International J.Math. Combin. Vol.3 (2008), 40-50

On the Basis Number of the Strong Product of Theta Graphs with Cycles

[1]M.M.M. Jaradat, [2]M.F. Janem and [2]A.J. Alawneh

(1.Department of Mathematics and physics of Qata University, Doha-Qatar.)

(2.Department of Mathematics and Statistics of Jordan University of Science and Technology, Irbid-Jordan.)

E-mail: mmjst4@qu.edu.qa, janemajdah@yahoo.com, ameen@just.edu.jo

Abstract: A basis $\mathcal{B}$ for the cycle space $\mathcal{C}(G)$ of a graph G is called a d-fold if each edge of G occurs in at most d of the cycles in the basis $\mathcal{B}$. A basis $\mathcal{B}$ for the cycle space $\mathcal{C}(G)$ of a graph G is Smarandachely if each edge of G occurs in at least 2 of the cycles in $\mathcal{B}$. The basis number, $b(G)$, of a graph G is defined to be the least integer d such that there is a d-fold basis of the cycle space of G. MacLane [20] made a connection between the the number of occurrence of edges of a graph in its cycle bases and the planarity of a graph, which is related with parallel bundles on planar map geometries, a kind of Smarandache geometries. In fact, he proved that a graph G is planar if and only if $b(G) \leq 2$. Jaradat [10] gave an upper bound of the basis number of the strong product of a graph with a bipartite graph in terms of the factors. In this work, we show that the basis number of the strong product of a theta graph with a cycle is either 3 or 4. Our result, improves Jaradat's upper bound in the case of specializing the factors by a theta graph and a cycle.

Key Words: Cycle space, cycle basis, Smarandache basis, basis number, strong product.

AMS(2000): 05C38, 05C75.

§1. Introduction

In graph theory, there are many numbers that give rise to a better understanding and interpretation of the geometric properties of a given graph such as the crossing number, the thickness, the genus, the basis number, etc.. The basis number of a graph is of a particular importance because MacLane, in [20], made a connection between the number of occurrences of edges of a graph in its cycle bases and the planarity of a graph; in fact, he proved that a graph is planar if and only if its basis number is at most 2. For the completeness, it should be mentioned that a basis $\mathcal{B}$ of the cycle space $\mathcal{C}(G)$ of a graph G is Smarandachely if each edge of G occurs in at least 2 of the cycles in $\mathcal{B}$

Product of graphs occur naturally in discrete mathematics as tools in combinatorial constructions. They give rise to important classes of graphs and deep structure problems. There are many graph products in the literature, such as, Cartesian product, strong product, lexicographic product, semi-strong product and semi-composition product. The extensive literature

[1]Received July 18, 2008. Accepted August 30, 2008.

on products that has evolved over the years presents a wealth of profound and beautiful results. This led Imrich and Klavzar to write a whole book on graph products [7].

The main purpose of this paper is to investigate the basis number of the strong product of a theta graph with a cycle. Our result improves the upper bounds that expected from applying Jaradat's theorems.

§2. Definitions and preliminaries

Unless otherwise specified, the graphs considered in this paper are finite, undirected, simple and connected. For a given graph G, we denote the vertex set of G by $V(G)$ and the edge set by $E(G)$.

For a given graph G, the set $\mathcal{E}$ of all subsets of $E(G)$ forms an $|E(G)|$-dimensional vector space over Z_2 with vector addition $X \oplus Y = (X\backslash Y) \cup (Y\backslash X)$ and scalar multiplication $1 \cdot X = X$ and $0 \cdot X = \emptyset$ for all $X, Y \in \mathcal{E}$. The cycle space, $\mathcal{C}(G)$, of a graph G is the vector subspace of $(\mathcal{E}, \oplus, \cdot)$ spanned by the cycles of G. Note that the non-zero elements of $\mathcal{C}(G)$ are cycles and edge disjoint union of cycles. It is known that for a connected graph G the dimension of the cycle space is the *cyclomatic number* or the *first Betti number*

$$\dim \mathcal{C}(G) = |E(G)| - |V(G)| + r \tag{1}$$

where r is the number of components in G.

The first important use of the basis number dates back to MacLane [20] when he made the connection between the basis number of a graph and the planarity. There after, in 1981, E. Schmeichel [21] formalized the definition of the basis number of a graph as follows: A basis $\mathcal{B}$ for $\mathcal{C}(G)$ is called a *cycle basis* of G. A cycle basis $\mathcal{B}$ of G is called a d-fold if each edge of G occurs in at most d of the cycles in $\mathcal{B}$. *The basis number,* $b(G)$, of G is the least non-negative integer d such that $\mathcal{C}(G)$ has a *d-fold* basis.

