



The Riemann Zeta Function and Its Analytic Continuation

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

The objective of this dissertation is to study the Riemann zeta function in particular it will examine its analytic continuation, functional equation and applications. We will begin with some historical background, then define of the zeta function and some important tools which lead to the functional equation. We will present four different proofs of the functional equation. In addition, the $\zeta(s)$ has generalizations, and one of these the Dirichlet L-function will be presented. Finally, the zeros of $\zeta(s)$ will be studied.

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1 Introduction

One of the oldest branches of mathematics is number theory. Many mathematicians have acknowledge the significance of complex analysis and applied it to number theory. The relationship between

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number theory and complex analysis is called Analytic Number Theory. The story begins formally with Euler's theorem, which was proved (1.1) to be divergent in 1737.

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \dots \tag{1.1}$$

Then, Euler proposed the Euler product formula:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

Following this Riemann considered the Euler's definition from the perspective of its complex argument. In 1859, Riemann extended the definition of Euler's zeta function from real variables to complex variables except a simple pole at $s = 1$ with residue 1. In his paper "On The Number of Primes Less Than a Given Magnitude". Riemann forced the zeta function to be a meromorphic function, and he proved the functional equation by using the analytical tools.

The Riemann zeta function is one of the most essential functions in mathematics. Its applications include many areas of study such as number theory and other sciences. This function was not created by Riemann but it is named after him since he developed the zeta function and proved the functional equation as well as demonstrating the significant relationship between the distribution of prime numbers and the Riemann zeta function.

Some of results required presentation because they will be used frequently throughout this dissertation.

Definition 1.1. For $Re(s) > 1$ the zeta function is defined as

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Definition 1.2. For $Re(s) > 1$ the Euler product is defined as

$$\zeta(s) = \prod \left(1 - \frac{1}{p^s}\right)^{-1}$$

Definition 1.3. For $Re(s) > 0$ the gamma function is defined as

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt$$

Definition 1.4. For $Re(s) > 1$ the Xi function is defined as

$$\xi(s) = \frac{1}{2} s(s-1) \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

The aim of this dissertation is to study the Riemann zeta function, in particular its analytic continuation, functional equation and applications.

This dissertation is divided into seven sections. The first section, briefly gives the historical background with particular attention paid to the development of the zeta function. In the second

section, some definitions and theorems related to the analytic continuation are provided along with some elementary asymptotic formula with out proofs. In the third section, provides several important analytic tools which help to prove the functional equation and play an important role in reaching the dissertation's goal. In the fourth section, the definition of the zeta function is stated in two ways, as the Dirichlet series and as the Euler product. In the fifth section, the functional equation is proved using four different methods. Then, In the sixth section, the zeta function has generalizations and one of these the Dirichlet L-function is presented with its analytic continuation. In addition, the function equation for the Dirichlet L-function is provided.Finally, studies the zeros of the zeta function and the Dirichlet L-function.

2 Background

In the 14th century ,the harmonic series , $\zeta(1)$ was proved to be divergent. Then,in the first half of the 18th century, Euler was the first mathematician to introduce the zeta function. Euler's zeta function is defined for any real number $s > 1$

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \dots$$

Euler only defined $\zeta(s)$ on \mathbb{R} . It means that Euler gave birth to $\zeta(s)$. Then, in the 19th century , Riemann improved the zeta function by using complex analysis. He extended the Euler definition from real variables to complex variables. Additionally, he proved the analytic continuation of the zeta function, hence obtaining the functional equation. Riemann considered the zeta function to be a complex function. He published his paper which is one of the most effective studies of modern mathematics in 1859. Moreover,by using complex analysis, Riemann connected the zeros of $\zeta(s)$ and the distribution of the prime numbers. However, before Riemann conjectures were proved or a results about primes extrapolated, he proved the two main consequences:

- (1) The zeta function can be meromorphic continued over the whole s-plane with a simple poles at $s = 1$,such that $\zeta(s) - \frac{1}{s-1}$ is an integral function.
- (2) The zeta function satisfies the reflection,

$$\pi^{-\frac{1}{2}s} \Gamma\left(\frac{1}{2}s\right) \zeta(s) = \pi^{-\frac{1}{2}s-1} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

The left hand side of this function is an even function on $s - \frac{1}{2}$, can be conclude the proprieties of the zeta function for $\sigma > 0$ from the proprieties for $\sigma > 1$ by the functional equation. Particularly, when $\sigma < 0$ the zeros of the zeta function are poles of $\Gamma\left(\frac{s}{2}\right)$ at negative even integers, these are called the trivial zeros. The remain in a plane, when $0 < \sigma < 1$ is called the critical strip. Riemann made several other remarkable conjectures

- (i) the zeta function has all its zeros in the critical strip
- (ii) $\xi(s)$ is an integral function defined as

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

which has no pole when $\sigma \leq 1/2$ and this function is an even function of $s - \frac{1}{2}$, also it has the product

$$\xi(s) = e^{A+Bs} \prod_{\lambda} \left(1 - \frac{s}{\lambda}\right) e^{s/\lambda}$$

such that A and B are absolutely constants and λ runs through the zeros of the zeta function in $0 \leq \sigma \leq 1$. Note that the most information is taken from Awan [1],Davenport [2] and Moros [3].

3 Analytic Number Theory

This section recall one of the most significant part of complex analysis that will be help to know the basic of the analytic number theory. Most of results will be taken from Brown et al.[4], Lang[5] and Titchmarsh[6].

3.1 Uniform convergence

Definition 3.1. If the partial sum $S_n(x) = \sum_{n=0}^{\infty} f_n(x)$ exists, the series $\sum_{n=0}^{\infty} f_n(x)$ is uniformly convergent in (a,b) , if every $\epsilon \geq 0$ such that ϵ is small. There exist a number n_1 depending on ϵ however, n_1 not depending on x

$$|S(x) - S_n(x)| < \epsilon$$

for $n > n_1$, all value of x in (a, b) .

Examples: (i) For $x \in [a, b]$, the power series $\sum_{n=0}^{\infty} x^n$ is uniformly convergent if $-1 < a < b < 1$.

(ii) Over any integral, the series $\sum_{n=0}^{\infty} \frac{\cos nx}{n^2}$ is uniformly convergent.

(iii) For $s \in [a, b]$, the Dirichlet series $\sum_{n=0}^{\infty} \frac{1}{n^s}$ is uniformly convergent if $1 < a < b$.

3.2 Identity theorem

If a region \mathbb{R} has two analytic and these functions have the same values for all points within \mathbb{R} , both functions are identical everywhere within \mathbb{R} . This result can be used to extend functions from the real axis to the complex plane.

3.3 Analytic functions

Definition 3.2. Suppose $B \subset \mathbb{C}$ is open set and let $f : B \rightarrow \mathbb{C}$ (i) let $b_0 \in B$, then the function f is differentiable at b_0 if

$$f'(b_0) = \lim_{b \rightarrow b_0} \frac{f(b) - f(b_0)}{b - b_0}$$

exist.

(ii) f is analytic on B , it has derivative at all points in B .

(iii) If the function f is analytic in a neighborhood of b_0 then f is analytic of b_0 .

(iv) The entire is a function on the whole complex plane.

Remark: If the function f has derivative, then f is continuous. However, if f is a continuous function, it does not have to be differentiable.

Proposition 3.1. Suppose $B \subset \mathbb{C}$ is open set and $f : B \rightarrow \mathbb{C}$, Then, f has a derivative at b_0 and, hence f is continuous at b_0 .

3.4 Analytic continuation

Definition 3.3. Suppose D_1 and D_2 are domains in the complex plane; the intersection of these domains, $D_1 \cap D_2$, is the set of all common points between D_1 and D_2 . The union of these domains, $D_1 \cup D_2$, is the set of all points in D_1 and D_2 .

Now, let $D_1 \cap D_2 \neq \phi$; this is also connected. The functions f_1, f_2 are analytic over the domains D_1, D_2 , respectively, such that $f_1 = f_2$ on $D_1 \cap D_2$. Hence, we can call f_2 an analytic continuation of f_1 in the second domain.

According to Brown et al.[4] the essential theorem from the theory of complex variables is:

Theorem 3.1. ([4] Theorem)

A function that is analytic in a domain D is uniquely determined over D by its values in a domain, or along a line segment, contained in D .

The definition of $F(b)$ analytic over $D_1 \cup D_2$

$$F(b) = \begin{cases} f_1(b) & \text{for } b \text{ is in } D_1 \\ f_2(b) & \text{for } b \text{ is in } D_2 \end{cases}$$

Since, $f_1 = f_2$ over $D_1 \cap D_2$, F is given by f_1 and f_2 over the domains D_1 and D_2 respectively. Based on the theorem this is a holomorphic function.

Since F is analytic in the union of D_1 and D_2 the function is uniquely determined by f_1 on the domain D_1 furthermore, it is uniquely determined by f_2 on a domain D_2 .

Thus, $F(b)$ is analytic continuation over the union of D_1 and D_2 of either f_1 or f_2 .

3.5 Some elementary asymptotic formula

Definition 3.4. (The "big oh" notation)

Where $h(x) > 0$ for all $x \geq a$, then $f(x)$ is big O of $h(x)$ Where $f(x) = O(h(x))$ and for $x \geq a$ $f(x)/h(x)$ is bounded. There exists a constant M greater than zero such that $|f(x)| \leq Mh(x)$ Moreover, it satisfies this equation:

$$f(x) = g(x) + O(h(x))$$

This means that $f(y) = O(h(y))$ for $l \geq a$ implies

$$\int_a^x f(y)dy = O\left(\int_a^x h(y)dy\right) \quad \text{for } x \geq a$$

The next theorem provides a number of asymptotic formula. In part (a) C is Euler's constant and it defined as

$$C = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log n \right)$$

. In part (b) the Riemann zeta function $\zeta(s)$ defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{when } s > 1$$

And by

$$\zeta(s) = \lim_{x \rightarrow \infty} \left(\sum_{n \leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} \right) \quad \text{when } 0 < s < 1$$

Theorem 3.2. If $X \geq 1$ we have that

$$(a) \quad \sum_{n \leq x} \frac{1}{n} = \text{Log}x + C + O\left(\frac{1}{x}\right)$$

$$\begin{aligned}
 (b) \quad \sum_{n \leq x} \frac{1}{n^2} &= \frac{x^{1-s}}{1-s} + \zeta(s) + O(x^{-s}) \quad \text{if } s > 0 \quad s \neq 1 \\
 (c) \quad \sum_{n > x} \frac{1}{n^s} &= O(x^{1-s}) \quad \text{if } s > 1 \\
 (d) \quad \sum_{n \leq x} n^\alpha &= \frac{x^{\alpha+1}}{\alpha+1} + O(x^\alpha) \quad \text{if } \alpha \geq 0
 \end{aligned}$$

Proof: See [7].

Note that these results are taken from Apostol[7]

4 Some Analytic Tools

As mentioned previously, Euler defined $\zeta(s)$ only for real variables. However, Riemann extended this definition to the domain of complex variables; to achieve this, Riemann studied the analytic continuation of $\zeta(s)$. The functional equations are extremely important in analytic continuation. To prove the functional equation requires certain some analytical tools and these will be provided and some of their properties discussed. This in turn will lead to several proofs of the functional equations. This section divides into six subsections, we will begin with the gamma function ,the theta function and the mellin transform. Then, the Dirichlet series and Abel summation (Partial summation) will be presented. Finally,the Fourier transformation will be laid out.

4.1 The gamma function

The gamma function is an important function for analytic number theory. It is useful function when dealing with the theory surrounding the zeta function. In particular, it is an extension of the factorial function to $Re(s) > 0$. It is also convergent.

Let we introduce this function :

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt \tag{4.1}$$

The first to denote this function as $\Gamma(s)$ was Legendre but the first to introduce it was Euler. However, Euler defined the function somewhat differently.

Proposition 4.1. ([8])

$$\Gamma(s + 1) = s\Gamma(s)$$

Proof

$$\begin{aligned}
 \Gamma(s + 1) &= \int_0^{\infty} e^{-t} t^{(s-1)+1} dt = \int_0^{\infty} e^{-t} t^s dt \\
 &= -e^{-t} t^s \Big|_0^{\infty} + s \int_0^{\infty} e^{-t} t^{s-1} dt = s\Gamma(s)
 \end{aligned}$$

For $\sigma > 0$ the term $(-e^{-t}t^s)|_0^\infty$ tends to zero. However, extending $\Gamma(s)$ to the entire complex plane needs the analytic continuation. The gamma function has several convenient properties support this.

Definition 4.1. (Euler's gamma-function) ([9] Definition A.3.1)

The Euler gamma-function is defined as

$$\frac{1}{\Gamma(s)} = s \exp(\gamma s) \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n}$$

such that γ is Euler's constant.

