Constructing triple categories of cybernetic processes

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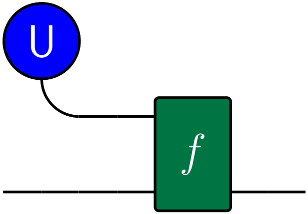
cybernetic systems

are 'parametrized systems': **plants** coupled to a **controller**.



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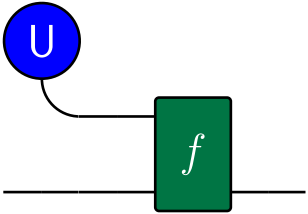
(Capucci, Gavranović, Hedges, and Rischel 2022)



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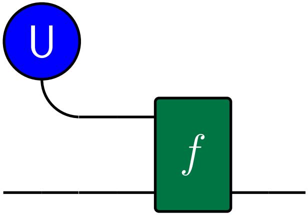

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Example: $\mathcal{U} = \mathcal{C} = \mathbf{Lens}(\mathbf{Set})$ and $\mathbf{Cont}\binom{X}{S} = \{\text{selection functions } S^X \to 2^X\}$

$\mathcal{U} = \mathcal{C} = \mathbf{Smooth}$ and $\mathbf{Cont}(X) = \{\text{linear maps } T^*X \to TX\}$

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$\mathbf{Cont} : \mathcal{U} \to \mathbf{Set}$

$\rightsquigarrow$

$$\mathbf{Para}(\odot_{\mathbf{Cont}}) = \left\{ P : \mathcal{U}, \mathsf{U} : \mathbf{Cont}(P), A \odot P \xrightarrow{f} B \right\}$$

symmetric monoidal bicategory of

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$\rightsquigarrow$ $\mathbf{Para}(\odot_{\mathbf{Cont}}) =$ **open gradient-based learners**

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e.g.

- Solutions concepts in game theory
- Trajectories/equilibria of learning agents
- Flows of controlled ODEs
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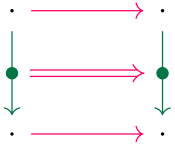
We want tools to treat compositionally behaviour *as well as* specification!

In **Categorical Systems Theory** (Myers 2021; Myers 2022) behaviour is handled compositionally using an extra dimension representing **morphisms between processes and systems**.



Ultimately, this trick allows to define **functorial (often corepresentable) notions of behaviour**!

Can we do the same for cybernetic systems?

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$(\mathcal{U}, \otimes, \mathbf{1})$

$(\mathcal{C}, \odot)$

$\mathbf{Cont} : \mathcal{U} \to \mathbf{Set}$

$\rightsquigarrow$

$(\mathbb{U}, \otimes, \mathbf{1})$ symmetric monoidal **double category** of **control processes**

$(\mathbb{C}, \mathbf{???})$ symmetric monoidal **???** of **plant processes**

$\mathbf{Cont} : \mathcal{U}^{\top} \xrightarrow{\text{uni. lax}} \mathbb{C}\mathbf{at}$ symmetric monoidal **doubly indexed category** of **control systems**

...and of course, **a $\mathbf{Para}$ construction!**

Results

In this talk, I will describe:

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- some behaviours we can represents in this way,
- **(bonus content)** a comparison of $\mathbb{P}\mathbf{ara}(\mathbb{A}\mathbf{rena})$ with $\mathbb{O}\mathbf{rg}$ (Shapiro and Spivak 2022)

Generalising Para

Generalising Para

The 'type signature' of the Para construction is that of a functor

$$\mathbf{Para} : \mathbb{P}\mathbf{sAct} \longrightarrow \mathbb{B}\mathbf{icat}$$

Generalising Para

For better results, we can replace bicategories with double categories:

$$\mathbb{P}\mathbf{ara}_{\mathbb{C}\mathbf{at}} : \mathbb{P}\mathbf{sAct}(\mathbb{C}\mathbf{at}) \longrightarrow \mathbb{P}\mathbf{sCat}(\mathbb{C}\mathbf{at})$$

$$\begin{array}{c} \mathcal{C} \times \mathcal{U} \\ \big\downarrow \odot \\ \mathcal{C} \end{array} \longmapsto \left\{ \begin{array}{ccc} A & \xrightarrow{h} & A' \\ (P,f) \downarrow \odot & \overset{\alpha}{\Longrightarrow} & \odot \downarrow (P',f') \\ B & \xrightarrow[k]{} & B' \end{array} \right\}$$