Latter on, Schmeichel [21] investigate the basis number of the known classes of graphs such as the complete graphs K_n and the complete bipartite graphs $K_{n,m}$. In fact, he proved that $b(K_n) = 3$, for $n \geq 5$ and $b(K_{n,m}) = 4$ for all $n, m \geq 5$ except a few numbers of graphs. Also, he proved that for any positive integer r, there exists a graph G with $b(G) \geq r$. After that, he joined Banks to prove that the basis number of n-cube is 4 for all $n \geq 7$ (see [6])

Since 1992, many researchers were attracted to study the basis number of graph products. The Cartesian product, $\Box$, was studied by Ali and Marougi [3] when they gave the following result:

Theorem 2.1 (Ali and Marougi) *If G and H are two connected disjoint graphs, then* $\mathrm{b}(G\Box H) \leq \max\{\mathrm{b}(G) + \triangle(T_H), \ \mathrm{b}(H) + \triangle(T_G)\}$ *where T_H and T_G are spanning trees of H and G, respectively, such that the maximum degrees $\triangle(T_H)$ and $\Delta(T_G)$ are minimum with respect to all spanning trees of H and G.*

Also, Alsardary and Wojciechowski [4] proved that for every $d \geq 1$ and $n \geq 2$, $b(K_n^d) \leq 9$ where K_n^d is a d times Cartesian product of the complete graph K_n.

Upper bounds of the strong product, $\boxtimes$, were obtained by Jaradat [11], [14] and [15] when he gave the following results:

Theorem 2.2(Jaradat) *Let G be a bipartite graph and H be a graph. Then* $b(H \boxtimes G) \leq \max\left\{b(G)+1, 2\Delta(G)+b(H)-1, \left\lfloor \frac{3\Delta(T_H)+1}{2} \right\rfloor, b(H)+2\right\}$.

Theorem 2.3(Jaradat) *Let G be a bipartite graph and C be a cycle. Then* $b(G \boxtimes C) \leq 4 + b(G)$.

The lexicographic product of two graphs G and H, $G[H]$, was studied by Jaradat and Al-zoubi [17] and Jaradat [13]. They obtained the following results:

Theorem 2.4 (Jaradat and Al-Zoubi) *For each two connected graphs G and H,* $b(G[H]) \leq \mathrm{Max}\{4, 2\Delta(G) + b(H), 2 + b(G)\}$.

Theorem 2.5(Jaradat *Let G, T_1 and T_2 be a graph, a spanning tree of G and a tree, respectively. Then,* $b(G[T_2]) \leq b(G[H]) \leq \max\{5, 4 + 2\Delta(T^G_{\min}) + b(H), 2 + b(G)\}$ *where T^G stands for the complement graph of a spanning tree T in G and $T_{\min}$ stands for a spanning tree for G such that* $\Delta(T^G_{\min}) = \min\{\Delta(T^G) | T$ *is a spanning tree of* $G\}$.

Ali [1], [2] gave an upper bound for the basis number of the semi-strong product, $\bullet$, and the direct product, $\times$, of some special graphs when he proved that $b(K_m \bullet K_n) \leq 9$ for any integers m, n and $b(C_n \times C_m) = 3$ for any two cycles C_n and C_m with $n, m \geq 3$. Also the following upper bound (among other results) were obtained by Jaradat [8], [9], [14] and [18]:

Theorem 2.6(Jaradat) *For each bipartite graphs G and H,* $b(G \times H) \leq 5 + b(G) + b(H)$.

Theorem 2.7 (Jaradat) *For each bipartite graphs G and H,*

$$b(G \bullet H) \leq \max\{b(G) + b(H) + \left\{\begin{array}{ll} 3, & \text{if both of } T_G \text{ and } T_H \text{ are paths,} \\ 4, & \text{if } T_H \text{ is a path,} \\ 5, & \text{if } T_G \text{ is a path,} \\ 6, & \text{if both of } T_G \text{ and } T_H \text{ are not paths.} \end{array}\right\}, \Delta(T_G) + b(H)\}$$

The wreath product, was studied by Jaradat and Al-Qeyyam (See [5], [12] and [16]).

For completeness, we recall that for any two graphs G and H, the strong product $G \boxtimes H$ is the graph with the vertex set $V(G \boxtimes H) = V(G) \times V(H)$ and the edge set $E(G \boxtimes H) = \{(u_1, u_2)(v_1, v_2) | u_1v_1 \in E(G)$ and $u_2 = v_2$ or $u_1 = v_1$ and $u_2v_2 \in E(H)$ or $u_1v_1 \in E(G)$ and $u_2v_2 \in E(H)\}$. The Cartesian product $G \square H$ is the graph with the vertex set $V(G \square H) = V(G) \times V(H)$ and the edge set $E(G \square H) = \{(u_1, u_2)(v_1, v_2) | u_1v_1 \in E(G)$ and $u_2 = v_2$ or $u_1 = v_1$ and $u_2v_2 \in E(H)\}$. Also, the direct product $G \times H$ is the graph with the vertex set $V(G \times H) = V(G) \times V(H)$ and the edge set $E(G \times H) = \{(u_1, u_2)(v_1, v_2) | u_1v_1 \in E(G)$ and $u_2v_2 \in E(H)\}$.

In the rest of this paper, $f_B(e)$ stand for the number of elements of B containing the edge e where $B \subseteq \mathcal{C}(G)$.

§3. The basis number of $\theta_n \boxtimes C_m$

In this section we investigate the basis number of the strong product of theta graphs and cycles. In fact we show that $3 \leq b(\theta_n \boxtimes C_m) \leq 4$. Throughout this section we assume that $1, 2, \ldots, n$ and $1, 2 \ldots, m$ to be the vertices of θ_n and C_m, respectively.