Theorem 4.1. ([9] Theorem A.3.1)

$$\Gamma(s) = \frac{1}{2} \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^s \left(1 + \frac{s}{n}\right)^{-1} \tag{4.2}$$

Corollary 4.2. ([9] Corollary A.3.3)

$$\Gamma(n+1) = n! \text{ for } n \in \mathbb{N}$$

Proposition 4.2. ([8] proposition)

The Gamma function $\Gamma(s)$ satisfies the extension formula:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$

Proof. Using Euler's gamma function

$$\begin{aligned} \frac{1}{\Gamma(s)} &= s \exp(\gamma s) \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n} \\ \frac{1}{\Gamma(s)} \frac{1}{\Gamma(-s)} &= -s^2 \exp(\gamma s) \exp(-\gamma s) \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n} \left(1 - \frac{s}{n}\right) e^{s/n} \end{aligned}$$

Since we have

$$\Gamma(-s) = -\Gamma\left(\frac{1-s}{s}\right)$$

We find that

$$\frac{1}{\Gamma(s)\Gamma(1-s)} = s \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{n^2}\right)$$

Using the infinite product formula, gives

$$\sin(\pi s) = s\pi \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{n^2}\right)$$

and hence

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$

□

Corollary 4.3. ([9] Corollary 1)

$$\Gamma(s) = \lim_{n \rightarrow \infty} \frac{(n-1)!n^s}{(s(s+1)\dots(s+n)-1)}$$

Corollary 4.4. ([9] Corollary 4)

$$\Gamma(1/2) = \sqrt{\pi}$$

Corollary 4.5. (doubling formula) ([9] Corollary 5)

$$\Gamma(s)\Gamma\left(s + \frac{1}{2}\right) = 2\sqrt{\pi} 2^{-2s} \Gamma(2s)$$

Proposition 4.3. The integral of the gamma function is uniformly convergent when $0 < a \leq \ell \leq b$, such that

$$\int_0^{\infty} t^{s-1} e^{-t} dt = \int_0^1 + \int_1^{\infty} = O\left(\int_0^1 t^{a-1} dt\right) + O\left(\int_1^{\infty} t^{b-1} e^{-t} dt\right) = O(1)$$

where the oh big depend on ℓ Thus, for $\ell > 0$ this integral shows a continuous function.

Proposition 4.4. If s is complex, the gamma function (4.1) is uniformly convergent $R(s) \leq a > 0$, if $s = \sigma + it$. Hence, $|t^{s-1}| = t^{\sigma-1}$.

The last propositions are taken from Chandrasekharan [10].

4.2 The theta function

Definition 4.2.

$$\theta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} \quad t > 0$$

Also, we can be written as

$$\theta(t) = 1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 t}$$

Definition 4.3. ([11] definition 4.4.1.)

If

$$w(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 t} \text{ and } \theta(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 t}$$

The relationship between $\theta(t)$ and $w(t)$ as $2w(t) = \theta(t) - 1$

Proposition 4.5. ([12] Proposition 1)

The theta function $\theta(t)$ satisfies the function equation:

$$\theta = \frac{1}{\sqrt{t}} \theta\left(\frac{1}{t}\right) \tag{4.3}$$

Before proving this propitiation it is necessary to we introduce:

The Poisson Summation Formula

If $\sum_{-\infty}^{\infty} f(t+n)$ uniformly converges for $0 \leq t \leq 1$, f is continuous and if $\sum_{-\infty}^{\infty} f(n)e^{2\pi nit}$ converges.

Hence, $\sum_{-\infty}^{\infty} f(t+n) = \sum_{-\infty}^{\infty} \widehat{f}(ne^{2\pi nit})$ where,

$$\widehat{f}(n) = \int_{-\infty}^{\infty} f(t)e^{-2\pi nit} dt$$

Proof. Let $\theta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}$ and $\sum_{n \in Z} f(n) = \sum_{k \in Z} \int_{-\infty}^{\infty} f(y) e^{-2\pi iky} dy$

We have

$$\begin{aligned} \theta(t) &= \sum_{n \in Z} f(n) = \sum_{k \in Z} \int_{-\infty}^{\infty} e^{-\pi y^2 t} e^{-2\pi iky} dy \\ &= \sum_{k \in Z} \int_{-\infty}^{\infty} e^{-\pi y^2 t - 2\pi iky} dy \\ &= \sum_{k \in Z} \int_{-\infty}^{\infty} e^{-\pi t[y^2 + 2i\frac{k}{t}y + i^2\frac{k^2}{t^2} - i^2\frac{k^2}{t^2}]} dy \\ &= \sum_{k \in Z} \int_{-\infty}^{\infty} e^{-\pi t(y+i\frac{k}{t})^2 - i^2\frac{k^2}{t^2}} dy \end{aligned}$$

It can be shown that

$$\begin{aligned} \theta(t) &= \sum_{k \in Z} \int_{-\infty}^{\infty} e^{-\pi k^2 \frac{1}{t}} e^{-\pi t(y+i\frac{k}{t})^2} dy \\ \theta(t) &= \sum_{k \in Z} e^{-\pi k^2 \frac{1}{t}} \int_{-\infty}^{\infty} e^{-\pi t(y+i\frac{k}{t})^2} dy \end{aligned}$$

let $y + i\frac{k}{t} = z$ and $dy = dz \quad |_{-\infty}^{\infty} \rightarrow \infty + i\frac{k}{t}$ and $-\infty + i\frac{k}{t}$

$$\theta(t) = \sum_{k \in Z} e^{-\pi n^2 \frac{1}{t}} \int_{-\infty + i\frac{k}{t}}^{\infty + i\frac{k}{t}} e^{-\pi tz^2} dz$$

Which implies that

$$\int_{-\infty + i\frac{k}{t}}^{\infty + i\frac{k}{t}} e^{-\pi tz^2} dz = \int_{-\infty}^{\infty} e^{-\pi tz^2} dz$$

and hence

$$\theta(t) = \sum_{k \in \mathbb{Z}} e^{-\pi k^2 \frac{1}{t}} \int_{-\infty}^{\infty} e^{-\pi t z^2} dz = \sum_{k \in \mathbb{Z}} e^{-\pi k^2 \frac{1}{t}} \sqrt{\frac{\pi}{\pi t}}$$

$$\sum_{n \in \mathbb{Z}} e^{-\pi n^2 t} = \sum_{n \in \mathbb{Z}} e^{-\pi k^2 \frac{1}{t}} \sqrt{\frac{1}{t}}$$

It has already been shown that

$$\theta(t) = \sum_{k \in \mathbb{Z}} e^{-\pi k^2 t}$$

It follows that

$$\theta(t) = \frac{1}{\sqrt{t}} \theta\left(\frac{1}{t}\right)$$

□

Corollary 4.6. ([12] Corollary 1)

suppose $\theta(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ then,

$$\theta\left(\frac{1}{t}\right) = -\frac{1}{2} + \frac{1}{2} t^{\frac{1}{2}} + \sqrt{t} \theta(t) \quad \text{when } t > 0$$

4.3 Mellin transform

The Mellin Transform is an integral transform defined as

$$f^*(s) = M(f)(s) = \int_0^{\infty} f(t)t^{s-1} dt$$

Examples: (1) As we know the gamma function is defined as

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt$$

Directly, we show that

$$\Gamma(s) = M(f)(s)$$

such that

$$f(t) = e^{-t}$$

It is obvious that this is analytically continues $\Gamma(s)$ to the right half plane $Re(s) > 0$.

(2) Suppose that $f(x) = 1/(e^x - 1)$ where $Re(s)$ greater than 1.

By mellin transform we have that

$$M(f)(s) = \int_0^{\infty} f(x)x^{s-1} dx$$

Since

$$\frac{1}{e^x - 1} = \left(\frac{e^x}{e^x}\right) \left(\frac{e^{-x}}{1 - e^{-x}}\right) = \sum_{n=1}^{\infty} e^{-nx}$$

Then

$$M(f)(s) = \int_0^{\infty} \sum_{n=1}^{\infty} e^{-nx} x^{s-1} dx$$

Setting $nx = t$ and $dt = ndx$

$$\begin{aligned} &= \sum_{n=1}^{\infty} \int_0^{\infty} \left(\frac{t}{n}\right)^{s-1} \frac{e^{-t}}{n} dt \\ &= \sum_{n=1}^{\infty} \frac{1}{n^s} \int_0^{\infty} t^{s-1} e^{-t} dt \end{aligned}$$

Hence, from the definition of the zeta function and the gamma function we find that

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \int_0^{\infty} t^{s-1} e^{-t} dt = \zeta(s)\Gamma(s)$$

Remark: Relating the theta function and the zeta function is required, particularly as there is a relationship between the Mellin function and the zeta function of the theta function. This relationship will appear in the third method of deriving the functional equation. Note that the Mellin function information is taken from Oosthuizen [13].

4.4 The dirichlet series

The next definition and theorems(4.7),(4.8) and (4.9) are taken from Karatsuba and Voronin [9].

Definition 4.4. A Dirichlet series is a statement of the form

$$f(n) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

such that $a_n \in C$ and $s = \sigma + it$.

Theorem 4.7. Let the series $\sum_{n=1}^{\infty} \frac{a_n}{n^{s_1}}$ is convergent, hence for every closed region consisted in the region $R(s) > R(s_1)$ the series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ is uniformly convergent. Thus, for $R(s) > R(s_1)$ the function $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ is analytic.

Theorem 4.8. Assume that in the region $\sigma_1 < \sigma < \infty$ for $\forall \sigma$

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

$$h(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

and

$$f(\sigma) = h(\sigma)$$

Hence, $a_n = b_n$ for $\forall n = 1, 2, 3, 4, \dots$

Theorem 4.9. If $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ and $h(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$ for $\sigma = \sigma_1$, these functions are definitely convergent and $|a_n| \leq b_n$, Hence

$$\int_{-L}^L |f(\sigma_1 + it)|^2 dt \leq 2 \int_{-2L}^{2L} |h(\sigma_1 + it)|^2 dt \quad \text{for any } L \geq 0$$

Theorem 4.10. ([7] Theorem 11.7)

Let the series $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ converges where $\sigma > \sigma_1$, f is multiplicative if

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \dots \right)$$

and f becomes completely multiplicative if

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \frac{1}{1 - f(p)p^{-s}}$$

4.5 Abel summation (partial summation)

Theorem 4.11. Suppose $\phi(x)$ is a function whose values belong to \mathbb{C} and this function has continuous derivative on the interval $[a, b]$; let c_m be arbitrary in \mathbb{C} and $C(x) = \sum_{a \leq m \leq x} c_m$. Then,

$$\sum_{a \leq n \leq b} c_n \phi(n) = - \int_a^b C(x) \phi'(x) dx + C(b) \phi(b)$$

Theorem 4.12. (Euler Summation Formula)

Assume $f(x)$ is complex valued and $f(x)$ has a continuous derivative on $[a, b]$ then,

$$\sum_{a \leq n \leq b} \phi(n) = \int_a^b \phi(x) dx + \int_a^b \lambda_1(x) \phi'(x) dx + \lambda_1(a) \phi(a) - \lambda_1(b) \phi(b)$$

where, $\lambda_1(x) = x - [x] - \frac{1}{2}$

Proof. From Theorem 4.11 , let $c_m = 1$ Then, $C[x] = [x] - [a]$

$$\begin{aligned} \sum_{a \leq n \leq b} \phi(n) &= ([b] - [a])\phi(b) - \int_a^b ([x] - [a])\phi'(x)dx \\ &= ([b] - [a])\phi(b) + [a](\phi(b) - \phi(a)) - \int_a^b [x]\phi'(x)dx \end{aligned}$$

We have $\lambda_1(x) = x - [x] - \frac{1}{2}$ also, we can be written as $[x] = x - \frac{1}{2} - \lambda_1(x)$

Therefore, we get

$$\begin{aligned} \sum_{a \leq n \leq b} \phi(n) &= ((b - \lambda_1(b) - (a - \lambda_1(a)))\phi(b) + (a - \frac{1}{2} - \lambda_1(a))(\phi(b) - \phi(a)) \\ &\quad - \int_a^b (x - \frac{1}{2} - \lambda_1(x))\phi'(x)dx \\ &= \lambda_1(a)\phi(a) - \lambda_1(b)\phi(b) + b\phi(b) - a\phi(b) + \lambda_1(a)\phi(b) + a\phi(b) \\ &\quad - a\phi(a) - \frac{1}{2}(\phi(b) - \phi(a)) - \lambda_1(a)\phi(b) - x\phi(x) \Big|_a^b + \frac{1}{2}\phi(b) - \phi(a) + \int_a^b \phi(x)dx + \int_a^b \lambda_1(x)\phi'(x)dx \end{aligned}$$

which yields the required result, after canceling.

□

The partial summation results are taken from Karatsuba and Voronin [9].

4.6 Fourier transform functions

If the function $f(x)$ is an even, then the Fourier integral represent as

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos xy \, dy \int_0^{\infty} \cos yt \, f(t)dt \tag{4.4}$$

this formula called the Fourier's cosine.

