(k)} = P_k(H_n, \cdots, H_n^{(k)}) \qquad (k \geq 2),$$

with

$$P_k(x_1, \cdots, x_k) = \sum_{n_1 + 2n_2 + \cdots + kn_k = k} \frac{1}{n_1! n_2! \cdots n_k!} \left(\frac{x_1}{1}\right)^{n_1} \left(\frac{x_2}{2}\right)^{n_2} \cdots \left(\frac{x_k}{k}\right)^{n_k}.$$

These polynomials are a (slight) modification of the Bell polynomials [4, 6]. A natural generalization of the ordinary Roman harmonic numbers, noted $H_{n,k}^{(r)}$, such that $H_{n,1}^{(r)} = H_n^{(r)}$ was also introduced in [8]. It is given by the following expression [8, Definition 2]:

$$H_{n,k}^{(r)} = \sum_{n \geq j_1 \geq \cdots \geq j_k \geq 1} \frac{1}{j_1 \, j_2 \cdots j_k^r} \qquad (k \geq 1, r \geq 1).$$

3. Overview of Some Known Formulas

In this section, we enumerate a number of more or less known identities, starting with the most classic examples and ending with the lesser-known, and make some comments about them.

The formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n! \, n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n) = \gamma \tag{1}$$

is a classical representation of γ dating back to Mascheroni and Euler which can be slightly modified as follows:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{(n+1)! \, n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \{\zeta(n) - 1\} = \gamma + \ln 2 - 1. \tag{2}$$

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[1]Introduced three decades ago by S. Roman, G-C. Rota and D. Loeb (see [11] for historical details).

A non-trivial generalization of Equation (1) is given by

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n^2} = \frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2) + \gamma_1 - \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\, \zeta(n+1)\,, \qquad (3)$$

where $\gamma_1 = \lim\limits_{n\to\infty} \left\{ \sum\limits_{j=1}^{n} \frac{\ln j}{j} - \frac{1}{2}\ln^2 n \right\}$ is the first Stieltjes constant. This nice identity is already known and appears in [7, 8].

The formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n!\, n} = \zeta(2) - 1 \qquad (4)$$

is a fairly known representation of $\zeta(2) = \dfrac{\pi^2}{6}$. It is in fact a special case of the more general formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_{n,k}}{n!\, n} = \zeta(k+1) - \frac{1}{k} \qquad (k \geq 1)\,,$$

which is called *Hermite's formula* in [4, 6]. A non-trivial generalization of Equation (4) is given by

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(2)}}{n!\, n} = \zeta(3) + \{\gamma + \ln(2\pi) - 12\ln A\}\,\zeta(2) + \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\, \zeta(n+2)\,, \qquad (5)$$

where $A = \lim\limits_{n\to\infty} \left\{ \dfrac{\prod_{k=1}^{n} k^k}{n^{\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}}\, e^{-\frac{n^2}{4}}} \right\} = \exp\left\{ \dfrac{1}{12} - \zeta'(-1) \right\}$ is the Glaisher-Kinkelin constant[2]. This identity is a direct consequence of [8, Equation (19)] and of the well-known relation:

$$\zeta'(2) = \{\gamma + \ln(2\pi) - 12\ln A\}\,\zeta(2)\,.$$

Furthermore, Equation (5) admits a kind of reciprocal which is given by [8, Equation (18)]. More precisely, this is the following identity:

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n!\, n^2} = \{12\ln A - \ln(2\pi)\}\,\zeta(2) - 1 - \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\, \zeta(n+2)\,, \qquad (6)$$

which is another non-trivial generalization of Equation (4).

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[2]This constant plays a role similar to the Stirling constant $\sqrt{2\pi}$ in the celebrated formula

$$\sqrt{2\pi} = \lim_{n\to\infty} \left\{ \frac{n!}{n^{n+\frac{1}{2}}\, e^{-n}} \right\} = \exp\left\{ -\zeta'(0) \right\}\,.$$

Remark 1. When k is greater than 2, no explicit formula, even conjectural, appears to be known for the sum $\sum_{n=1}^{\infty} \dfrac{\lambda_n H_n^{(k)}}{n!\,n}$, nor for the reciprocal sum $\sum_{n=1}^{\infty} \dfrac{\lambda_n H_n}{n!\,n^k}$ (see, however, Remark 4 below where an interpretation of these sums in terms of Ramanujan summations is given).