Definition 3.1 *A theta graph θ_n is defined to be a cycle to which we add a new edge that joins two non-adjacent vertices. We may assume 1 and δ are the two vertices of θ_n of degree 3.*

Applying Theorem 2.2 for the case $H = \theta_n$ and G to be a cycle of even length C_m, we get $b(\theta_n \boxtimes C_m) \leq \max\{2, 5, 3, 4\} = 5$. Also, applying the same theorem by considering $H = C_m$ and G to be a theta graph that contains no odd cycles θ_n, we get $b(C_m \boxtimes \theta_n) \leq \max\{3, 6, 3, 4\} = 6$. Moreover, by specializing G in Theorem 2.3 to θ_n that contains no odd cycle, then $b(\theta_n \boxtimes C_m) \leq 6$. However, these upper bounds will be reduced to 4 as we will see in Theorem 3.6. Now, for this purpose, we consider the following cycles: For each $j = 1, 2, \ldots, m-2$, set

$$\begin{aligned} \mathcal{A}_1^{(j)} &= (1,j)(2,j+1)(1,j+2)(\delta,j+1)(1,j), \\ \mathcal{A}_2^{(j)} &= (\delta,j)(\delta-1,j+1)(\delta,j+2)(1,j+1)\,(\delta,j)\,, \end{aligned}$$

and let

$$\mathcal{A}_1 = \bigcup_{j=1}^{m-2} \mathcal{A}_1^{(j)} \quad \text{and} \quad \mathcal{A}_2^{(i)} = \bigcup_{j=1}^{m-2} \mathcal{A}_2^{(j)}.$$

The following result will be useful in our main result.

Lemma 3.2 *Every linear combination of cycles of $\mathcal{A}_1 \cup \mathcal{A}_2$ contains at least one edge of $\{(1,j)(\delta,j+1),(1,j+1)(\delta,j) | 1 \leq j \leq m-2\}$.*

Proof Consider $\mathcal{O}$ to be a linear combinations of cycles of $\mathcal{A}_1 \cup \mathcal{A}_2^{(i)}$. Then

$$\mathcal{O} = \bigoplus_{j=1}^{s_1} \mathcal{A}_1^{(1_j)} \oplus \bigoplus_{j=1}^{s_2} \mathcal{A}_2^{(2_j)}$$

where $\mathcal{A}_1^{(1_j)} \in \mathcal{A}_1$, $\mathcal{A}_2^{(2_j)} \in \mathcal{A}_2$, $1_1 < 1_2 < \cdots < 1_{s_1}$ and $2_1 < 2_2 < \cdots < 2_{s_2}$. Now, let $t_1 = \min\{1_1, 2_1\}$. We now consider the following two cases.

Case 1. $t_1 = 1_1$. Then by the definition of $\mathcal{A}_1$, $\mathcal{A}_1^{(1_1)}$ contains the edge $(1, 1_1)(\delta, , 1_1 + 1)$ where $1_1 \leq m-2$. Since $E(\mathcal{A}_1^{(j)}) \cap E(\mathcal{A}_1^{(i)}) = \varnothing$, $(1, 1_1)(\delta, , 1_1 + 1) \notin \mathcal{A}_1^{(1_j)}$ for each $1 \leq j \leq s_1$. Also, since $1_1 \leq 2_1, (1, 1_1)(\delta, , 1_1 + 1) \notin \mathcal{A}_2^{(2_j)}$ for each $1 \leq j \leq s_2$. Therefore, $(1, 1_1)(\delta, , 1_1 + 1) \in \mathcal{O}$.

Case 2. $t_1 = 2_1$. Then we argue more or less as in Case 1, to have that $(1, 2_1 + 1)(\delta, , 2_1) \in \mathcal{O}$ where $2_1 \leq m-2$. □

Now, for $j = 1, 2, \ldots, m-1$, consider the following set of cycles:

$$\mathcal{K}_j = (1,j)(\delta,j)\,(1,j+1)\,(\delta,j+1)(1,j),$$

and let

$$\mathcal{K} = \bigcup_{j=1}^{m-1} \mathcal{K}_j.$$

Lemma 3.3 *Every linear combination of cycles of* $\mathcal{K}$ *contains at least one edge of* $\{(1,j)(\delta,j)|1 \leq j \leq m-1\}$.

Proof Let

$$\mathcal{O} = \sum_{i=1}^{s} \mathcal{K}_{j_i}(\text{mod}2)$$

where $\mathcal{K}_{j_i} \in \mathcal{K}$ and $j_1 < j_2 < \cdots < j_s \leq m-1$. Then by the definition of $\mathcal{K}$,

$$E(\mathcal{K}_{j_1}) \cap E(\cup_{i=2}^{s} \mathcal{K}_{j_i}) \subseteq \{(1, j_1+1)(\delta, j_1+1)\}.$$

But, $(1,j_1)(\delta,j_1) \in E(\mathcal{K}_{j_1})$. Hence, $(1,j_1)(\delta,j_1) \in \mathcal{O}$. □

Lemma 3.4 Let θ_n be a graph of order $n \geq 4$ and C_m be a cycle of order $m \geq 3$. Then $b\,(\theta_n \boxtimes C_m) \geq 3$.