Similarly, If the function $f(x)$ is an odd, then we obtain that

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin xy \, dy \int_0^{\infty} \sin yt \, f(t)dt \tag{4.5}$$

this formula called the Fourier's sine. If we say

$$h(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\infty} \cos xt \, f(t) \, dt$$

Then, (4.4) provides:

$$f(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\infty} \cos xt \, h(t) \, dt$$

Hence, the functions $f(x)$ and $h(x)$ have a reciprocal relationship. Likewise, from 4.5 we find that the reciprocal formula

$$g(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\infty} \sin xt f(t) dt \tag{4.6}$$

and

$$f(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\infty} \cos xt g(t) dt$$

4.6.1 Integration of fourier integral

By integrating (4.4) we have

$$\int_0^{\xi} f(x) dx = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \xi y}{y} dy \int_0^{\infty} \cos yt f(t) dt$$

It holds over the interval $(0, \infty)$ for any integral function $f(t)$

$$\begin{aligned} & \int_0^Y \frac{\sin \xi y}{y} dy \int_0^{\infty} \cos yt f(t) dt \\ &= \int_0^{\infty} f(t) dt \int_0^Y \frac{\sin \xi y \cos yt}{y} dy \end{aligned}$$

For every y and t , the uniform convergent and inner product on the right is bounded. Therefore, it holds that

$$\begin{aligned} & \frac{1}{2} \int_0^Y \frac{\sin(\xi + t)y}{y} dy + \frac{1}{2} \int_0^Y \frac{(\sin \xi - t)y}{y} dy \\ &= \frac{1}{2} \int_0^{Y(\xi+t)} \frac{\sin u}{u} du \pm \frac{1}{2} \int_0^{Y(\xi-t)} \frac{\sin u}{u} du \end{aligned}$$

Now, we write the sign of $\xi - t$ and

$$\int_0^U \frac{\sin u}{u} du$$

This formula is a bounded function of U . Therefore, by using Lebesgue's convergence theorem as Y goes to infinity

$$\int_0^{\infty} \frac{\sin \xi y \cos yt}{y} dy = \frac{\pi}{2}$$

Note that These results were obtained from Titchmarsh[6].

5 The Riemann Zeta Function $\zeta(s)$

5.1 Definition of $\zeta(s)$

The Riemann zeta function can be defined by a simple formula in two representations, the first representation of $\zeta(s)$ by the Dirichlet series

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for } s = \sigma + it, \sigma > 1 \quad (5.1)$$

If $\ell > 1$ and $\sigma \geq \ell$, since $\left| \frac{1}{n^s} \right| = \left| \frac{1}{n^\sigma} \right|$ Then,

$$\sum_{n=1}^{\infty} \left| \frac{1}{n^s} \right| = \sum_{n=1}^{\infty} \left| \frac{1}{n^\sigma} \right| \leq \sum_{n=1}^{\infty} \left| \frac{1}{n^\ell} \right| < \infty$$

Hence, it uniformly converges. Moreover, the Riemann zeta function is analytic for $\sigma > 1$.

5.2 The euler product

Theorem 5.1. *The second way of defining $\zeta(s)$ is based on Euler's work defined by*

$$\zeta(s) = \prod \left(1 - \frac{1}{p^s} \right)^{-1}$$

for $s = \sigma + it, \sigma > 1$ where, this product is taken over every prime number.

Proof. Suppose $X \geq 2$, the function $\zeta_X(s)$ is defined as

$$\zeta_X(s) = \prod_{p \leq X} \left(1 - \frac{1}{p^s} \right)^{-1} \quad (5.2)$$

with all factors on the right hand side, allowing the term to be presented by a geometric series formula

$$\frac{1}{1 - p^s} = \sum_{\ell=0}^{\infty} p^{\ell s}$$

such that every geometric series is convergent. This allows term by term multiplication and substitute the right hand side with this equation:

$$\zeta_X(s) = \prod_{p \leq X} \sum_{\ell=0}^{\infty} \frac{1}{p^{\ell s}} \sum_{\ell_1=0}^{\infty} \dots \sum_{\ell_j=0}^{\infty} \frac{1}{(p_1^{\ell_1} \dots p_j^{\ell_j})^s} \quad (5.3)$$

such that $p_1 < p_2 < \dots < p_j$ and p_j represents all the prime numbers up to X .

We use a fundamental theorem of arithmetic (unique factorization of integers) and we see that every positive integer number $n \leq X$ can be represented as

$$n = p_1^{\ell_1} \dots p_j^{\ell_j}$$

when the ℓ_1, \dots, ℓ_j are positive integers. As a result, can be taken the right side of (5.3) as

$$\sum_{n \leq X} \frac{1}{n^s} + \sum_{n > X} \frac{1}{n^s} \quad (5.4)$$

where the ' stands for summation over those natural numbers (n greater than X) whose primes divisors are all less than or equal to X . The upper bound of this sum is

$$\begin{aligned} \left| \sum'_{n>X} \frac{1}{n^s} \right| &\leq \sum'_{n>X} \frac{1}{n^\sigma} < \sum_{n>X} \frac{1}{n^\sigma} \leq \frac{1}{X^\sigma} + \int_X^\infty \frac{du}{u^\sigma} \\ &= \frac{1}{X^\sigma} + \frac{u^{-\sigma+1}}{-\sigma+1} \Big|_X^\infty \\ &= \frac{1}{X^\sigma} + \frac{X^{\sigma-1}}{\sigma-1} \leq \frac{\sigma}{\sigma-1} X^{\sigma-1} \end{aligned}$$

The final formula provides an upper bound for the sum. Combining the definition of the zeta function $\zeta_X(s)$ with (5.2),(5.1)and(5.4) gives

$$\zeta_X(s) = \prod_{p \leq X} \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{n \leq X} \frac{1}{n^s} + O\left(\frac{\sigma}{\sigma-1} X^{1-\sigma}\right)$$

If X goes to $+\infty$ we obtain $X^{1-\sigma}$ goes to $+\infty$ such that $\sigma > 1$. Thus, we proved that

$$\sum_{n \leq X} \frac{1}{n^s} = \prod_{p \leq X} \left(1 - \frac{1}{p^s}\right)^{-1}$$

□

Now, we will present an important properties of Riemann zeta function in terms of analytic continuation.

Theorem 5.2. ([12] Theorem 4)

The function $\zeta(s) - \frac{1}{s-1}$ extends to the right half of the plane for $\sigma > 0$ and this function is analytic when $\sigma > 0$.

Proof. Consider $f_n(s) = \zeta(s) - \frac{1}{s-1}$

Initially, we will prove $f_n(s)$ to be convergent, then we will show that

$$f_n(s) = \zeta(s) - \frac{1}{s-1}$$

From equation (5.1) we have that

$$\begin{aligned} f_n(s) &= \sum_{n=1}^\infty \frac{1}{n^s} - \int_n^{n+1} \frac{1}{x^s} dx \\ &= \sum_{n=1}^\infty \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s}\right) dx \end{aligned}$$

Then

$$|f_n(s)| = \left| \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s}\right) dx \right| \leq \max_{n \leq x \leq n+1} \left| \frac{1}{n^s} - \frac{1}{x^s} \right|$$

However, we know that

$$\frac{1}{n^s} - \frac{1}{x^s} = \int_n^x s u^{-s-1} du$$

Hence

$$\left| \frac{1}{n^s} - \frac{1}{x^s} \right| \leq |s| \int_n^{n+1} u^{-\sigma-1} du \leq \frac{|s|}{\sigma} \left(\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} \right)$$

We obtain

$$\sum_{n=1}^{\infty} |f_n(s)| \leq \frac{|s|}{\sigma} \sum_{n=1}^{\infty} \left(\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} \right) = \frac{|s|}{\sigma}$$

Since, $f_n(s)$ is an entire function. So, $\sum_{n=1}^{\infty} |f_n(s)|$ converges. In addition, $\sum_{n=1}^{\infty} |f_n(s)|$ is analytic function for $\sigma > 1$.

$$\begin{aligned} \sum_{n=1}^{\infty} f_n(s) &= \sum_{n=1}^{\infty} \left(\frac{1}{n^s} - \int_n^{n+1} \frac{1}{x^s} dx \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{x^s} dx \\ &= \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \left(\frac{(n+1)^{-s+1} - n^{-s+1}}{-s+1} \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n^s} - \frac{1}{s-1} \end{aligned}$$

By equation (5.1) we get

$$= \zeta(s) - \frac{1}{s-1}$$

Thus, for $\sigma > 1$ $\zeta(s) = \sum_{n=1}^{\infty} f_n(s) + \frac{1}{s-1}$

For $\sigma > 0$ $f_n(s)$ and $\frac{1}{s-1}$ are analytic, then $\sum_{n=1}^{\infty} f_n(s) + \frac{1}{s-1}$ is an analytic continuation of the zeta function. □

Theorem 5.3. ([12] Theorem 5)

Suppose

$$\xi(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) \tag{5.5}$$

such that $\xi(s)$ is an analytic continuation on the entire plane and that $\xi(s)$ satisfies

$$\xi(s) = \xi(1-s)$$

Proof. Using the definition of the zeta function and the gamma function. Since

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty x^{\frac{s}{2}-1} e^{-x} dx \tag{5.6}$$

and

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} \tag{5.7}$$

Applying (5.6) and (5.7) in (5.5) gives

$$\begin{aligned} \xi(s) &= \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{s}{2}} \left(\int_0^\infty x^{\frac{s}{2}-1} e^{-x} dx \right) \left(\sum_{n=1}^\infty \frac{1}{n^s} \right) \\ &= \sum_{n=1}^\infty \int_0^\infty \left(\frac{x}{n^2 \pi} \right)^{-s/2} e^{-x} dx \end{aligned}$$

Consider that $y = \frac{x}{\pi n^2}$ then $x = y\pi n^2$

$$\xi(s) = \sum_{n=1}^\infty \int_0^\infty y^{-\frac{s}{2}-1} e^{-\pi n^2 y} dy = \int_0^\infty y^{\frac{s}{2}-1} \psi(y) dy$$

Dividing the integral

$$\xi(s) = \int_0^1 y^{\frac{s}{2}-1} \psi(y) dy + \int_1^\infty y^{\frac{s}{2}-1} \psi(y) dy$$

changing the variable $y = \frac{1}{v}$

then

$$\xi(s) = \int_0^1 v^{-\frac{s}{2}-1} \psi(y) dy + \int_1^\infty y^{\frac{s}{2}-1} \psi(y) dy$$

We use the corollary (4.1.5)

$$\xi(s) = \int_1^\infty v^{-\frac{s}{2}-1} \left(-\frac{1}{2} + \frac{1}{2} v^{\frac{1}{2}} + v^{\frac{1}{2}} \psi(v) \right) + \int_1^\infty y^{\frac{s}{2}-1} \psi(y) dy$$

Dividing the integral

$$\begin{aligned} &= -\frac{1}{2} \int_1^\infty v^{-\frac{s}{2}-1} dv + \frac{1}{2} \int_0^\infty v^{-\frac{s}{2}-\frac{1}{2}} dv + \int_0^\infty v^{-\frac{s}{2}-1} \psi(v) dv + \int_1^\infty y^{\frac{s}{2}-1} \psi(y) dy \\ &= -\frac{1}{s} + \frac{1}{s-1} + \int_0^\infty v^{\frac{s}{2}-1} \psi(v) dv + \int_1^\infty y^{\frac{s}{2}-1} \psi(y) dy \end{aligned}$$

Assume that

$$K(s) = \int_0^\infty y^{-\frac{s}{2}-1} \psi(y) dy + \int_1^\infty y^{\frac{s}{2}-1} \psi(y) dy$$

Therefore we obtain

$$\xi(s) = -\frac{1}{s} + \frac{1}{s-1} + K(s)$$

We obvious that $K(s)$ is an entire function. In addition, $\frac{1}{s}$ and $\frac{1}{s-1}$ are analytic when $s \neq 0, 1$, then $\xi(s) = -\frac{1}{s} + \frac{1}{s-1} + K(s)$ is an analytic continuation of $\xi(s)$. □

6 Functional Equations and Analytic Continuation

One of the aims in this dissertation is to prove the functional equation of $\zeta(s)$. In this section, four different proofs of the functional equation are provided. Note that these proofs are taken from Davenport [2], Titchmarsh [14] and Chandrasekharan [10]. More methods are available in [14].

6.1 First method

Theorem 6.1. ([14] Theorem 2.1)

The function $\zeta(s)$ is regular for all value of s except $s=1$, where there is a simple pole with residue 1. It satisfies the functional equation.

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{1}{2} s \pi \Gamma(1-s) \zeta(1-s) \tag{6.1}$$

Proof. We will prove this theorem depending on the summation formula.

Assume $\phi(x)$ to be a function which is continuously differentiable on $[a, b]$ a; $[x]$ denotes the greatest integer less than x .

Using Euler summation formula gives

$$\begin{aligned} \sum_{a < m \leq b} \phi(m) &= \int_a^b \phi(x) dx + \int_a^b (x - [x] - \frac{1}{2}) \phi'(x) dx + (a - [a] - \frac{1}{2}) \phi(a) \\ &\quad - (b - [b] - \frac{1}{2}) \phi(b) \end{aligned} \tag{6.2}$$

With respect to $(a, b]$; this formula is clearly additive. It is sufficient to assume that $n \leq a < b \leq n + 1$. Then

$$\int_a^b (x - n - \frac{1}{2}) \phi'(x) dx = (b - n - \frac{1}{2}) \phi(b) - (a - n - \frac{1}{2}) \phi(a) - \int_a^b \phi(x) dx \tag{6.3}$$

Take $\sigma > 1$, where $a = 1$, and b goes to ∞ and add 1 to both sides hence we obtain:

$$\zeta(s) = s \int_1^{\infty} \frac{x - [x] + \frac{1}{2}}{x^{s+1}} dx + \frac{1}{s-1} + \frac{1}{2} \tag{6.4}$$

Titchmarsh noted that the numerator $x - [x] + \frac{1}{2}$ is bounded and, when $\sigma > 0$, the integral in (6.4) is convergent. This integral is uniformly convergent in any finite region to the right of $\sigma = 0$. Hence, for $\sigma > 0$, it defines the analytic function of s regularly. The right hand side gives the analytic function $\zeta(s)$ up to $\sigma = 0$ and it is clear that a simple pole at $s=1$ with residue 1.