4. New Supplementary Formulas

The following binomial formula is nothing else than a variant of [2, Proposition 1] which is an elementary but efficient tool to generate several series identities with Cauchy and harmonic numbers. For appropriate analytic functions f with moderate growth, we have the relation:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} f(k) = \int_0^1 f(x)\,dx\,. \tag{7}$$

For instance, using adequate functions f, we can easily derive from Equation (7) the identities given by Equations (1) and (4) above (see [2, Example 4]). We now present a number of interesting new identities that can also be deduced from this method.

Applying Equation (7) with $f(x) = \dfrac{\psi(x+1) + \gamma}{x+1}$, we can use the binomial identity [1, Equation (9.32)]

$$\frac{H_n}{n+1} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{H_k}{k+1}\,,$$

to derive the following identity:

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{(n+1)!} = \frac{1}{2}\zeta(2) + \ln 2 - 1 + \sum_{n=3}^{\infty} \frac{(-1)^n}{n} \left\{ \sum_{k=2}^{n} (\zeta(k) - 1) \right\}, \tag{8}$$

which is a refinement of a formula previously given by Boyadzhiev [2, Example 5]. Moreover, subtracting Equation (8) from Equation (4) leads to the new identity

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{(n+1)!\,n} = \frac{1}{2}\zeta(2) - \ln 2 - \sum_{n=3}^{\infty} \frac{(-1)^n}{n} \left\{ \sum_{k=2}^{n} (\zeta(k) - 1) \right\}, \tag{9}$$

which can be seen as a modification of Equation (4) quite similar to Equation (2).

Applying Equation (7) with $f(x) = \dfrac{\psi(x+1) + \gamma}{x}$, we can use the binomial identity [1, Equation (5.22)]

$$H_n^{(2)} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{H_k}{k}\,,$$

to obtain the following new identity:

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(2)}}{n!} = \int_0^1 \frac{\psi(x+1) + \gamma}{x}\, dx = \zeta(2) - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n+1)\,. \qquad (10)$$

Moreover, writing $H_n^{(2)} = H_{n-1}^{(2)} + \dfrac{1}{n^2}$ in the left member of Equation (10), we can deduce from Equation (3) yet another interesting new identity:

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+1} H_n^{(2)}}{(n+1)!} = \frac{1}{2}\zeta(2) - \frac{1}{2}\gamma^2 - \gamma_1\,. \qquad (11)$$

Remark 2. If ζ_H denotes the harmonic zeta function defined by

$$\zeta_H(s) = \sum_{n=1}^{\infty} \frac{H_n}{n^s} \qquad (\Re(s) > 1)\,,$$

the following nice identity [7, Equation (14)]:

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n}\, \zeta_H(n) = \frac{1}{2}\zeta(2) + \frac{1}{2}\gamma^2 + \gamma_1\,,$$

allows us to rewrite Equation (11) above as follows:

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+1} H_n^{(2)}}{(n+1)!} = \zeta(2) - \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\, \zeta_H(n)\,. \qquad (12)$$

Remark 3. Very recently, using a more powerful method, Young [12] has established the identity below which significantly generalizes Equation (10):

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(k)}}{n!} = \zeta(k) - \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\, \zeta(n+1,\underbrace{1,\ldots,1}_{k-2}) \qquad (k \geq 2)\,.$$

However, it should be noted that this formula cannot be extended to the case $k = 1$. Indeed, the series $\displaystyle\sum_{n \geq 1} \frac{\lambda_n H_n}{n!}$ is divergent since $\dfrac{\lambda_n H_n}{n!} \sim \dfrac{1}{n \ln n}$.

5. Interpretation as Ramanujan Summation

If $\sum_{n\geq 1}^{\mathcal{R}} f(n)$ denotes the $\mathcal{R}$-sum of the series $\sum_{n\geq 1} f(n)$ (i.e. the sum of the series in the sense of Ramanujan's summation method, following the masterful exposition in [3]), then, under certain appropriate conditions of growth and analyticity, we can make use of the following binomial transformation formula [8, Equation (10)]:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} k f(k) = \sum_{n\geq 1}^{\mathcal{R}} f(n) \,. \tag{13}$$

This formula allows us to give an interesting interpretation in terms of Ramanujan summation of almost all the series identities mentioned above.