Proof Assume that $\theta_n \boxtimes C_m$ has a 2-fold basis $\mathcal{B}$. Since the girth of $\theta_n \boxtimes C_m$ is 3, we have that

$$\begin{aligned} 3\,|\mathcal{B}| &\leq 2\,|E(\theta_n \boxtimes C_m)| \\ 3(3m(n+1)+1) &\leq 2(3m(n+1)+nm) \\ 9mn+9m+3 &\leq 6mn+6m+2nm \\ mn+3m+3 &\leq 0 \\ m(n+3)+3 &\leq 0 \end{aligned}$$

which is a contradiction. Hence $\mathcal{B}(\theta_n \boxtimes C_m)$ is a 3-fold basis. □

The following result of Jaradat and et al. will be needed in our coming result:

Proposition 3.5 (Jaradat and et al)*Let* A *and* B *be two linearly independent sets of cycles such that* $E(A) \cap E(B)$ *subset of an edge set of a forest or an empty set. Then* $A \cup B$ *is linearly independent.*

The following cycles which were introduced in [11] will be used frequently in the coming results.

$$\begin{aligned}
\mathcal{L}_{ab} &= \Big\{\mathcal{L}^{(j)} = (a, v_j)(b, v_{j+1})(a, v_{j+1})(a, v_j) | j = 1, 2, 3, \cdots, m-1\Big\} \\
&\quad \cup \Big\{\mathcal{L}^{(n)} = (a, v_n)(b, v_1)(a, v_1)(a, v_n)\Big\} \\
\mathcal{T}_{ab} &= \Big\{\mathcal{T}^{(j)} = (a, v_j)(a, v_{j+1})(b, v_j)(a, v_j) | j = 1, 2, 3, \cdots, m-1\Big\} \\
&\quad \cup \Big\{\mathcal{T}^{(n)} = (a, v_n)(a, v_1)(b, v_n)(a, v_n)\Big\} \\
\mathcal{S}_{ab} &= \Big\{\mathcal{S}^{(j)} = (a, v_{j+1})(b, v_j)(b, v_{j+1})(a, v_{j+1}) | j = 1, 2, 3, \cdots, m-1\Big\} \\
&\quad \cup \Big\{\mathcal{S}^{(n)} = (a, v_1)(b, v_n)(b, v_1)(a, v_1)\Big\}.
\end{aligned}$$

Also

$$\mathcal{F}_n = \begin{cases} (a, v_1)(b, v_2)(a, v_3)(b, v_4) \ldots (a, v_{n-1})(b, v_n)(a, v_1) \text{ if } m \text{ is even,} \\ (a, v_1)(b, v_1)(a, v_2)(b, v_3) \ldots (a, v_{n-1})(b, v_n)(a, v_1) \text{ if } m \text{ is odd.} \end{cases}$$

Let

$$\mathcal{B}_{ab} = \mathcal{L}_{ab} \cup \mathcal{T}_{ab} \cup \mathcal{S}_{ab} \text{ and } \mathcal{B}^*_{ab} = \mathcal{B}_{ab} - \{\mathcal{S}^{(m)}\} \cup \{\mathcal{F}_m\}$$

Moreover, by Theorem 2.6 of [11], we have that

$$\dim \mathcal{C}(C_n \boxtimes C_m) = 3mn + 1. \tag{2}$$

Note that $\theta_n \boxtimes C_m$ is decomposable into $(C_n \boxtimes C_m) \cup (1\alpha \Box N_m) \cup (1\alpha \times C_m)$ where N_m is the null graph with vertex set $V(C_m)$. Thus,

$$\begin{aligned}
\dim \mathcal{C}(\theta_n \boxtimes C_m) &= \dim \mathcal{C}(C_n \boxtimes C_m) + m + 2m, & (3)\\
&= 3mn + 3m + 1. & (4)
\end{aligned}$$

Now, we state and prove our main result.

Theorem 3.6 *For any graph θ_n of order $n \geq 4$ and cycle C_m of order $m \geq 3$, we have* $3 \leq b(\theta_n \boxtimes C_m) \leq 4$.

Proof By Lemma 3.4, it is sufficient to exhibit a 4-fold basis, $\mathcal{B}$, for $\mathcal{C}(\theta_n \boxtimes C_m)$. According to the parity of m, n and δ (odd or even), we consider the following cases.

Case 1. m and n are even and δ is odd. Then define

$$\mathcal{B}(\theta_n \boxtimes C_m) = \left(\bigcup_{i=1}^{n-1} \mathcal{B}_{a_i a_{i+1}}\right) \cup \mathcal{B}^*_{a_n a_1} \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{K} \cup \{C\} \cup \{C_1, C_2, C_3, C_4, C_5\}$$

where $B_{a_i a_{i+1}}$ and $B^*_{a_n a_1}$ are as in above and

$$C = (1,1)(2,2)(3,1)(4,2)...(n-1,1)(n,2)(1,1).$$

Also,

$$\begin{aligned}
C_1 &= (1,m-1)(2,m)(3,m)(4,m)...(\delta,m)\,(1,m-1)\,.\\
C_2 &= (1,1)(1,m)(\delta,m)(1,1).\\
C_3 &= (\delta,1)(\delta+1,2)(\delta,3)...(\delta,m-1)(1,m)\,(\delta,1)\,.\\
C_4 &= (1,1)\,(2,m)\,(3,1)\,(4,m)\,...\,(\delta,1)\,(1,m)\,(2,1)\,(3,m)\,...\,(\delta,m)\,(1,1)\,.\\
C_5 &= (1,m)(2,1)(3,m)(4,1)...(\delta,m)(1,m).
\end{aligned}$$