For $0 < \sigma < 1$ we have

$$\int_0^1 \frac{[x] - x}{x^{s+1}} dx = - \int_0^1 x^{-s} dx = \frac{1}{s-1}$$

and

$$\frac{s}{2} = \int_1^{\infty} \frac{dx}{x^{s+1}} = \frac{1}{2}$$

By (6.4) we can write that, for $0 < \sigma < 1$

$$\zeta(s) = \int_0^{\infty} \frac{[x] - x}{x^{s+1}} dx \tag{6.5}$$

For $\sigma > -1$, the equation (6.4) provides the analytic continuation of the zeta function.

If $f(x) = [x] - x + \frac{1}{2}$ and $f_1(x) = \int_1^x f(y) dy$

Since, $\int_n^{n+1} f(y) dy = 0 \quad \forall n$ integers, then $f_1(x)$ is also bounded. Thus,

$$\int_{x_1}^{x_2} \frac{f(x)}{x^{s+1}} dx = \left[\frac{f_1(x)}{x^{s+1}} \right]_{x_1}^{x_2} + (s+1) \int_{x_1}^{x_2} \frac{f_1(x)}{x^{s+2}} dx \tag{6.6}$$

which goes to zero as $x_1 \rightarrow \infty$ and $x_2 \rightarrow \infty$.

If $\sigma > -1$, the integral (6.4) is convergent.

$$s \int_0^1 \frac{[x] - x + \frac{1}{2}}{x^{s+1}} dx = \frac{1}{s-1} + \frac{1}{2} \text{ for } (\sigma < 0) \tag{6.7}$$

Therefore,

$$\zeta(s) = s \int_0^{\infty} \frac{[x] - x + \frac{1}{2}}{x^{s+1}} dx \text{ for } -1 < \sigma < 0 \tag{6.8}$$

$[x] - x + \frac{1}{2}$ has a Fourier series expansion

$$[x] - x + \frac{1}{2} = \sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{\pi n} \tag{6.9}$$

If x is not integer, we will substitute the series (6.9) in equation (6.8) to gives

$$\begin{aligned} \zeta(s) &= \frac{s}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} \frac{\sin 2n\pi x}{x^{s+1}} dx \\ &= \frac{s}{\pi} \sum_{n=1}^{\infty} \frac{(2n\pi)^s}{n} \int_0^{\infty} \frac{\sin y}{y^{s+1}} dy \end{aligned}$$

$$\zeta(s) = \frac{s}{\pi} (2n\pi)^s - \Gamma(-s) \sin s\pi \zeta(1-s) \tag{6.10}$$

This permitted for $-1 < \sigma < 0$, but, for $\sigma < 0$ the right hand side is analytic for all value of s. Therefore, this gives the analytic continuation (not just for $-1 < \sigma < 0$), which means it provides the analytic continuation throughout the plane.

This is sufficient to prove

$$\lim_{\lambda \rightarrow \infty} \sum_{n=1}^{\infty} \frac{1}{n} \int_{\lambda}^{\infty} \frac{\sin 2n\pi x}{x^{s+1}} dx = 0 \quad (-1 < \sigma < 0)$$

to justify the term-wise integration.

The series in (6.9) is bounded convergent, so the term-wise integration is absolutely justified over any finite range. Then

$$\begin{aligned} \int_{\lambda}^{\infty} \frac{\sin 2nx\pi}{x^{s+1}} dx &= \left[\frac{-\cos 2n\pi x}{2n\pi(x^{s+1})} \right]_{\lambda}^{\infty} - \frac{s+1}{2n\pi} \int_{\lambda}^{\infty} \frac{\cos 2n\pi x}{x^{s+2}} dx \\ &= O\left(\frac{1}{n\lambda^{\sigma+1}}\right) + O\left(\frac{1}{n} \int_{\lambda}^{\infty} \frac{dx}{x^{\sigma+2}}\right) = O\left(\frac{1}{n\lambda^{\sigma+1}}\right) \end{aligned}$$

Thus

$$\lim_{\lambda \rightarrow \infty} \sum_{n=1}^{\infty} \frac{1}{n} \int_{\lambda}^{\infty} \frac{\sin 2n\pi x}{x^{s+1}} dx = 0$$

as a result of (6.4) we obtain

$$\begin{aligned} \lim_{s \rightarrow 1} \left(\zeta(s) - \frac{1}{s-1} \right) &= \int_1^{\infty} \frac{[x] - x + \frac{1}{2}}{x^2} dx + \frac{1}{2} \\ &= \lim_{n \rightarrow \infty} \int_1^n \frac{[x] - x}{x^2} dx + 1 \end{aligned}$$

Taking the integral

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \sum_{m=1}^{n-1} m \int_m^{m+1} \left(\frac{dx}{x^2} - \log n + 1 \right) \\ &= \lim_{n \rightarrow \infty} \sum_{m=1}^{n-1} \frac{1}{m+1} + 1 - \log n = \gamma \end{aligned}$$

Thus, we get

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|)$$

(near s=1)

□

Note:

We can write the function (6.1) as different ways, if we change s into $1 - s$ we obtain that

$$(1 - s) = 2^{1-s} \pi^{1-s} \cos \frac{1}{2} s \pi \Gamma(s) \zeta(s) \tag{6.11}$$

It could be written

$$\zeta(s) = \chi(s) \zeta(1 - s) \tag{6.12}$$

where

$$\chi(s) = 2^s n^{s-1} \sin \frac{1}{2} 2\pi \Gamma(1 - s) = \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2} - \frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}$$

and

$$\chi(s) \chi(1 - s) = 1$$

$$\xi(s) = \frac{1}{2} s(s - 1) \pi^{-\frac{1}{2}s} \Gamma\left(\frac{1}{2}s\right) \zeta(s)$$

it is exactly proved from (6.11) and (6.12)

$$\xi(s) = \xi(1 - s)$$

if $s = \frac{1}{2} + iz$ and $z \in \mathbb{C}$

$$\xi\left(\frac{1}{2} + iz\right) = \xi\left(1 - \left(\frac{1}{2} + iz\right)\right)$$

$$\Xi(z) = \left(\frac{1}{2} + iz\right)$$

$$\Xi(z) = \Xi(-z)$$

* Hardy applied a similar argument in the first method , However not to it self, to this function

$$(1 - 2^{1-s}) \zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$$

Hardy's proof: one can see the proof in Titchmarsh[14].

6.2 Second method

The fundamental formula is:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx \quad \text{for } \sigma > 1 \tag{6.13}$$

Proof. to prove this formula, start with

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx \tag{6.14}$$

By replacing the variable $x = nx$

$$\sum_{n=1}^{\infty} \frac{\Gamma(s)}{n^s} = \sum_{n=1}^{\infty} \int_0^{\infty} x^{s-1} e^{-nx} dx \tag{6.15}$$

is created.

$$= \int_0^{\infty} \left[\sum_{n=1}^{\infty} e^{-nx} \right] x^{s-1} dx \text{ for } \sigma > 1$$

Since,

$$\sum_{n=1}^{\infty} \left[\int_0^{\infty} x^{s-1} e^{-nx} dx \right] \leq \sum_{n=1}^{\infty} \int_0^{\infty} x^{\sigma-1} e^{-nx} dx = \sum_{n=1}^{\infty} \frac{\Gamma(\sigma)}{n^{\sigma}} < \infty \tag{6.16}$$

this is convergent for $\sigma > 1$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\Gamma(s)}{n^s} &= \int_0^{\infty} \frac{e^{-x}}{1 - e^{-x}} \cdot \left(\frac{e^x}{e^x} \right) x^{s-1} dx = \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx \\ \Gamma(s) \sum_{n=1}^{\infty} \frac{1}{n^s} &= \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx \end{aligned}$$

By the definition of $\zeta(s)$ we find that

$$\Gamma(s)\zeta(s) = \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx$$

Now, we assume the complex integral

$$I(s) = \int_c \frac{z^{s-1}}{e^z - 1} dz$$

such that the contour C containing of the positive real axis from ∞ to ρ for $0 < \rho < 2\pi$, the circle $|z| = \rho$ and the positive real axis from ρ to ∞ . On the circle,

$$\begin{aligned} |z^{s-1}| &= |e^{(s-1) \log z}| = |e^{((\sigma-1)+it)(\log|z|-i \arg z)}| \\ &= e^{(\sigma-1)\log|z|-t \arg z} \\ &\leq |z|^{\sigma-1} e^{2\pi|t|} \\ |e^z - 1| &> A|z| \end{aligned}$$

Therefore, for $\sigma > 1$ the integral round this circle goes to 0 as $\rho \rightarrow 0$

Hence on letting $\rho \rightarrow 0$, we obtain

$$\begin{aligned} I(s) &= - \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx + \int_0^{\infty} \frac{(xe^{2\pi i})^{s-1}}{e^x - 1} dx \\ &= -\Gamma(s)\zeta(s) + e^{2\pi i s} \Gamma(s)\zeta(s) \end{aligned}$$

$$= \Gamma(s)\zeta(s)(e^{2\pi is} - 1)$$

by using the result

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$

Then

$$\begin{aligned} I(s) &= \frac{\zeta(s)}{\Gamma(1-s)} 2\pi i \frac{e^{2\pi is}}{e^{\pi is} - e^{-\pi is}} \\ &= \frac{\zeta(s)}{\Gamma(1-s)} 2\pi i e^{\pi is} \end{aligned}$$

Hence,

$$\zeta(s) = \frac{e^{-i\pi s}\Gamma(1-s)}{2\pi i} \int_c \frac{z^{s-1}}{e^z - 1} dz$$

This integral is convergent in any finite region of the s-plane. Therefore, it defines an integral function. The last formula gives the analytic continuation of the zeta function. The poles of $\Gamma(1-s)$ are only possible poles, $s = 1, 2, 3, \dots$. We see that $\zeta(s)$ is regular for all values of s except $s = 1$. Thus, the only possible pole is at $s = 1$.

$$I(1) = \int_c \frac{dz}{e^z - 1} = 2\pi i$$

and

$$\Gamma(1-s) = -\frac{1}{s-1} + \dots$$

Thus the residue of $\zeta(s)$ at the pole is 1.

If $s \in \mathbb{N}$ and since

$$\frac{z}{e^z - 1} = 1 - \frac{1}{2}z + B_1 \frac{z^2}{2!} - B_2 \frac{z^3}{4!} + \dots$$

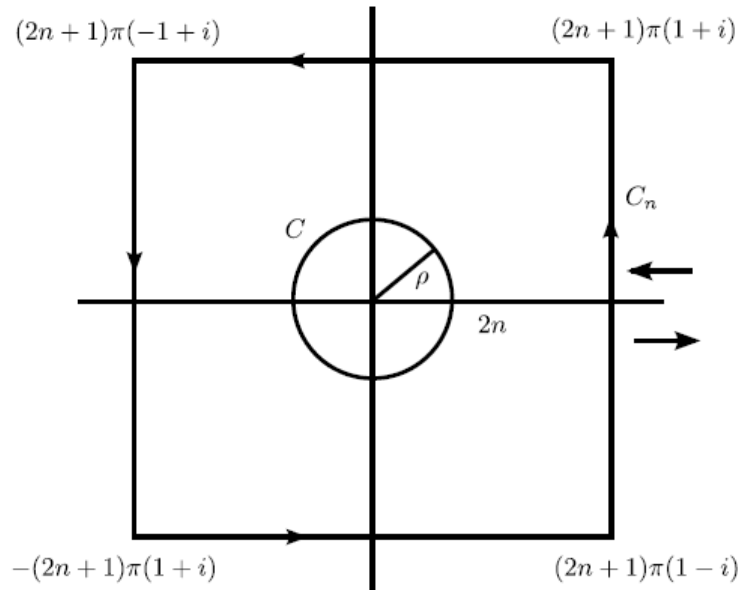
where $(B_1, B_2 \dots \text{Bernoulli numbers})$.