where $(P, f) : A \odot P \to B$ in $\mathcal{C}$

$\alpha : P \to P'$ in $\mathcal{U}$

and $(\alpha \odot h) \,⨟\, f' = f \,⨟\, k$

Generalising Para

Now it's easy to see how to move beyond $\mathbb{C}\mathbf{at}$: we're looking for a functor

$$\color{teal}{\mathbb{P}\mathbf{ara}_{\mathbb{K}}} : \mathbb{P}\mathbf{sAct}(\mathbb{K}) \longrightarrow \mathbb{P}\mathbf{sCat}(\mathbb{K})$$

where $\mathbb{K}$ is a suitably complete (TBD) 2-category

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How do we actually define this functor in generality?

Constructing Para

For starters, $\mathbb{P}\mathbf{ara}_{\mathbb{C}\mathbf{at}}(\odot)_1$ is a comma category:

$$\mathbb{P}\mathbf{ara}_{\mathbb{C}\mathbf{at}}(\odot)_1 = \left\{ \begin{array}{ccc} A & \xrightarrow{h} & A' \\ (P,f)\ \Big\downarrow\odot & \overset{\alpha}{\Longrightarrow} & \odot\Big\downarrow\ (P',f') \\ B & \xrightarrow[k]{} & B' \end{array} \right\} = \left\{ \begin{array}{ccc} A \odot P & \xrightarrow{\alpha \odot h} & A' \odot P' \\ f\Big\downarrow & & \Big\downarrow f' \\ B & \xrightarrow[k]{} & B' \end{array} \right\} = \odot/\mathcal{C}$$

so we can easily reproduce that in a $\mathbb{K}$ with comma objects.

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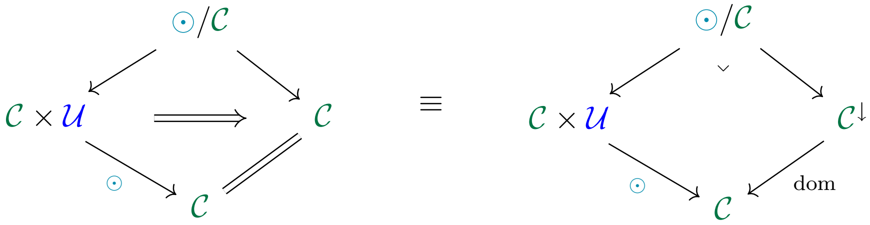
What about the rest of the pseudocategory structure on $\mathbb{P}\mathbf{ara}_{\mathbb{K}}(\odot)$?

Constructing Para

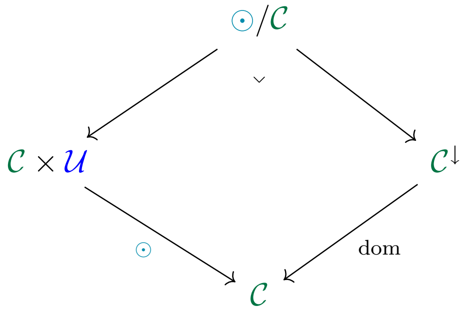
If $\mathbb{K}$ has $\mathbb{C}\mathbf{at}$-powers & pullbacks, we have:



comma squares $\equiv$ $\downarrow$-power $+$ pullback

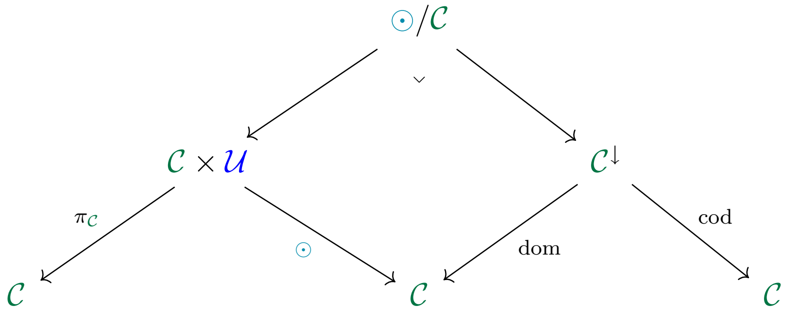
Constructing Para

Moreover this...



Constructing Para

Moreover this... comes from a composition of spans!