Let $\mathcal{B}_1 = \bigcup_{i=1}^{n-1} \mathcal{B}_{a_i a_{i+1}} \cup \mathcal{B}^*_{a_n a_1} \cup \{C\}$. Note that $\mathcal{B}_1 =$B$(C_n \boxtimes C_m)$ is a basis for $\mathcal{C}(C_n \boxtimes C_m)$ (see Theorem 2.6, Case 1 of [11]). Thus, $\mathcal{B}_1$ is linearly independent. Note that C_5 contains the edge $(\delta,m)(1,m)$ which does not appear in any cycle of $\mathcal{B}_1$. Hence, $\mathcal{B}_1 \cup \{C_5\}$ is linearly independent. Now, C_2 contains the edge $(\delta,m)(1,1)$ which does not appear in any cycle of $\mathcal{B}_1 \cup \{C_5\}$. So, $\mathcal{B}_1 \cup \{C_5, C_2\}$ is linearly independent. Similarly, the cycle C_4 contains the edge $(\delta,1)(1,m)$ which does not appear in any cycle of $\mathcal{B}_1 \cup \cup\{C_5, C_2\}$. Thus, $\mathcal{B}_1 \cup \{C_2, C_4, C_5\}$ is linearly independent. Also, C_3 contains the edge $(\delta, m-1)(1,m)$ which does not appear in any cycle of $\mathcal{B}_1 \cup \{C_2, C_4, C_5\}$. Therefore, $\mathcal{B}_1 \cup \{C_2, C_3, C_4, C_5\}$ is linearly independent. Finally, C_1 contains the edge $(1,m-1)(\delta,m)$ which does not appear in any cycle of $\mathcal{B}_1 \cup \{C_2, C_3, C_4, C_5\}$. Thus, $\mathcal{B}_1 \cup \{C_1, C_2, C_3, C_4, C_5\}$ is linearly independent. By Lemma 3.2, any linear combination of cycles of $\mathcal{A}_1 \cup \mathcal{A}_2$ contains at least one edge of $\{(1,j)(\delta,j+1), (1,j+1)(\delta,j) | 1 \le j \le m-2\}$ which does not occur in any cycle of $\mathcal{B}_1 \cup \{C_1, C_2, C_3, C_4, C_5\}$. Thus, $\mathcal{B}_1 \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \{C_1, C_2, C_3, C_4, C_5\}$ is linearly independent. Similarly, by Lemma 3.2, any linear combination of cycles of $\mathcal{K}$ contains at least one edge of $\{(1,j)(\delta,j) | 1 \le j \le m-1\}$, which does not occur in any cycle of $\mathcal{B}_1 \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \{C_1, C_2, C_3, C_4, C_5\}$. Therefore, $\mathcal{B}(\theta_n \boxtimes C_m)$ is linearly independent. Note that

$$\begin{aligned}
|\mathcal{B}(\theta_n \boxtimes C_m)| &= |\mathcal{B}_1| + |\mathcal{K}| + |\mathcal{A}_1| + |\mathcal{A}_2| + \sum_{i=1}^{5} |C_i|\\
&= 3mn + 1 + |\mathcal{K}| + |\mathcal{A}_1| + |\mathcal{A}_2| + \sum_{i=1}^{5} |C_i|\\
&= 3mn + 1 + (m-1) + (m-2) + (m-2) + 5\\
&= 3mn + 3m + 1\\
&= 3m(n+1) + 1\\
&= \dim \mathcal{C}(\theta_n \boxtimes C_m),
\end{aligned}$$

where the last equality follows from equation (4). Therefore, B$(\theta_n \boxtimes C_m)$ is a basis for $\mathcal{C}(\theta_n \boxtimes C_m)$. To complete the proof of this case, we show that B$(\theta_n \boxtimes C_m)$ is a 3-fold basis. Let $e \in E(\theta_n \boxtimes C_m)$. Then 1) if $e = (1,m-1)(2,m)$, then $f_{\mathcal{B}_1}(e) = 1$, $f_{\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{K}}(e) = 0$ and $f_{\{C_i\}_{i=1}^5}(e) = 1$. 2) If $e \in \{(i,m)(i+1,m) | i = 2,3,\ldots,m-1\}$, then $f_{\mathcal{B}_1}(e) = 2$, $f_{\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{K}}(e) = 0$ and $f_{\{C_i\}_{i=1}^5}(e) = 1$. 3) If $e = (1,m)(\delta,m)$ or $(1,1)(\delta,1)$, then $f_{\mathcal{B}_1}(e) = 0$, $f_{\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{K}}(e) = 1$ and $f_{\{C_i\}}(e) \le 2$. 4) If $e = (1,1)(1,m)$, then $f_{\mathcal{B}_1}(e) = 2$, $f_{\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{K}}(e) = 0$ and $f_{\{C_i\}_{i=1}^5}(e) = 1$. 5) If $e \in \{(i,1)(i+1,m), (i+1,1)(i,m) | i = 1,2,\ldots,n-1\} \cup \{(1,1)(n,m), (1,m)(\delta,1)\}$, then $f_{\mathcal{B}_1}(e) = 0$, $f_{\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{K}}(e) = 0$ and $f_{\{C_i\}_{i=1}^5}(e) \le 2$. 6) If $e \in \{(1,j)(2,j+1) | j = 1,2,\ldots,m-2\} \cup \{(\delta-1,j)(\delta,j+1) | j = 1,2,\ldots,m-1\}$, then $f_{\mathcal{B}_1}(e) = 1$, $f_{\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{K}}(e) = 1$ and $f_{\{C_i\}_{i=1}^5}(e) = 0$. 7) If