Then, $\zeta(0) = -\frac{1}{2}$, $\zeta(-2m) = 0$, $\zeta(1-2m) = \frac{(-1)^m B_m}{2m}$ such that $m = 1, 2, 3, \dots$

Let the integral

$$\int \frac{z^{s-1}}{e^z - 1} dz$$

We take the integral along the contour C_n as in the diagram,



There is an integral between C and C_n which has poles at the points $\pm 2i\pi, \dots, \pm 2in\pi$ the residue at $-2mi\pi$ is $(2m\pi e^{\frac{3}{2}i\pi})^{s-1}$ and the residue at $2mi\pi$ is $(2m\pi e^{\frac{1}{2}i\pi})^{s-1}$. taking together

$$\begin{aligned} & (2m\pi e^{\frac{1}{2}i\pi})^{s-1} + (2m\pi e^{\frac{3}{2}i\pi})^{s-1} \\ &= (2m\pi)^{s-1} e^{i\pi(s-1)} 2 \cos \frac{1}{2}\pi(s-1) \\ &= -2(2m\pi)^{s-1} e^{i\pi s} \sin \frac{1}{2}\pi(s-1) \end{aligned}$$

The theorem of residue gives

$$I(s) = \int_{C_n} \frac{z^{s-1}}{e^z - 1} dz + 4\pi e^{i\pi s} \sin \frac{1}{2}\pi s \sum_{m=1}^n (2m\pi)^{s-1}$$

Now, consider $\sigma > 0$ as n goes to infinity. On the contours C_n , the function $\frac{1}{(e^z - 1)}$ is bounded.

$$|z^{s-1}| = O(|z|^{\sigma-1})$$

Therefore, the integral round C_n goes to zero and we get

$$\begin{aligned} I(s) &= 4\pi i e^{i\pi s} \sin \frac{1}{2}\pi s \sum_{n=1}^{\infty} (2n\pi)^{s-1} \\ &= 4\pi i e^{i\pi s} \sin \frac{1}{2}\pi s (2\pi)^{s-1} \Gamma(1-s) \end{aligned}$$

Note that the diagram taken from Chandrasekharan [10].

* When $Re(s) < 0$ and therefore by analytic continuation for $\forall s$

By taking

$$\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) = \frac{\sqrt{\pi}}{2^{s-1}}\Gamma(s)$$

and changing the variables from s to $1-s$ we obtain that

$$\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{1-s+1}{2}\right) = \frac{\sqrt{\pi}}{2^{1-s-1}}\Gamma(s)$$

Then

$$\Gamma\left(1-\frac{s}{2}\right)2^{-s} = \sqrt{\pi}\Gamma(1-s)$$

Using this result

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$

Therefore, we have

$$\begin{aligned} \Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right) &= \frac{\pi}{\sin \frac{s\pi}{2}} \\ \Gamma(1-s) &= 2^s \Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}\right)^{-1} \sqrt{\pi} \left(\sin \frac{\pi s}{2}\right)^{-1} \end{aligned}$$

Hence

$$\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) = \pi^{-\frac{1}{2}s-1}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

□

6.3 Third method

Proof. This method proves the equation (6.17) by the gamma function

$$\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) = \pi^{-\frac{1}{2}s-1}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \tag{6.17}$$

Definition of the gamma function

$$\Gamma(s) = \int_0^{\infty} t^{s-1}e^{-t} dt \tag{6.18}$$

Replace s by $\frac{s}{2}$

$$\Gamma\left(\frac{s}{2}\right) = \int_0^{\infty} t^{\frac{s}{2}-1}e^{-t} dt$$

Let $t = \pi n^2 x$ and $dt = \pi n^2 dx$, Then

$$\Gamma\left(\frac{s}{2}\right) = \int_0^{\infty} (\pi n^2 x)^{\frac{s}{2}-1} e^{-\pi n^2 x} \pi n^2 dx$$

$$\Gamma\left(\frac{s}{2}\right) = \int_0^{\infty} \pi^{\frac{s}{2}} n^s x^{\frac{s}{2}} e^{-\pi n^2 x} dx$$

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \frac{1}{n^s} = \int_0^{\infty} x^{\frac{s}{2}-1} e^{-\pi n^2 x} dx$$

We add sum from both sides

$$\sum_{n=1}^{\infty} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \frac{1}{n^s} = \sum_{n=1}^{\infty} \int_0^{\infty} x^{\frac{s}{2}-1} e^{-\pi n^2 x} dx$$

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \sum_{n=1}^{\infty} \frac{1}{n^s} = \int_0^{\infty} x^{\frac{s}{2}-1} \sum_{n=1}^{\infty} e^{-\pi n^2 x} dx$$

Since he $\zeta(s)$ defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Thus we have

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) = \int_0^{\infty} x^{\frac{s}{2}-1} \sum_{n=1}^{\infty} e^{-\pi n^2 x} dx \tag{6.19}$$

From definition (4.2)

$$w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x} dx \tag{6.20}$$

and

$$\begin{aligned} \theta(x) &= \sum_{n \in \mathbb{Z}} e^{-\pi n^2 x} = 1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 x} \\ &= 1 + 2 w(x) \end{aligned}$$

We apply (6.20)in(6.19)

$$\int_0^{\infty} x^{\frac{s}{2}-1} \sum_{n=1}^{\infty} e^{-\pi n^2 x} dx = \int_0^{\infty} x^{\frac{s}{2}-1} w(x) dx$$

Dividing the integral

$$\int_0^{\infty} x^{\frac{s}{2}-1} w(x) dx = \int_1^{\infty} x^{\frac{s}{2}-1} w(x) dx + \int_0^1 x^{\frac{s}{2}-1} w(x) dx$$

$$\theta(x) = \frac{1}{\sqrt{x}} \theta\left(\frac{1}{x}\right)$$

or

$$\begin{aligned} 2w(x) + 1 &= \frac{1}{\sqrt{x}} \left[2w\left(\frac{1}{x}\right) + 1 \right] \\ 2w(x) &= \frac{1}{\sqrt{x}} \left[2w\left(\frac{1}{x}\right) + 1 \right] - 1 \\ w(x) &= \frac{1}{\sqrt{x}} \left[\frac{2w\left(\frac{1}{x}\right) + 1}{2} \right] - \frac{1}{2} \end{aligned}$$

It follows that

$$w(x) = \frac{1}{\sqrt{x}} w\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2} \tag{6.21}$$

$$\begin{aligned} \int_0^1 x^{\frac{s}{2}-1} w(x) dx &= \int_0^1 x^{\frac{s}{2}-1} \left(\frac{1}{\sqrt{x}} w\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2} \right) dx \\ &= \int_0^1 \left[x^{\frac{s}{2}-\frac{3}{2}} w\left(\frac{1}{x}\right) + \frac{1}{2} \left(x^{\frac{s}{2}-\frac{3}{2}} - x^{\frac{s}{2}-1} \right) \right] dx \\ &= \int_0^1 x^{\frac{s}{2}-\frac{3}{2}} w\left(\frac{1}{x}\right) dx + \frac{1}{2} \int_0^1 \left[x^{\frac{s}{2}-\frac{3}{2}} - x^{\frac{s}{2}-1} \right] dx \\ &= \int_0^1 x^{\frac{s}{2}-\frac{3}{2}} w\left(\frac{1}{x}\right) dx + \frac{1}{2} \left[\frac{1}{\frac{s}{2}-\frac{1}{2}} x^{\frac{s}{2}-\frac{1}{2}} - \frac{1}{\frac{s}{2}} x^{\frac{s}{2}} \right]_0^1 dx \\ &= \int_0^1 x^{\frac{s}{2}-\frac{3}{2}} w\left(\frac{1}{x}\right) dx + \frac{1}{s(s-1)} \end{aligned}$$

Now, if we change variable $x = \frac{1}{u}$ and $dx = -\frac{1}{u^2} du$ also, $|_0^1 \rightarrow |_\infty^1$

Hence

$$\begin{aligned} &= \frac{1}{s(s-1)} + \int_\infty^1 \left(\frac{1}{u}\right)^{\frac{s}{2}-\frac{3}{2}} w(u) \left(-\frac{du}{u^2}\right) \\ &= \frac{1}{s(s-1)} + \int_1^\infty \left(\frac{1}{x}\right)^{\frac{s}{2}-\frac{3}{2}} w(x) \left(-\frac{dx}{x^2}\right) \\ \int_0^1 x^{\frac{s}{2}-1} w(x) dx &= \int_1^\infty x^{-\frac{s}{2}-\frac{1}{2}} w(x) dx + \frac{1}{s(s-1)} \\ \int_0^\infty x^{\frac{s}{2}-1} w(x) dx &= \int_1^\infty x^{\frac{s}{2}-1} w(x) dx + \int_0^1 x^{\frac{s}{2}-1} w(x) dx \\ \int_0^\infty x^{\frac{s}{2}-1} w(x) dx &= \frac{1}{s(s-1)} + \int_1^\infty x^{\frac{s}{2}-1} w(x) dx + \int_1^\infty x^{-\frac{s}{2}-\frac{1}{2}} dx \end{aligned}$$

Since we have

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^\infty x^{\frac{s}{2}-1} w(x) dx$$

Then

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty \left(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}} \right) w(x) dx$$

For any s , this integral is convergent and this formula provides the analytic continuation. If we replace s by $s - 1$ the right side is unchanged. Hence, we obtain

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

□

6.4 Fourth method

Proof. This method depends on self-reciprocal function. If $\sigma > 1$ could be written as

$$\zeta(s)\Gamma(s) = \int_0^1 \left(\frac{1}{e^x - 1} - \frac{1}{x}\right) x^{s-1} dx + \frac{1}{s-1} + \int_1^\infty \frac{x^{s-1}}{e^x - 1} dx \tag{6.22}$$

For σ greater than zero and analytic continuation (6.13) holds. Moreover, if $0 < \sigma < 1$ we have

$$\frac{1}{s-1} = - \int_1^\infty \frac{x^{s-1}}{x} dx \tag{6.23}$$

Therefore

$$\zeta(s)\Gamma(s) = \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x}\right) x^{s-1} dx \quad \text{for } 0 < \sigma < 1 \tag{6.24}$$

Since the function $f(x) = \frac{1}{e^{x\sqrt{2\pi}} - 1} - \frac{1}{x\sqrt{2\pi}}$ is self-reciprocal for the sine transforms

by (4.6)

$$f(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty f(y) \sin xy \, dy$$

Let $x = \xi\sqrt{2\pi}$ and $\xi = \frac{x}{\sqrt{2\pi}}$ in (6.24) gives

$$\begin{aligned} \zeta(s)\Gamma(s) &= (\sqrt{2\pi})^s \int_0^\infty f(\xi) \left(\frac{x}{\sqrt{2\pi}}\right)^{s-1} d\xi \\ \zeta(s)\Gamma(s) &= (2\pi)^{\frac{s}{2}} \int_0^\infty f(\xi) \xi^{s-1} d\xi \\ &= (2\pi)^{\frac{s}{2}} \sqrt{\frac{2}{\pi}} \int_0^\infty \xi^{s-1} d\xi \int_0^\infty f(y) \sin \xi y \, dy \end{aligned}$$

Changing order of integration, then

$$= 2^{\frac{1}{2}s + \frac{1}{2}} \pi^{\frac{1}{2}s - \frac{1}{2}} \int_0^\infty f(y) \, dy \int_0^\infty \xi^{s-1} \sin \xi y \, d\xi$$

Put $\xi = u$

$$= 2^{\frac{1}{2}s + \frac{1}{2}} \pi^{\frac{1}{2}s - \frac{1}{2}} \int_0^\infty f(y) y^{-1} \, dy \int_0^\infty u^{s-1} \sin u \, du$$

$$= 2^{\frac{1}{2}s+\frac{1}{2}} \pi^{\frac{1}{2}s-\frac{1}{2}} (2\pi)^{\frac{1}{2}s-\frac{1}{2}} \Gamma(1-s) \zeta(1-s) \frac{\pi}{2 \cos \frac{1}{2}\pi s \Gamma(1-s)}$$

Now, we know that the integral $\int_0^{\infty} f(y) \sin \xi y \, dy$ is uniformly convergent when $0 < \delta \leq \xi \leq \Delta$ and it is enough to prove this function equal to zero.

$$\lim_{\delta \rightarrow 0} \lim_{\Delta \rightarrow \infty} \int_0^{\infty} f(y) \, dy \left(\int_0^{\delta} + \int_{\Delta}^{\infty} \right) \xi^{s-1} \sin \xi y \, d\xi = 0$$

$$\int_0^{\delta} \xi^{s-1} \sin \xi y \, d\xi = \int_0^{\delta} O(\xi^{s-1} \xi y) d\xi = O(\delta^{\sigma+1} y)$$

Furthermore, we have

$$= y^{-s} \int_0^{\delta y} u^{s-1} \sin u \, du = O(y^{-\sigma})$$

Since,

$$f(y) = O(1)$$

as y tends to 0 and equal $O(y^{-1})$ as y tends to ∞ , we get that

$$\int_0^{\infty} f(y) dy \int_0^{\delta} \xi^{s-1} \sin \xi y \, d\xi = \int_0^1 O(\delta^{\sigma+1} y) dy + \int_1^{\frac{1}{\delta}} O(\delta + 1) dy + \int_{\frac{1}{\delta}}^{\infty} O(y^{-\delta-1}) dy$$

$$= O(\xi^{\delta}) \rightarrow 0$$

The same way, when Δ goes to 0.

□

7 Generalization of $\zeta(s)$

7.1 The dirichlet L-function

There are many ways in which the Riemann zeta function has been generalized and this dissertation will introduce only one of these generalizations. This section will give the definition of the Dirichlet L-function, an analytic continuation of $L(s, \chi)$ and the functional equation of $L(s, \chi)$.