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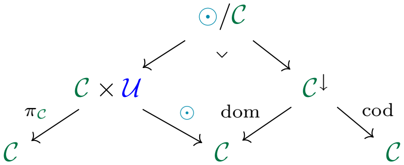
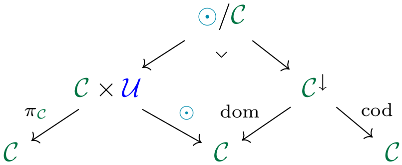


These spans encode some relevant structure:

Constructing Para



These spans encode some relevant structure:

- both spans are **pseudomonads** in $\mathbb{S}\mathbf{pan}(\mathbb{K})$, in particular the pseudomonad structure on $\mathcal{C} \xleftarrow{\pi_{\mathcal{C}}} \mathcal{C} \times \mathcal{U} \xrightarrow{\odot} \mathcal{C}$ coincides with the $\mathcal{U}$-pseudoaction on $\mathcal{C}$,

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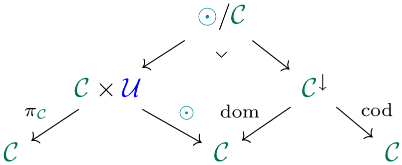

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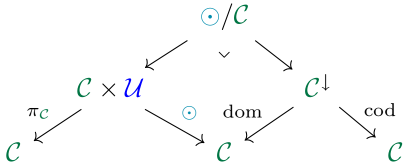
- the resulting composite $\mathcal{C} \leftarrow \odot/\mathcal{C} \rightarrow \mathcal{C}$ is the **underlying graph** of $\mathbb{P}\mathbf{ara}(\odot)$:

$$\begin{array}{ccccc} \mathcal{C} & \longleftarrow & \odot/\mathcal{C} & \longrightarrow & \mathcal{C} \\ A & \longleftarrow\!\shortmid & (P, A \odot P \xrightarrow{f} B) & \longmapsto & B \end{array}$$

Constructing Para



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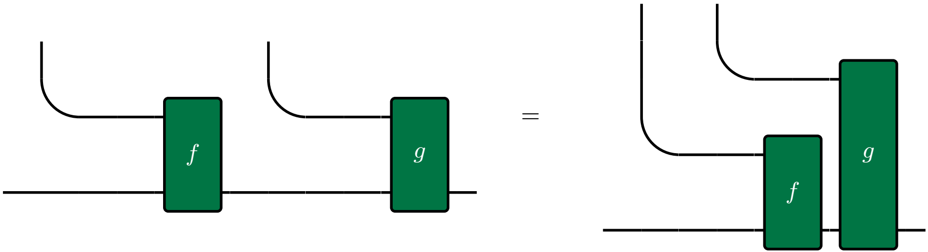
$$A \longmapsfrom (P, A \odot P \xrightarrow{f} B) \longmapsto B$$

Since $\mathbb{P}\mathbf{sCat}(\mathbb{K}) \cong \mathbb{P}\mathbf{sMnd}(\mathbb{S}\mathbf{pan}(\mathbb{K}))$ (at least on objects), **we get the full pseudocategory structure $\mathbb{P}\mathbf{ara}(\odot)$ if we can show $\mathcal{C} \leftarrow \odot/\mathcal{C} \rightarrow \mathcal{C}$ is a pseudomonad** too.

Constructing Para

Such a pseudomonad structure corresponds to a composition law for parametric morphisms, which we know:

$$(P,\ A \odot P \xrightarrow{f} B) \fatsemi (Q,\ B \odot Q \xrightarrow{g} C) = (PQ,\ A \odot (PQ) \xrightarrow{\delta_A} (A \odot P) \odot Q \xrightarrow{f \odot P} B \odot Q \xrightarrow{g} C)$$



Constructing Para

Abstractly, such a pseudomonad structure on $\mathcal{C} \leftarrow \odot/\mathcal{C} \rightarrow \mathcal{C}$ is obtained from a **pseudodistributive law**[1] between $\mathcal{C} \overset{\pi_{\mathcal{C}}}{\leftarrow} \mathcal{C} \times \mathcal{U} \overset{\odot}{\rightarrow} \mathcal{C}$ and $\mathcal{C} \overset{\mathrm{dom}}{\leftarrow} \mathcal{C}^{\downarrow} \overset{\mathrm{cod}}{\rightarrow} \mathcal{C}$.

$$\mathcal{C}/\pi_{\mathcal{C}} \xrightarrow{\quad\text{dist}\quad} \odot/\mathcal{C}$$

$$(P,\ A \xrightarrow{f} B) \longmapsto (P,\ A \odot P \xrightarrow{f \odot P} B \odot P)$$