$e \in \{(1,j+1)(2,j),(\delta-1,j+1)(\delta,j)|j=1,2,\ldots,m-1\}$, then $f_{\mathcal{B}_1}(e)=2$, $f_{\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{K}}(e)=1$ and $f_{\{C_i\}_{i=1}^5}(e)=0$. 8) If $e=(1,m-1)(\delta,m)$ or $(1,m)(\delta,m-1)$, then $f_{\mathcal{B}_1}(e)=0$, $f_{\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{K}}(e)=1$ and $f_{\{C_i\}_{i=1}^5}(e)=1$. 9) If $e \in \{\,(1,j)(\delta,j+1),(1,j+1)(\delta,j)|j=1,2,\ldots,m-2\}$, then $f_{\mathcal{B}_1}(e)=0$, $f_{\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{K}}(e)\leq 2$ and $f_{\{C_i\}_{i=1}^5}(e)=0$. 10) If $e \in \{\,(\delta,j)(\delta+1,j+1),\{(\delta,j+1)(\delta+1,j)|j=1,2,\ldots,m-1\}$, then $f_{\mathcal{B}_1}(e)\leq 2$, $f_{\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{K}}(e)=0$ and $f_{\{C_i\}_{i=1}^5}(e)\leq 1$. if e is not of the above form, then $f_{\mathcal{B}_1}(e)\leq 3$, $f_{\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{K}}(e)=0$ and $f_{\{C_i\}_{i=1}^5}(e)=0$. From all of the above, we have that $f_{\mathcal{B}(\theta_n\boxtimes C_m)}(e)\leq 3$.

Case 2. m and δ are even and n is odd. Then define

$$\mathcal{B}(\theta_n\boxtimes C_m)=\left(\bigcup_{i=1}^{n-1}\mathcal{B}_{a_ia_{i+1}}\right)\cup\mathcal{B}^*_{a_na_1}\cup\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{K}\cup\{C^*,C_1,C_2,C_3,C_4,C_5\}$$

where

$$C^*=(1,1)(2,2)(3,1)(4,2)\ldots(n,1)(1,1)$$

and $\mathcal{B}_{a_ia_{i+1}},\mathcal{B}^*_{a_na_1}$, $\mathcal{A}_1,\mathcal{A}_2,\mathcal{K},C_1,C_2$ and C_3 are as defined in Case 1 and

$$\begin{aligned} C_4 &= (1,1)(2,m)(3,1)(4,m)...(\delta,m)(1,1),\\ C_5 &= (1,m)(2,1)(3,m)(4,1)...(\delta,1)(1,m). \end{aligned}$$

By the same argument as in Case 1 of Theorem 2.6 of [11], we show that $\left(\bigcup_{i=1}^{n}\mathcal{B}_{a_ia_{i+1}}\right)\cup\mathcal{B}^*_{a_na_1}\cup\{C^*\}$ is linearly independent. Following, more or less, the same proof of Case 1 by replacing C with C^*, we can show that $\mathcal{B}(\theta_n\boxtimes C_m)$ is a 4-fold basis for $\mathcal{C}(\theta_n\boxtimes C_m)$.

Case 3. m,n and δ are even. Then we define

$$\mathcal{B}(\theta_n\boxtimes C_m)=\left(\bigcup_{i=1}^{n-1}\mathcal{B}_{a_ia_{i+1}}\right)\cup\mathcal{B}^*_{a_na_1}\cup\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{K}\cup\{C,C_1,C_2,C_3,C_4,C_5\},$$

where $\mathcal{B}_{a_ia_{i+1}},\mathcal{B}^*_{a_na_1}$, $\mathcal{A}_1,\mathcal{A}_2,\mathcal{K},C_1,C_2$, C_3,C_4 and C_5 are as defined in Case 2 and C is as in Case 1. By following, word by word, the proof of Case 2 after replacing C^* by C we get that $\mathcal{B}(\theta_n\boxtimes C_m)$ is a 4-fold basis.

Case 4. m is even and δ and n are odd. By relabeling the vertices of θ_n in the opposite direction, we get a similar case to Case 2.