Recall the basic definition:

Definition 7.1. (Dirichlet characters). ([15] definition 6.2)

Let ℓ be a non-negative integer. A Dirichlet character modulo ℓ is an arithmetic function χ with the following properties:

- 1) $\chi(n + \ell) = \chi(n)$ for all $n \in \mathbb{N}$
- 2) $\chi(nm) = \chi(n)\chi(m)$ for all $n, m \in \mathbb{N}$
- 3) $\chi(n) = 0$ if and only if $(n, \ell) \neq 1$.

Definition 7.2. ([9] definition 4.2)

The function $L(s, \chi)$ was introduced by Dirichlet in 1837 , it defined as

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

This function is a holomorphic function for the real part of complex variable s greater than 1.

Moreover, the L-function can be written as an Euler's product for $\zeta(s)$

$$L(s, \chi) = \prod \left(1 - \frac{\chi(p)}{p^s} \right)^{-1} \tag{7.1}$$

For $Re(s) > 1$

The proof of (7.1) is precisely the same proof of theorem (5.1)

Assume $\chi(n) = \chi_1(n)$ where $\chi_1(n)$ is the principal character modulo ℓ and if

$$\chi_1(n) = \begin{cases} 1 & \text{if } (\ell, n) = 1 \\ 0 & \text{if } (\ell, n) > 1 \end{cases} \tag{7.2}$$

$$\tag{7.3}$$

For the real part $Re(s) > 1$, the $L(s, \chi)$ connected with a character χ modulo ℓ .

$$L(s, \chi_1) = \prod_p \left(1 - \frac{\chi_1(p)}{p^s} \right)^{-1}$$

Applying (7.2) in (7.1) gives

$$\begin{aligned} &= \prod_{(p, \ell)=1} \left(1 - \frac{1}{p^s} \right)^{-1} \\ &= \prod_{p|\ell} \left(1 - \frac{1}{p^s} \right) \prod_p \left(1 - \frac{1}{p^s} \right)^{-1} \end{aligned}$$

By this relation

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s} \right)^{-1}$$

We obtain that

$$L(s, \chi_1) = \zeta(s) \prod_{p|\ell} \left(1 - \frac{1}{p^s} \right)$$

The only difference between $L(s, \chi_1)$ and $\zeta(s)$ by a simple factor.

7.2 Analytic continuation of $L(s, \chi_1)$ for $Re(s) > 0$

By following next lemma , the analytic continuation is obtained.

Lemma 7.1. ([9] Lemma 4.1.1)

Assume

$$S(x) = \sum_{n \leq x} \chi(n)$$

such that $\chi(n)$ is non principal character modulo ℓ . Then, for $\text{Re}(s)$ greater than zero we have that

$$L(s, \chi) = s \int_1^{\infty} S(x)x^{-s-1} dx$$

Proof. Suppose $\text{Re}(s) > 1$ and $N \geq 1$

We use the partial summation and we get

$$\sum_{n \leq N} \chi(n)n^{-s} = - \int_1^N c(x)(x^{-s})' + c(N) N^{-s}$$

Taking derivative of (x^{-s})

$$\begin{aligned} &= - \int_1^N c(x)((-s)x^{-s-1}) + c(N) N^{-s} \\ &= s \int_1^N c(x)x^{-s-1} + c(N) N^{-s} \\ &= 1 + s \int_1^N c(x)x^{-s-1} + c(N) N^{-s} \end{aligned}$$

Such that $c(x) = S(x) - 1$ and $x \geq |c(x)|$, let N goes to infinity we get that

$$\begin{aligned} L(s, \chi) &= 1 + s \int_1^{\infty} c(x)x^{-s-1} dx \\ &= 1 + s \int_1^{\infty} (S(x) - 1)x^{-s-1} dx \\ &= s \int_1^{\infty} S(x)x^{-s-1} dx \end{aligned}$$

The lemma is proved. □

Corollary 7.2. ([9] Corollary 4.1.1)

$$L(s, \chi) = s \int_1^{\infty} S(x)x^{-s-1} dx$$

provides an analytic continuation of L -function $L(s, \chi)$ for $\text{Re}(s) > 0$.

Theorem 7.3. (Analytic properties of L-functions)([15] Theorem 6.9)

Suppose χ be any Dirichlet character modulo ℓ and $L(s, \chi)$ be connected L-function then:

(i) If a Dirichlet characters $\chi \neq \chi_1$, such that χ_1 is the principal character modulo ℓ therefore, $L(s, \chi)$ is analytic in $\text{Re}(s) > 0$.

(ii) If a Dirichlet characters $\chi = \chi_1$, therefore, $L(s, \chi)$ has a simple pole at $s = 1$ and is analytic at all remain points in $\text{Re}(s) > 0$.

7.3 Functional equation for $L(s, \chi)$

Theorem 7.4. ([9] Theorem 4.3.1)

Assume

$$\delta = \begin{cases} 0 & \text{if } \chi(-1) = 1 \\ 1 & \text{if } \chi(-1) = -1 \end{cases} \quad (7.4)$$

such that χ is primitive character modulo k

The function (ξ, χ) defined as

$$\xi(s, \chi) = (\pi k^{-1})^{-(s+\delta/2)} \Gamma\left(\frac{s+\delta}{2}\right) L(s, \chi)$$

Then, we have the identity

$$\xi(1-s, \bar{\chi}) = \frac{i^\delta \sqrt{k}}{g(\chi)} \xi(s, \chi)$$

Before prove theorem (7.4), we present an important Lemma for this theorem

Lemma 7.5. ([9] Lemma 4.2)

Suppose that χ is primitive character modulo k , if χ is even character define $\theta(\tau, \chi)$ by setting

$$\theta(\tau, \chi) = \sum_{n=-\infty}^{\infty} \chi(n) \exp(-\pi\tau n^2/k) \quad , \tau > 0 \quad (7.6)$$

if χ is an odd character defined

$$\theta_1(\tau, \chi) = \sum_{n=-\infty}^{\infty} n\chi(n) \exp(-\pi\tau n^2/k) \quad , \tau > 0 \quad (7.7)$$

These functions $\theta(\tau, \chi)$ and $\theta_1(\tau, \chi)$ satisfy the following functional equations

$$g(\bar{\chi}) \theta(\tau, \chi) = \sqrt{k\tau^{-1}} \theta(\tau^{-1}, \bar{\chi}) \quad (7.8)$$

$$g(\bar{\chi}) \theta_1(\tau, \chi) = \sqrt{k\tau^{-3}} \theta(\tau^{-1}, \bar{\chi}) \quad (7.9)$$

where $g(\chi)$ is Gauss sum

$$g(\chi) = \sum_{a=1}^k \exp(2\pi ia/k)$$

Proof. Now, we will proof theorem (7.4)

Firstly, let χ is an even character as in (7.4), then

$$\pi^{-s/2} k^{s/2} \Gamma\left(\frac{s}{2}\right) L(s, \chi) = \int_0^{\infty} \tau^{s/2-1} \left(\sum_{n=1}^{\infty} \chi(n) \exp(-\pi\tau n^2/k) d\tau \right) \quad (7.10)$$

We use (7.6) and definition (4.2), then we have that

$$\sum_{n=1}^{\infty} \chi(n) \exp(-\pi\tau n^2/k) = \frac{1}{2} \theta(\tau, \chi) \quad (7.11)$$

Applying (7.11) in (7.10), we obtain

$$\pi^{-s/2} k^{s/2} \Gamma\left(\frac{s}{2}\right) L(s, \chi) = \frac{1}{2} \int_0^{\infty} \tau^{s/2-1} \theta(\tau, \chi) d\tau$$

Dividing the integral into two parts, changing the variable from τ to τ^{-1} , consequently, we have

$$\begin{aligned} \pi^{-s/2} k^{s/2} \Gamma\left(\frac{s}{2}\right) L(s, \chi) &= \frac{1}{2} \int_1^{\infty} \tau^{s/2-1} \theta(\tau, \chi) d\tau + \frac{1}{2} \int_1^{\infty} \tau^{-s/2-1} \theta(\tau^{-1}, \chi) d\tau \\ &= \frac{1}{2} \int_1^{\infty} \tau^{s/2-1} \theta(\tau, \chi) d\tau + \frac{1}{2} \frac{\sqrt{k}}{g(\bar{\chi})} \int_1^{\infty} \tau^{-s/2-1} \theta(\tau, \bar{\chi}) d\tau \end{aligned} \quad (7.12)$$

Therefore, (7.12) gives an analytic continuation of L-function $L(s, \chi)$ onto the whole s-plane. $L(s, \chi)$ is regular because $\Gamma\left(\frac{s}{2}\right) \neq 0$.

Using the fact

$$g(\chi)g(\bar{\chi}) = g(\chi)\overline{g(\chi)} = k$$

Since, $\chi(-1) = 1$ we show that if we changed the variables s by $1 - s$ and χ by $\bar{\chi}$. The right hand side of (7.12) multiplied by $\frac{\sqrt{k}}{g(\chi)}$. this provides the theorem (7.4), in the first case(7.4).

Now, let χ is odd character $\chi(-1) = -1$, by (7.7) in lemma (7.5) we have that

$$\pi^{-s+1/2} k^{s+1/2} \Gamma\left(\frac{s+1}{2}\right) n^{-s} = \int_0^{\infty} n \exp(-\pi\tau n^2/k) \tau^{\frac{s}{2}-\frac{1}{2}} d\tau$$

Multiplying both of sides by $\sum_{n=1}^{\infty} \chi(n)$.

As a result, for $\text{Re}(s)$ greater than 1 we have

$$\pi^{-s+1/2} k^{s+1/2} \Gamma\left(\frac{s+1}{2}\right) \sum_{n=1}^{\infty} \chi(n)/n^{-s} = \int_0^{\infty} \tau^{\frac{s}{2}-\frac{1}{2}} \sum_{n=1}^{\infty} \chi(n) n \exp(-\pi\tau n^2/k) d\tau$$

By the definition of $L(\chi, s)$ and the equation (7.7) we find

$$\pi^{-s+1/2} k^{s+1/2} \Gamma\left(\frac{s+1}{2}\right) L(s, \chi) = \frac{1}{2} \int_0^{\infty} \theta_1(\tau, \chi) \tau^{\frac{s}{2}-\frac{1}{2}} d\tau$$

Dividing the integral into two parts

$$= \int_1^{\infty} \theta_1(\tau, \chi) \tau^{\frac{s}{2}-\frac{1}{2}} d\tau + \frac{1}{2} \frac{i\sqrt{k}}{g(\bar{\chi})} \int_1^{\infty} \theta_1(\tau, \chi) \tau^{-s/2} \quad (7.13)$$

Therefore, (7.13) provides an analytic continuation of $L(s, \chi)$ onto the whole s -plane, $L(s, \chi)$ is regular; moreover, if s is changed by $1 - s$ and χ by $\bar{\chi}$, the right hand side of (7.13) is multiplied by $\frac{i\sqrt{k}}{g(\chi)}$.

Using the fact

$$g(\chi)g(\bar{\chi}) = -k$$

This provides the theorem (7.4) in the second case (7.5). □

8 The Zeros of $\zeta(s)$

According the theory of the Riemann zeta function, the critical strip is the region $0 \leq \sigma \leq 1$ of the complex plane. The critical line is the line $\sigma = 1/2$. The trivial zeros of the Riemann zeta function are the real zeros of $\zeta(s)$ at $-2, -4, -6, \dots, -2n..$ these are negative even numbers. The non-trivial zeros are the complex of $\zeta(s)$. This section will give the theorems for trivial and non-trivial zeros. However, before this it will provide a few important basic theorems which help to reach the desired results. These results were obtained from Gleinig and Bars [16] and Stein and Shakarchi [17]. Chapter 7 in [17] also has more details.

Theorem 8.1. ([17] Theorem 1.3)

The function $\Gamma(s)$ initially defined for $Re(s) > 0$ has an analytic continuation to a meromorphic function on \mathbb{C} whose only singularities are simple poles at the negative integers.

Lemma 8.2. ([18] Lemma 2.1)

The function $1/\Gamma(s)$ is an entire function of s in \mathbb{C} with simple zeros, for s that are negative even integers; this function has zeros nowhere else.

Theorem 8.3. (Trivial zeros) ([17] Theorem 1.1)

Only the zeros of the Riemann zeta function outside of the critical strip are at $s = -2, -4, -6, -8, \dots, -2n, \dots$

Proof. The third method of deriving the functional equation gives

$$\pi^{-\frac{1}{2}s} \Gamma\left(\frac{1}{2}s\right) \zeta(s) = \Pi^{-\frac{1}{2}s-1} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

and the fact that $\zeta(s)$ has no zero when $Re(s) > 1$, along with the fact $\Gamma(s)$ has no zero. It follow that, $\zeta(1-s)$ can only be zero when $\Gamma((1-s)/2)$ has a simple pole at $s = -2, -4, -6, \dots, -2n..$.We find that when $(1-s)/2$ is negative even integer. Therefore, when $1-s$ is negative even integer. It follows that for $Re(s) < 0$ the $\zeta(s)$ can be only zero if $s = -2, -4, -6, \dots, -2n..$. □

Theorem 8.4. (Non-Trivial zeros) ([16] Theorem p.g. 19)

The Riemann zeta function only has zeros at $s = -2, -4, -6, \dots -2n, \dots$ and the complex numbers λ_n that lie on $(0 < Re(s) < 1)$ the critical strip. Moreover, on the critical strip the zeros are situated symmetrically respecting to the real axis and to the point $1/2$.