[1](Gambino and Lobbia 2021)

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$$\begin{array}{ccc} \mathcal{C}/\pi_\mathcal{C} & \xrightarrow{\quad \text{dist} \quad} & \odot/\mathcal{C} \\ (P,\ A \xrightarrow{f} B) & \longmapsto & (P,\ A \odot P \xrightarrow{f \odot P} B \odot P) \end{array}$$

In fact a pseudomonad $\mathcal{C} \xleftarrow{p} \mathcal{E} \xrightarrow{\odot} \mathcal{C}$ distributes over $\mathcal{C} \xleftarrow{\mathrm{dom}} \mathcal{C}^{\downarrow} \xrightarrow{\mathrm{cod}} \mathcal{C}$ **as soon as** p **is a fibration** in $\mathbb{K}$:

$$\begin{array}{ccc} \mathcal{C}/p & \xrightarrow{\qquad\qquad \text{dist} \qquad\qquad} & \odot/\mathcal{C} \\ (P : \mathcal{E}_B, A \xrightarrow{f} B) & & (f^*P : \mathcal{E}_A, A \odot (f^*P) \xrightarrow{f \odot P} B \odot P) \\ & \searrow^{\mathcal{E}_f} \qquad \nearrow^{\odot^{\downarrow}} & \\ & (f^*P : \mathcal{E}_A, A \xrightarrow{f} B) & \end{array}$$

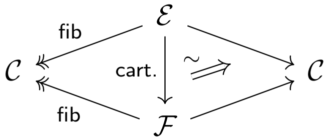
[1](Gambino and Lobbia 2021)

Fibred actions

Hence our generalised $\mathbb{P}\mathbf{ara}$ construction naturally consumes **fibred actions**:

Definition

Let $\mathbb{K}$ be a 2-cosmos.[2] We call $\mathbf{f}\mathbb{S}\mathbf{pan}^{\cong}(\mathbb{K})$ the tricategory of $\mathbb{K}$-spans whose left leg is a cloven fibration. Two-cells are cartesian triangles on the left and pseudocommutative triangles on the right:



Definition

A **fibred action** is a pseudomonad in $\mathbf{f}\mathbb{S}\mathbf{pan}^{\cong}(\mathbb{K})$.

[2] See (Bourke and Lack 2023), for our purposes: admitting $\mathbb{C}\mathbf{at}$-powers and (strict) pullbacks and equipped with a pullback-stable class of isofbrations

Fibred actions

A fibred action is an action whose actor ($\mathcal{E}$) depends on the actee ($\mathcal{C}$):

$$\mathcal{C} \overset{p}{\twoheadleftarrow} \mathcal{E} \overset{\odot}{\longrightarrow} \mathcal{C} \qquad \rightsquigarrow \qquad \odot : (A : \mathcal{C}) \times \mathcal{E}_A \longrightarrow \mathcal{C}$$

Example

$\mathcal{C} \overset{\text{dom}}{\twoheadleftarrow} \mathcal{C}^{\downarrow} \overset{\text{cod}}{\to} \mathcal{C}$ it's the chief example: morphisms act on their domains by sending them to their codomains:

$$A \odot (A \xrightarrow{P} B) = B, \quad A \odot (A \overset{1_A}{=\!=} A) = A,$$

$$(A \odot (A \xrightarrow{P} B)) \odot (A \xrightarrow{Q} C) = A \odot (A \xrightarrow{P} B \,;\, A \xrightarrow{Q} C)$$

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Example

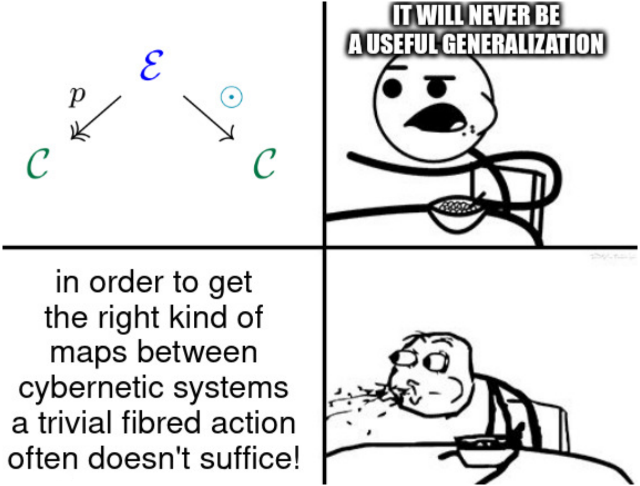
Assume $(\mathcal{C}, \times, 1)$ is a **cartesian pseudomonoid** in $\mathbb{K}$, then we can form the 'simple fibred action' $\mathcal{C} \overset{\text{fst}}{\twoheadleftarrow} S(\mathcal{C}) \overset{\times}{\to} \mathcal{C}$.