Case 5. m is odd and n and δ are even. Then we define

$$\mathcal{B}(\theta_n\boxtimes C_m)=\left(\bigcup_{i=1}^{n-1}\mathcal{B}_{a_ia_{i+1}}\right)\cup\mathcal{B}^*_{a_na_1}\cup\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{K}\cup\{C\}\cup\{C_1,C_2,C_3,C_4,C_5\}$$

where $\mathcal{B}_{a_ia_{i+1}}$, $\mathcal{B}^*_{a_na_1}$ and C are as defined in Case 1. Also, $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{K}$, C_2, C_4,C_5, are as in Case 3, and

$$\begin{aligned} C_1 &= (1,m)(2,m)(3,m)(4,m-1)\,(5,m)\,(6,m-1)\ldots(\delta,m-1)\,(1,m),\\ C_3 &= (1,m-1)(2,m)(3,m-1)(4,m)...(\delta,m)(1,m-1). \end{aligned}$$

Let $\mathcal{B}_1 = \bigcup_{i=1}^{n-1} \mathcal{B}_{a_ia_{i+1}} \cup \mathcal{B}^*_{a_na_1} \cup \{C\}$. Note that $\mathcal{B}_1 =$B$(C_n \boxtimes C_m)$ is a basis for $\mathcal{C}(C_n \boxtimes C_m)$ (see Theorem 2.6 Case 2 of [11]). Thus, $\mathcal{B}_1$ is linearly independent. Note that $E(\mathcal{B}_1) \cap E(C_4) = \{(1,1)(2,m),(2,m)(3,1),\ldots,(\delta-1,1)(\delta,m)\}$ which is an edge set of a path. Thus, by Proposition 3.5, $\mathcal{B}_1 \cup \{C_4\}$ is linearly independent. Similarly, $E(\mathcal{B}_1 \cup \{C_4\}) \cap E(C_5) = \{(1,m)(2,1),(2,1)(3,m),\ldots,(\delta-1,m)(\delta,1)\}$ which is an edge set of a path. Thus, by Proposition 3.5, $\mathcal{B}_1 \cup \{C_4, C_5\}$ is linearly independent. Also, $E(\mathcal{B}_1 \cup \{C_4, C_5\}) \cap E(C_2) = \{(1,m)(1,1),(1,1)(\delta,m)\}$ which is an edge set of a forest. Thus, $\mathcal{B}_1 \cup \{C_2, C_4, C_5\}$ is linearly independent. Now, since $E(C_1) \cap E(C_3) = \varnothing$ and

$$\begin{aligned} E(C_1 \cup C_3) \cap E(\mathcal{B}_1 \cup \{C_2, C_4, C_5\}) &= \{(i,m-1)(i+1,m) | 1 \le i \le \delta - 1\} \cup \\ \{(i,m)(i+1,m-1)|2 &\le i \le \delta-1\} \cup \{(1,m)(2,m),(2,m)(3,m)\}, \end{aligned}$$

which is an edge set of a tree, we have that $\mathcal{B}_1 \cup \{C_1, C_2, C_3, C_4, C_5\}$ is linearly independent. Now, by a similar argument as in Case 1, we can show that B$(\theta_n \boxtimes C_m)$ is a 4-fold basis.

Case 6. m and n are odd and δ is even. Then we define

$$\mathcal{B}(\theta_n \boxtimes C_m) = \left(\bigcup_{i=1}^{n-1} \mathcal{B}_{a_ia_{i+1}}\right) \cup \mathcal{B}^*_{a_na_1} \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{K} \cup \{C^*, C_1, C_2, C_3, C_4, C_5\},$$

where $\mathcal{B}_{a_ia_{i+1}}, \mathcal{B}^*_{a_na_1}$, $\mathcal{A}_1, \mathcal{A}_2, \mathcal{K}, C_1, C_2$, C_3, C_4 and C_5 are as defined in Case 5 and C^* is as in Case 2. To this end, we use the same argument as in Case 5 to show that $\mathcal{B}(\theta_n \boxtimes C_m)$ is a 4-fold basis.

Case 7. m,n and δ are odd. Then we define

$$\mathcal{B}(\theta_n \boxtimes C_m) = \left(\bigcup_{i=1}^{n-1} \mathcal{B}_{a_ia_{i+1}}\right) \cup \mathcal{B}^*_{a_na_1} \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{K} \cup \{C^*, C_1, C_2, C_3, C_4, C_5\},$$

where $\mathcal{B}_{a_ia_{i+1}}, \mathcal{B}^*_{a_na_1}$ are as defined above, $\mathcal{K}, \mathcal{A}_1$, $\mathcal{A}_2$ ad C_2 are as in Case 5 and C^* is as in Case 2. Also, we set,

$$\begin{aligned} C_1 &= (1,m-1)(2,m)(3,m-1)(4,m)...(\delta-1,m)(\delta,m)(1,m-1), \\ C_3 &= (1,m)(2,m-1)(3,m)(4,m-1)...(\delta-1,m-1)(\delta,m-1)(1,m), \\ C_4 &= (1,m)(2,m)(3,1)(4,m)(5,1)...(\delta,1)(1,m), \\ C_5 &= (1,1)(2,1)(3,m)(4,1)(5,m)...(\delta,m)(1,1). \end{aligned}$$