Proof. Define the Möbius function $\mu(s)$ as

$$\mu(s) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^\ell & \text{if } n = p_1 \dots p_\ell \text{ and } p_1 \dots p_\ell \text{ are distinct primes} \\ 0 & \text{Otherwise} \end{cases}$$

The relation between this function and Euler Product is

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = \sum_{n=1}^{\infty} \mu(n)n^{-s}$$

Hence

$$\begin{aligned} \left| \frac{1}{\zeta(s)} \right| &= \left| \sum_{n=1}^{\infty} \mu(n)n^{-s} \right| < \left| \sum_{n=1}^{\infty} n^{-\sigma} \right| < \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{x^\sigma} dx + 1 \\ &= \int_1^{\infty} \frac{1}{x^\sigma} dx + 1 = \frac{x^{-\sigma+1}}{-\sigma+1} \Big|_1^{\infty} = [-(1)^{-\sigma+1}/(-\sigma+1)] \\ &= \frac{1}{\sigma-1} + 1 = \sigma/\sigma - 1 \end{aligned}$$

From the inequality , we prove the zeta function has no zero when $Re(s)$ greater than 1. In addition, for $Re(s)$ less than zero and using the Riemann's function equation, gives $\zeta(s)$ with no zero apart from the zeros at $s = -2, -4, -6, \dots - 2n\dots$

□

The next theorems,lemmas and proof in this section are taken from Stein and Shakarchi [17] and Riffer-Renert [18].

Theorem 8.5. *The $\zeta(s)$ has no zero when $Re(s)$ equals 1.*

The proof of this theorem need some work with additional theorems and requires the statement of several important lemmas. An essential theorem about zeros and poles is also given.

Lemma 8.6. *If the real part $Re(s)$ greater than 1, then*

$$\log \zeta(s) = \sum_{p,\ell} \frac{p^{-\ell s}}{\ell} = \sum_{n=1}^{\infty} \frac{C_n}{n^s}$$

such that $C_n \geq 0$.

Lemma 8.7. *If φ in \mathbb{R} , we have that*

$$0 \leq 3 + 4 \cos \varphi + \cos 2\varphi$$

Lemma 8.8. *If the real part $Re(s)$ greater than 1 and let $t \in \mathbb{R}$ then*

$$0 \leq \log |\zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it)|$$

Theorem 8.9. ([17] Theorem 1.1)

Suppose f be holomorphic in a connected open set Ω such that it has a zero at $s_0 \in \Omega$ and does not vanish identically in Ω . Then there exists a neighborhood $U \subset \Omega$ of s_0 a non-vanishing holomorphic function g on U , and a unique positive integer n such that

$$f(s) = (z - z_0)^n g(s)$$

for all $s \in U$.

Theorem 8.10. ([17] Theorem 1.1)

If f has a pole at $s_0 \in \Omega$ then in a neighborhood of that point there exist a non-vanishing holomorphic function h and a unique positive integer n such that

$$f(s) = (z - z_0)^{-n} h(s)$$

Proof. Now, we will prove theorem (8.5) by contradiction.

let we have a point $s_0 = (1 + it)$ such that $\zeta(1 + it) = 0$ for $t \neq 0$, we know that the zeta function is holomorphic at the point $(1 + it)$, it must disappear at least to order 1 at $(1 + it)$, by theorem (8.9) there exist a constant C , it is greater than 0

Therefore,

$$\frac{|\zeta(\sigma + it)|^4}{C(\sigma - 1)^4} \leq 1 \quad \text{as } \sigma \text{ goes to } 1$$

Likewise, by theorem (8.10) the zeta function has a simple pole at $s = 1$ and there exist a constant C' , it is greater than 0 So

$$\frac{|\zeta(\sigma)|^3}{C'(\sigma - 1)^3} \leq 1 \quad \text{as } \sigma \text{ goes to } 1$$

At the end, since the zeta function is holomorphic at $\sigma + 2it$, $|\zeta(\sigma + 2it)|$ is remains bounded as σ tends to 1.

Hence,

$$\lim_{\sigma \rightarrow 1} |\zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it)| = 0$$

However, this is contradiction with Lemma(8.8). Since the logarithm of \mathbb{R} between zero and one is non-positive. Thus, the Riemann zeta function has no zero on the real line $Re(s) = 1$. □

8.1 Weierstrass product for $\zeta(s)$ and $\mathbf{L}(s, \chi)$

Lemma 8.11. $\xi(s)$ and $\xi(s, \chi)$ are entire functions of order 1.

To understand and prove this lemma we need to review some theorems and corollary. We take these results from Karatsuba and Voronin [9].

Theorem 8.12. If $\xi(s)$ is an entire function and

$$\xi(s) = \xi(1 - s) \tag{8.1}$$

Theorem 8.13. Let $G(s)$ is an entire function such that $G(s)$ does not equal zero. The set of zeros of $G(s)$ is denoted by P , and the multiplicity of zero of G at λ is denoted by $v_\lambda(G)$. Assume that

$$\log \max_{|s|=R} |G(s)| \ll_\epsilon (R + 1)^{1+\epsilon}$$

for $\forall \epsilon > 0$ and $R > 0$. Then

(i)

$$f(s) = \prod_{\lambda \in P, \lambda \geq 0} \left(\left(1 - \frac{s}{\lambda}\right) e^{s/\lambda} \right)^{v_\lambda(G)}$$

On every compact subset of the s -plane, this product is uniformly convergent, therefore it defines an entire function $f(s)$.

(ii)

$$G(s) = s^{v_0(G)} \exp(As + b) f(s)$$

such that A and B are belong to \mathbb{C}

(iii)

$$\sum_{\lambda \in P, \lambda \neq 0} v_\lambda(G)/|\lambda|^2$$

This series is convergent. (iv)

$$\sum_{\lambda \in P, \lambda \neq 0} v_\lambda(G)/|\lambda|$$

If this series is convergent. Then $\exists c > 0$, c is absolutely constant. one has

$$\log \max_{|s|=R} |G(s)| \leq c(R + 1)$$

Corollary 8.14.

$$\zeta(s) = \sum_{n=1}^N \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{1}{2} N^{-s} + s \int_N^\infty \frac{\lambda(u) du}{u^{s+1}}$$

Such that N belong to \mathbb{N} and $\lambda = \frac{1}{2} - [u]$.

Theorem 8.15. (Stirling formula)

$$(1) \log \Gamma(s) = \left(\frac{s-1}{2} \right) \log s - s + \log \sqrt{2\pi} + O(|s|^{-1})$$

such that δ greater than 0 and args between $\delta - \pi$ and $\pi - \delta$. The constant O -big depends on δ .

$$(2) \log \Gamma(s) = \left(\frac{s-1}{2} \right) \log s - s + \log \sqrt{2\pi} + \int_0^\infty \frac{\lambda(u)}{u+s} du$$

such that $\lambda = \frac{1}{2} - [u]$.

Proof. Now, we want to prove that $\xi(s)$ is an entire function of order 1. According to theorem (8.12) we have that $\xi(s) = \xi(1-s)$, it will suffices to bound $|\xi(s)|$ for $Re(s) \geq \frac{1}{2}$.

By the corollary (8.14), when $Re(s) \geq \frac{1}{2}$, $N = 1$ and since $|\zeta(s)| = O(|s|)$, $|s(s-1)| = O(|s^2|)$ we have that

$$\zeta(s)s(s-1) = O(|s^3|)$$

Moreover, By theorem (8.15) and since $\log|\Gamma(s)| \leq |\log\Gamma(s)|$ obviously

$$|\Gamma(s)| \leq \exp(C|s| \log|s|) \quad , |\zeta(s)| < (C|s|) \quad \text{whenever } Re(s) \geq \frac{1}{2}$$

and

$$|\pi^{-\frac{s}{2}}| \leq \exp(C|s|)$$

Then,

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{8.2}$$

The order of this function at most 1, we obvious that

$$\log|\Gamma(s)| = \log\Gamma(|s|) = |s|\log|s| + O(|s|)$$

and

$$\log\xi(|s|) = \frac{1}{2}|s|\log|s| + O(|s|)$$

However, when s goes to $+\infty$ i.e. $s > 1$ one has

$$\log \Gamma(s) \sim s \log s \quad \text{i.e.} \quad \log \xi(s) \sim s \log s$$

Therefore, the $\xi(s)$ is a function of order 1. The proof of $\xi(s, \chi)$ is similar. If we use (7.1), we obtain that $\xi(s, \chi)$ is an entire function of order 1. \square

* If we use the lemma(8.11) and $s \rightarrow +\infty$

$$|\xi(s)| \geq e^{cs} \quad , \quad \xi(s, \chi) \geq e^{cs}$$

such that c is constant, $c > 0$.

Applying theorem (8.13). Each of the functions $\xi(s)$ and $\xi(s, \chi)$ could be represented as

$$\xi(s) = e^{A+Bs} \prod_{n=1}^{\infty} \left(1 - \frac{s}{\lambda_n}\right) e^{s/\lambda_n} \tag{8.3}$$

where A and B are constants. The series $\sum_{n=1}^{\infty} |\lambda_n|^{-1-\epsilon}$ converges for $\forall \epsilon > 0$ and the series $\sum_{n=1}^{\infty} |\lambda_n|^{-1}$ diverges. The product (8.3) has infinitely many factors, therefore $\xi(s)$ and $\xi(s, \chi)$ have infinitely many zeros. One can see by a simple argument, when the λ_n lie in $0 \leq \sigma \leq 1$ where $s = \sigma + it$. i.e. Assume we take the case of the function $\xi(s)$. By these formulas

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(s/2)\zeta(s)$$

and

$$\xi(s) = \xi(s-1)$$

it is enough to see that $\xi(s) \neq 0$ when $\sigma > 1$. This will follow if we show that $\zeta(s) \neq 0$ when $\sigma > 1$.

By theorem (4.1) and when $\sigma > 1$ such that $s = \sigma + it$, we have that

$$\begin{aligned} \zeta(s) &= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right)^{-1} \\ \frac{1}{\zeta(s)} &= \prod_p \left(1 - \frac{1}{p^s}\right) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \\ \frac{1}{|\zeta(s)|} &= \left|\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right| < \sum_{n=1}^{\infty} \frac{1}{n^\sigma} < 1 + \int_1^{\infty} \frac{du}{u^\sigma} \end{aligned}$$

Taking the integral

$$= 1 + \frac{u^{-\sigma+1}}{-\sigma+1} \Big|_1^\infty = 1 + \frac{1}{\sigma-1} = \frac{\sigma}{\sigma-1}$$

Since

$$\frac{1}{|\zeta(s)|} = \frac{\sigma}{\sigma-1}$$

Hence

$$|\zeta(s)| > \frac{\sigma-1}{\sigma}$$

The proof of $\xi(s)$ is the same as the proof of $\xi(s, \chi)$. We use (8.1) it easy to show that $\xi(0)$ does not equal 0 and $\xi(1)$ also does not equal 0. Since, $\Gamma(s/2) \neq 0$. Therefore, all of zeros of the product (8.3) are zeros of the Riemann zeta function $\zeta(s)$, for $\xi(s, \chi)$. Note that we take the proof from Karatsuba and Voronin [9] and Kumchev [19].

8.2 Theorems relating to the zeros of $\zeta(s)$

This section gives the consequences of the functional equation. It will look at the logarithmic derivative $\zeta'(s)/\zeta(s)$ to discover the poles and zeros of the Riemann zeta function. Moreover, it focuses on the zeros for $\sigma > 1$; where $s = \sigma + it$, there are no zero. By using (6.20) in the third method, the zeros of the Riemann zeta function in $\sigma > 0$ are found to be simple zeros at $-2, -4, -6, \dots, 2n, \dots$. In this section most results are taken from Karatsuba and Voronin [9] and Kumchev [19].