Objects of $S(\mathcal{C})$ are pairs $\begin{pmatrix} A \\ B \end{pmatrix}$ of objects in $\mathcal{C}$ and morphisms are maps

$$S(\mathcal{C})\left(\begin{pmatrix} A \\ B \end{pmatrix}, \begin{pmatrix} C \\ D \end{pmatrix}\right) = \mathcal{C}(A, C) \times \mathcal{C}(A \times B, D)$$

The action behaves like the self-action $\mathcal{C} \times \mathcal{C} \overset{\times}{\to} \mathcal{C}$ but maps between scalars are different!

Fibred actions: a crucial generalization!



This is crucial, e.g. to make trajectories of controlled ODEs corepresentable.

Recap

When $\mathbb{K}$ is a 2-cosmos (suitably complete 2-category), we have a functor:

$$\mathbb{P}\mathbf{ara}_{\mathbb{K}} : \mathbb{P}\mathbf{sMnd}(\mathbf{f}\mathbb{S}\mathbf{pan}^{\cong}(\mathbb{K})) \longrightarrow \mathbb{P}\mathbf{sMnd}(\mathbf{f}\mathbb{S}\mathbf{pan}^{\cong}(\mathbb{K}))$$

which (on carriers) is:

$$\mathbb{P}\mathbf{ara}_{\mathbb{K}} \left(\begin{array}{ccc} & \mathcal{E} & \\ {\scriptstyle p} \swarrow & & \searrow {\scriptstyle \odot} \\ \mathcal{C} & & \mathcal{C} \end{array} \right) := \begin{array}{ccc} & \odot/\mathcal{C} & \\ {\scriptstyle \mathrm{dom}} \swarrow & & \searrow {\scriptstyle \mathrm{cod}} \\ \mathcal{C} & & \mathcal{C} \end{array}$$

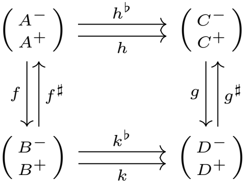
To avoid coherence hell for the pseudodistributive law, one has to toil away a bit more: this leads, for instance, to replace $\mathbb{P}\mathbf{sMnd}$ with a (conjectural) Kleisli completion for a certain kind of enriched bicategories (Garner and Shulman 2016). This is a very cool story categorical story, and yields another extra bit of generality!

DJM sketched it in his CT2023 talk.

Applications

The process theory $\mathbb{A}\mathbf{rena}(q)$

To each fibration $q : \mathcal{B} \to \mathcal{C}$ corresponds a double category $\mathbb{A}\mathbf{rena}(q)$ (Myers 2021) so defined:



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$$\begin{array}{ccc} \begin{pmatrix} A^- \\ A^+ \end{pmatrix} & \overset{h^\flat}{\underset{h}{\rightrightarrows}} & \begin{pmatrix} C^- \\ C^+ \end{pmatrix} \\ f \downarrow \uparrow f^\sharp & & g \downarrow \uparrow g^\sharp \\ \begin{pmatrix} B^- \\ B^+ \end{pmatrix} & \overset{k^\flat}{\underset{k}{\rightrightarrows}} & \begin{pmatrix} D^- \\ D^+ \end{pmatrix} \end{array}$$

$\begin{pmatrix} A^- \\ A^+ \end{pmatrix}, \dots, \begin{pmatrix} D^- \\ D^+ \end{pmatrix}$ are **bundles** (objects in $\mathcal{B}$)

$\begin{pmatrix} h^\flat \\ g \end{pmatrix}, \begin{pmatrix} k^\flat \\ k \end{pmatrix}$ are **charts** (maps in $\mathcal{B}$)

$\begin{pmatrix} f^\sharp \\ f \end{pmatrix}, \begin{pmatrix} g^\sharp \\ g \end{pmatrix}$ are **lenses** (maps in $\mathcal{B}^\vee$)

the square exists if both squares (int. and ext.) commute

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Note: when q is symmetric monoidal (resp. cartesian monoidal), so is $\mathbb{A}\mathbf{rena}(q)$.