As in Case 2, we can see that $\mathcal{B}_1 = \left(\bigcup_{i=1}^{n-1} \mathcal{B}_{a_ia_{i+1}}\right) \cup \mathcal{B}^*_{a_n\mathcal{A}_1^{(i)}} \cup C^*$ is linearly independent. Now, the cycle C_1 contains the edge $(\delta,m)(1,m-1)$ which does not appear in any cycle of $\mathcal{B}_1$. Hence $\mathcal{B}_1 \cup \{C_1\}$ is linearly independent. Also, C_4 contains the edge $(\delta,1)(1,m)$ which does not appear in any cycle of $\mathcal{B}_1 \cup \{C_1\}$. Thus, $\mathcal{B}_1 \cup \{C_1, C_4\}$ is linearly independent. C_5 contains the edge $(1,1)(\delta,m)$ which does not appear in any cycle of $\mathcal{B}_1 \cup \{C_1, C_4\}$. So, $\mathcal{B}_1 \cup \{C_1, C_4, C_5\}$ is linearly independent. C_2 contains the edge $(1,m)(\delta,m)$ which does not appear in any cycle of $\mathcal{B}_1 \cup \{C_1, C_4, C_5\}$. Hence $\mathcal{B}_1 \cup \{C_1, C_2, C_4, C_5\}$ is linearly independent. Finally, C_3 contains the edge $(1, m)(\delta, m-1)$ which does not appear in any cycle of $\mathcal{B}_1 \cup \{C_1, C_2, C_4, C_5\}$. Therefore, $\mathcal{B}_1 \cup \{C_1, C_2, C_3, C_4, C_5\}$ is linearly independent. To this end, to complete this case, we use the same argue as in Case 1.

Case 8. m and δ are odd and n is even. By relabeling the vertices of θ_n in the opposite direction, we get a similar case to Case 7. □

By noting that $C_m \boxtimes \theta_n$ is isomorphic to $\theta_n \boxtimes C_m$, we get the following result:

Corollary 3.1 *For any graph θ_n of order $n \geq 4$ and cycle C_m of order $m \geq 3$, we have* $3 \leq b(C_m \boxtimes \theta_n) \leq 4$.

References

[1] A.A. Ali, The basis number of the direct product of paths and cycles, *Ars Combin.* 27 (1988), 155-163.

[2] A.A. Ali, The basis number of complete multipartite graphs, *Ars Combin.* 28 (1989), 41-49.

[3] A.A. Ali and G.T. Marougi, The basis number of Cartesian product of some graphs, *The J. of the Indian Math. Soc.* 58 (1992), 123-134.

[4] A.S. Alsardary and J. Wojciechowski, The basis number of the powers of the complete graph, *Discrete Math.* 188 (1998), 13-25.

[5] M.K. Al-Qeyyam and M.M.M. Jaradat, On the basis number and the minimum cycle bases of the wreath product of some graphs II, (To appear in JCMCC).

[6] J.A. Banks and E.F. Schmeichel, The basis number of n-cube, *J. Combin. Theory Ser. B* 33 (1982), no. 2, 95-100.

[7] W. Imrich and S. Klavzar, *Product Graphs: Structure and Recognition*, Wiley, New York, 2000.

[8] M.M.M. Jaradat, On the basis number of the direct product of graphs, *Australas. J. Combin.* 27 (2003), 293-306.

[9] M.M.M. Jaradat, The basis number of the direct product of a theta graph and a path, *Ars Combin.* 75 (2005), 105-111.

[10] M.M.M. Jaradat, An upper bound of the basis number of the strong product of graphs, *Discussion Mathematicea Graph Theory* 25 (2005), 391-406.

[11] M.M.M. Jaradat, The basis number of the strong product of trees and cycles with some graphs, *J. Combin. Math. Combin. Comput.* 58 (2006), 195–209.

[12] M.M.M. Jaradat, On the basis number and the minimum cycle bases of the wreath product of some graphs I, *Discussions Mathematicae Graph Theory* 26 (2006) 113-134.

[13] M.M.M. Jaradat, A new upper bound of the basis number of the lexicographic product of graphs, To appear in Ars Combin.

[14] M.M.M. Jaradat, An upper bound of the basis number of the semi-strong product of cycles with bipartite graphs, *Bulletin of the Korean Mathematical Society* 44 (2007) no. 3, 385-394.

[15] M. M. M. Jaradat, The basis number of the strong product of paths and cycles with bipartite graphs, *Missouri Journal of Mathematical Sciences* 19 (2007) no. 3, 219-230.

[16] M.M.M. Jaradat and M.K. Al-Qeyyam, On the basis number and the minimum cycle bases of the wreath product of wheels. *International Journal of Mathematical combinatorics*, Vol. 1 (2008), 52-62.

[17] M.M.M. Jaradat and M.Y. Alzoubi, Un upper bound of the basis number of the lexicographic products of graphs, *Australas. J. Combin.* 32 (2005), 305-312.

[18] M.M.M. Jaradat and MY. Alzoubi, On the basis number of the semi-strong product of bipartite graphs with cycles, *Kyungpook Math. J.* 45 (1) (2005), 45-53.

[19] M.M.M. Jaradat, M.Y. Alzoubi and E.A. Rawashdeh, The basis number of the Lexicographic product of different ladders, *SUT Journal of Mathematics* 40, no. 2, 91-101 (2004).

[20] S. MacLane, A combinatorial condition for planar graphs, *Fundamenta Math.* 28 (1937), 22-32.

[21] E.F. Schmeichel, The basis number of a graph, *J. Combin. Theory Ser. B*, 30 (1981), 123-129.