8.2.1 Results of the functional equation for the Riemann zeta function $\zeta(s)$

Since,

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{8.4}$$

and

$$\xi(s) = \xi(1-s) \tag{8.5}$$

such that $s = \sigma + it$ also, by equation (8.3) can be written as

$$\xi(s) = e^{A+Bs} \prod_{n=1}^{\infty} \left(1 - \frac{s}{\lambda_n}\right)^{s/\lambda_n} \tag{8.6}$$

Combining (8.4) and (8.6), we obtain that

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \xi(s) = e^{A+Bs} \prod_{n=1}^{\infty} \left(1 - \frac{s}{\lambda_n}\right)^{s/\lambda_n} \tag{8.7}$$

By analytic continuation of $\zeta(s)$, the $\zeta(s)$ can represent as

$$\zeta(s) = \sum_{n=1}^N \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{1}{2}N^{-s} + s \int_N^{\infty} \frac{\lambda(u)du}{u^{s+1}}$$

For $\sigma > 0$ and N equal 1 we find that

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + s \int_1^{\infty} \frac{\lambda(u)du}{u^{s+1}} \tag{8.8}$$

Multiplying both sides by $(s-1)$ for (8.4) and (8.7), then take limit when s tends to 1, we get

$$\lim_{s \rightarrow 1} (s-1)\zeta(s) = 1$$

and

$$\lim_{s \rightarrow 1} \xi(s) = \frac{1}{2}(\pi)^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}$$

From (8.4) we have that

$$\xi(0) = \xi(1) = \frac{1}{2}$$

Since, the inverse of $\Gamma(s)$ has expansion

$$(\Gamma(s))^{-1} = s e^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n}$$

Consequently, $\lim_{s \rightarrow 0} s\Gamma(s) = 1 e^A = \xi(0) = -\zeta(0) = \frac{1}{2}$ then we can say that $\zeta(0) = -\frac{1}{2}$
Hence,we obtain

$$\zeta(0) = -\frac{1}{2}$$

and

$$\xi(1) = \frac{1}{2}$$

Now, for $\sigma > 1$, we get

$$\begin{aligned} (1 - 2^{1-s})\zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} - 2 \sum_{n=1}^{\infty} \frac{1}{(2n)^s} \\ (1 - 2^{1-s})\zeta(s) &= \zeta(s) - 2^{1-s}\zeta(s) \\ &= \sum_{n=1}^{\infty} \frac{1}{n^s} - 2^{1-s} \sum_{n=1}^{\infty} \frac{1}{n^s} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^s} - 2 \sum_{n=1}^{\infty} \frac{1}{2n^s} \\ &= 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \dots \end{aligned}$$

For $s > 0$, the last series is convergent for $s > 0$, we have

$$(1 - 2^{1-s})\zeta(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \dots$$

This demonstrates that $\zeta(s) \neq 0$ for all real positive s . Therefore, the λ_n in (8.6) is complex numbers. It means that the zeros of $\zeta(s)$ is also the zeros of $\xi(s)$,are does not belong to \mathbb{R} .

Besides of the λ_n , the Riemann zeta function $\zeta(s)$ has other zeros by

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

Since $s = \sigma + it$, if $Re(s) = \sigma$ less than zero then, $Re(1-s)$ greater than one, and

$\pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$ does not vanish, However, $\Gamma(s/2)$ has poles at $s/2 = -1, -2, -3, \dots$. The Riemann zeta function $\zeta(s)$ vanishes when $s = -2, -4, -6, \dots$.

Next that by (8.1) and

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \frac{1}{s(s-1)} + \int_1^{\infty} (s^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}) w(x) dx$$

It follow that

$$\overline{\xi(s)} = \xi(\bar{s})$$

Theorem 8.16. *The Riemann zeta function satisfies that*

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \left(\frac{1}{s+2n} - \frac{1}{2n} \right) + \sum_{n=1}^{\infty} \left(\frac{1}{s-\lambda_n} - \frac{1}{\lambda_n} \right) - \frac{1}{s-1} + c \tag{8.9}$$

such that c is constant and λ_n runs through all of the complex zeros of the zeta function.

Corollary 8.17. *The number of λ_n where λ_n is the zeros of $\zeta(s)$ such that $T \leq |Im\lambda_n| \leq T+1$ less than $c \log T$. By $O(\log T)$, the λ_n in the region $0 \leq Re(s) \leq 1$ and $T \leq Im(s) \leq T+1$ is bounded.*

Corollary 8.18. *For T greater than or equal 2 then we have that*

$$O(\log T) = \sum_{|T-\gamma_n|>1} \frac{1}{|T-\gamma_n|}$$

Corollary 8.19. *Assume T greater than or equal 2 and let $\lambda_n = \beta_n + i\gamma_n$ such that $n = 1, 2, 3, 4, \dots$ it's complex zeros of the zeta function hence,*

$$\sum_{n=1}^{\infty} \frac{1}{1+(T-\gamma_n)^2} \leq c \log T$$

Corollary 8.20. *Let $-1 \leq \sigma \leq 2$ where $s = \sigma + it$ then,*

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{1}{s-\lambda_n} - \frac{1}{s-1} + O(\log|t|+2)$$

Proof. $m = |t| + 2$ and for $-1 \leq \sigma \leq 2$ where $s = \sigma + it$ we have that

$$\left| \sum_{n=1}^{\infty} \left(\frac{1}{2n+s} - \frac{1}{2n} \right) \right| \leq \sum_{n \leq m} \left(\frac{1}{n} + \frac{1}{n} \right) + \sum_{n \leq m} \frac{|s|}{n^2} = O(\log m)$$

By theorem (8.15) we have that

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \left(\frac{1}{s-\lambda_n} + \frac{1}{\lambda_n} - \frac{1}{s-1} \right) + O \log(m) \tag{8.10}$$

Subtracting (8.10) when $s = 2 + it$

$$\sum_{n=1}^{\infty} \left(\frac{1}{s-\lambda_n} - \frac{1}{2+it-\lambda_n} \right) + O \log(m)$$

Then If $|\gamma_n - t| > 1$ hence, we obtain that

$$\left| \frac{1}{\sigma+it-\lambda_n} - \frac{1}{2+it-\lambda_n} \right| \leq \frac{2}{(\gamma_n-t)^2} \leq \frac{3}{(\gamma_n-t)^2}$$

□

8.2.2 Results of the functional equation for $L(s, \chi)$

The basic consequences focusing on the zeros of L-function are similar to the zeros of the Riemann zeta function.

If χ is an even character and theorem (7.3) we have that

$$\sqrt{k}(\pi k^{-1})^{-s/2}\Gamma\left(\frac{s}{2}\right)L(s, \chi) = (\pi k^{-1})^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)L(1-s, \bar{\chi})$$

By (7.1) we have

$$L(s, \chi) = \prod \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \text{ for } \sigma > 1$$

$$|L(s, \chi)| = \left| \prod \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \right| = \left| \left(\sum_{n=1}^{\infty} \frac{\mu(n)\chi(n)}{n^s} \right)^{-1} \right|$$

Then

$$\frac{1}{|L(s, \chi)|} = \left| \prod \left(1 - \frac{\chi(p)}{p^s}\right) \right| = \left| \sum_{n=1}^{\infty} \frac{\mu(n)\chi(n)}{n^s} \right| < \sum_{n=1}^{\infty} \frac{1}{n^\sigma}$$

$$< 1 + \int_1^{\infty} \frac{1}{u^\sigma} du = 1 + (u^{-\sigma+1} / -\sigma + 1) \Big|_1^{\infty} = \sigma / (\sigma - 1)$$

Thus

$$|L(s, \chi)| = (\sigma - 1) / \sigma$$

When $Re(s)$ greater than 0, therefore that $Re(1-s)$ greater than or equal 1. We find that the only zeros of $L(s, \xi)$ are poles of $\Gamma(s/2)$ at negative even integers.

If χ is an odd character then

$$i\sqrt{k}(\pi k^{-1})^{-(s+1)/2}\Gamma\left(\frac{s+1}{2}\right)L(s, \chi) = g(\chi)(\pi k^{-1})^{-(2-s)/2}\Gamma\left(\frac{2-s}{2}\right)L(1-s, \bar{\chi})$$

As a result, when $\sigma < 0$ such that $s = \sigma + it$, the zeros of $L(s, \chi)$ are poles of $\Gamma\left(\frac{s+1}{2}\right)$ at negative odd integers. These zeros called trivial zeros and by subsection (8.1) we know that $L(s, \chi)$ has infinitely many non-trivial zeros.

8.2.3 The theorem of de la vallée-poussin of $\zeta(s)$

Theorem 8.21. ([9] Theorem 6.2.4)

The Riemann zeta function does not have any zeros in the region of the s -plane and c is constant which is greater than zero, then we have

$$\sigma \geq 1 - \frac{c}{\log(|t| + 2)}$$

Proof. Assume $\lambda = \beta + i\gamma$ is zero of the zeta function, hence $\gamma_1 < |\gamma|$ such that $\gamma_1 > 0$ Since,

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)}$$

such that $\text{Re}(s)$ greater than one Then, we have

$$-Re \left(\frac{\zeta'(s)}{\zeta(s)} \right) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma} \cos(t \log n)$$

Since, $\varphi \in \mathbb{R}$ then

$$3 + 4 \cos \varphi + \cos 2\varphi = 2(1 + \cos \varphi)^2 \geq 0$$

Thus

$$3 \left(\frac{\zeta'(\sigma)}{\zeta(\sigma)} \right) - 4 \left(\frac{\zeta'(\sigma + it)}{\zeta(\sigma + it)} \right) + \left(-Re \frac{\zeta'(\sigma + 2it)}{\zeta(\sigma + 2it)} \right) \geq 0 \tag{8.11}$$

For every term on the left hand side in(8.8) we obtain an upper bound and by corollary (8.17) and theorem (8.16) we get that for $s = \sigma$ and $1 < \sigma \leq 2$

$$A_1 + \frac{1}{\sigma - 1} > -\frac{\zeta'(\sigma)}{\zeta(\sigma)}$$

such that A_1 is constant and it's greater than zero. Using Theorem (8.16)

$$-Re \left(\frac{\zeta'(\sigma + it)}{\zeta(\sigma + it)} \right) < -\sum_{n=1}^{\infty} Re \left(\frac{1}{s - \lambda_k} + \frac{1}{\lambda_k} \right) + A_2 \log(|t| + 2)$$

such that A_2 is constant and it's greater than zero. Since, $\lambda_k = \beta_k + it, \beta_k > 0$

$$Re \frac{1}{s - \lambda_k} = Re \frac{1}{\sigma - \beta_k + i(t - \gamma_k)} = \frac{\sigma - \beta_k}{(\sigma - \beta_k)^2 + (t - \gamma_k)^2} > 0$$

$$Re \frac{1}{\lambda_k} = \frac{\beta_n}{\beta_k^2 + \gamma_k^2} \geq 0$$

Then

$$-Re \left(\frac{\zeta'(\sigma + it)}{\zeta(\sigma + it)} \right) < -\frac{\sigma - \beta}{(\sigma - \beta)^2 + (t - \gamma)^2} + A_2 \log(|t| + 2)$$

We substitute $-\sigma - \beta/(\sigma - \beta)^2 + (t - \gamma)^2$ by a weaker one. In addition, assume $t = 2t$ then

$$-Re \left(\frac{\zeta'(\sigma + it)}{\zeta(\sigma + it)} \right) < +A_2 \log(2|t| + 2)$$

Now, the terms on the left hand side in (8.11)are bounded. Hence, we obtain

$$\frac{3}{\sigma - 1} - \frac{4(\sigma - \beta)}{(\sigma - \beta)^2 + (t - \gamma)^2} + A \log(|t| + 2) \geq 0$$

such that A is greater than one and it is constant, for $|t| > \gamma_1, t$ and σ when $1 < \sigma \leq 2$

We choose

$$\sigma = 1 + \frac{1}{2A \log(\gamma) + 2} \quad \text{such that } t = \sigma$$

We have that

$$\frac{3}{\sigma - 1} + A \log(|\gamma| + 2) \geq \frac{4}{\sigma - \beta}$$

Finally, we get

$$\beta \leq 1 - \frac{1}{14A \log(|\gamma| + 2)}$$

□

Note that the proof above is taken from ([9]) and ([19]).

8.2.4 The theorem of de la vallée-poussin of $L(s, \chi)$

Theorem 8.22. ([9], Theorem 7.2.2)

If χ is a complex character modulo ℓ . Thus, there are no zeros in the region, .

$$\sigma \geq 1 - \frac{c}{\log \ell(|t| + 2)}$$

where $s = \sigma + it$.

If χ is a real character modulo ℓ . Thus, there are no zeros in the region,

$$\sigma \geq 1 - \frac{c}{\log \ell(|t| + 2)}, |t| > 0$$

where $s = \sigma + it$.

9 Conclusion

This dissertation has demonstrated the most important aspects of the zeta function such as its analytic continuation, proved the functional equation and introduced its applications. Additionally, some of interesting properties of the zeta function have been proved:

(i) The function $\zeta(s) - [1/(s - 1)]$ extends to the right half plane for $\sigma > 0$ and this function is analytic when $\sigma > 0$.

(ii) let $\xi(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$ such that $\xi(s)$ is an analytic continuation on the entire plane.

We also prove the functional equation by four methods. The first method proving this theorem depended on the summation formula. The function $\zeta(s)$ is regular for all value of s except $s=1$, where there is a simple pole with residue 1, it satisfies the functional equation:

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{1}{2} s \pi \Gamma(1 - s) \zeta(1 - s)$$

The second method proved the fundamental formula by using the theorem of residue and contour integration. The third method proved this relation by using gamma and theta functions.

$$\pi^{-\frac{1}{2}s} \Gamma\left(\frac{1}{2}s\right) \zeta(s) = \pi^{-\frac{1}{2}s-1} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

The last method depended on a self-reciprocal function. It proved by using Fourier transform functions. The Dirichlet L-function was then introduced. It is noticeable that it has similar results to the $\zeta(s)$ results. Moreover, the zeros of $\zeta(s)$ have been discussed particularly, trivial zeros, non-trivial zeros and the fact that $\zeta(s)$ has no zero when $Re(s) > 1$.

Following this dissertation, many other parts of the Riemann zeta function and its analytic continuation remain to be studied. One suggestion for interested researchers is to look at the famous Riemann Hypothesis, while another would be to look at convexity as it applies to the Riemann zeta function.

Competing Interests

Authors have declared that no competing interests exist.

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