From the binomial identities

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} = 1 \quad \text{and} \quad \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{k}{k+1} = \frac{1}{n+1} \,,$$

we derive, respectively, the formulas

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n} \tag{14}$$

and

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{(n+1)!\,n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n+1} \,, \tag{15}$$

whose expressions are given by Equations (1) and (2).

From the reciprocal binomial identities

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} H_k = \frac{1}{n} \quad \text{and} \quad \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{1}{k} = H_n \,,$$

we derive respectively the reciprocal formulas

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n^2} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n}{n} \,, \tag{16}$$

and

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n}{n!\,n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^2} \,, \tag{17}$$

whose expressions are given by Equations (3) and (4).

From the reciprocal binomial identities

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{H_k}{k} = H_n^{(2)} \quad \text{and} \quad \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} H_k^{(2)} = \frac{H_n}{n} \,,$$

we derive, respectively, the reciprocal formulas

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n^{(2)}}{n!\,n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n}{n^2} \tag{18}$$

and

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n!\, n^2} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n^{(2)}}{n}, \tag{19}$$

whose expressions are given by Equations (5) and (6).

Remark 4.   1) Formulas (14) and (16) are two particular cases of the more general relation [8, Equation (12)]:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n^k} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_{n,k-1}}{n} \qquad (k \geq 1).$$

2) Formulas (17) and (18) are two particular cases of the more general relation [8, Equation (11)]:

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(k)}}{n!\, n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_{n,k-1}}{n^2} \qquad (k \geq 1).$$

3) Formula (19) is also a special case of the more general formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n!\, n^k} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_{n,k-1}^{(2)}}{n} \qquad (k \geq 2),$$

where $H_{n,k}^{(r)}$ are the generalized Roman harmonic numbers introduced in [8].

From the the self-reciprocal binomial identity

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{H_k}{k+1} = \frac{H_n}{n+1},$$

we derive the formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{(n+1)!\, n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n}{n(n+1)}, \tag{20}$$

whose expression is given by Equation (9).

Applying Equation (13) to the function

$$f(x) = \frac{\psi(x+1) + \gamma - 1}{x(x-1)},$$

whose values at positive integers are

$$f(n) = \frac{H_n - 1}{n(n-1)} \qquad (n \geq 2)$$

and

$$f(1) = \zeta(2) - 1 = \lim_{x \to 1} \frac{\psi(x+1) + \gamma - 1}{x(x-1)},$$

then, using the binomial identity

$$\sum_{k=2}^{n} (-1)^{k-1} \binom{n}{k} \frac{1 - H_k}{k-1} = n H_n^{(2)} - n,$$

which is the reciprocal of the identity [1, Equation (5.24)]:

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} k H_k^{(2)} = \frac{1 - H_n}{n-1} \qquad (n \geq 2),$$

allows us to deduce the nice formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(2)}}{n!} = \zeta(2) - \sum_{n \geq 1}^{\mathcal{R}} \frac{H_n - 1}{n(n-1)}, \tag{21}$$

where the $\mathcal{R}$-summed sequence in the right member of Equation (21) has the value $\zeta(2) - 1$ when $n = 1$; this is the interpretation of Equation (10) in terms of Ramanujan summation. Furthermore, subtracting Equation (16) from Equation (21) allows us to derive yet another formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+1} H_n^{(2)}}{(n+1)!} = \zeta(2) - \sum_{n \geq 1}^{\mathcal{R}} \frac{H_{n-1}}{n-1}, \tag{22}$$

where the $\mathcal{R}$-summed sequence in the right member of Equation (22) has the value $\zeta(2)$ when $n = 1$; this provides an interpretation of Equation (12) in terms of Ramanujan summation.

Remark 5. Very recently, using a more sophisticated method, Young [12] has established the relation below which is a substantial improvement of Equation (21):

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(k)}}{n!} = \zeta(k) - \sum_{n \geq 1}^{\mathcal{R}} \frac{H_{n,k-1} - H_{n,k-2}}{n(n-1)} \qquad (k \geq 2).$$

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