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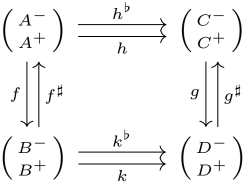

$\begin{pmatrix} A^- \\ A^+ \end{pmatrix}, \ldots, \begin{pmatrix} D^- \\ D^+ \end{pmatrix}$ are **bundles** (objects in $\mathcal{B}$)

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$\begin{pmatrix} f^\sharp \\ f \end{pmatrix}, \begin{pmatrix} g^\sharp \\ g \end{pmatrix}$ are **lenses** (maps in $\mathcal{B}^\vee$)

the square exists if both squares (int. and ext.) commute

Note: when q is symmetric monoidal (resp. cartesian monoidal), so is $\mathbb{A}\mathbf{rena}(q)$.

Example

Let $q = \mathrm{cod} : \mathbf{Set}^{\downarrow} \to \mathbf{Set}$, then objects of $\mathbb{A}\mathbf{rena}(\mathrm{cod})$ are (equivalent to) polynomials, the maps are still known as lenses and charts; and the double category we obtain is cartesian monoidal.

The process theory $\mathbb{A}\mathbf{rena}(q)$

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$$\begin{array}{ccc}
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f \downarrow \uparrow f^\sharp & & g \downarrow \uparrow g^\sharp \\
\begin{pmatrix} B^- \\ B^+ \end{pmatrix} & \overset{k^\flat}{\underset{k}{\rightrightarrows}} & \begin{pmatrix} D^- \\ D^+ \end{pmatrix}
\end{array}$$

$\begin{pmatrix} A^- \\ A^+ \end{pmatrix}, \dots, \begin{pmatrix} D^- \\ D^+ \end{pmatrix}$ are **bundles** (objects in $\mathcal{B}$)

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the square exists if both squares (int. and ext.) commute

Note: when q is symmetric monoidal (resp. cartesian monoidal), so is $\mathbb{A}\mathbf{rena}(q)$.

Example

Let $q = \mathrm{subm} : \mathbf{Smooth}^{\downarrow} \to \mathbf{Smooth}$, then objects of $\mathbb{A}\mathbf{rena}(q)$ are submersions of smooth manifolds, the maps are lenses and charts; and the double category we obtain is cartesian monoidal.

The process theory $\mathbb{A}\mathbf{rena}$

Let's consider q cartesian monoidal, so that $\mathbb{A}\mathbf{rena}$ is cartesian monoidal too and we can define the simple fibred action for it:

$$\mathbb{A}\mathbf{rena} \xleftarrow{\;\mathsf{fst}\;} S(\mathbb{A}\mathbf{rena}) \xrightarrow{\;\times\;} \mathbb{A}\mathbf{rena}$$

We claim

- $\mathbb{P}\mathbf{ro}\mathbb{T}\mathbf{h}$ ($= \mathbb{S}\mathbf{ym}\mathbb{M}\mathbf{on}\mathbb{D}\mathbf{bl}\mathbf{Cat}^{v}$) is a 2-cosmos (see (Bourke and Lack 2023, Theorem 4.4)),

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Thus we can define $\mathbb{P}\mathbf{ara}_{\mathbb{P}\mathbf{ro}\mathbb{T}\mathbf{h}}$ and apply it to $\mathbb{A}\mathbf{rena} \xleftarrow{\text{fst}} S(\mathbb{A}\mathbf{rena}) \xrightarrow{\times} \mathbb{A}\mathbf{rena}$.

The cybernetic process theory $\mathbb{P}\mathbf{ara}(\mathbb{A}\mathbf{rena})$

$\mathbb{P}\mathbf{ara}(\mathbb{A}\mathbf{rena}) := \mathbb{P}\mathbf{ara}_{\mathbb{P}\mathbf{ro}\mathbb{T}\mathbf{h}}(\mathbb{A}\mathbf{rena} \xleftarrow{\mathsf{fst}} S(\mathbb{A}\mathbf{rena}) \xrightarrow{\times} \mathbb{A}\mathbf{rena})$ is a pseudocategory object in $\mathbb{S}\mathbf{ym}\mathbb{M}\mathbf{on}\mathbb{D}\mathbf{bl}\mathbf{Cat}^{v}$, hence a **symmetric monoidal triple category**:

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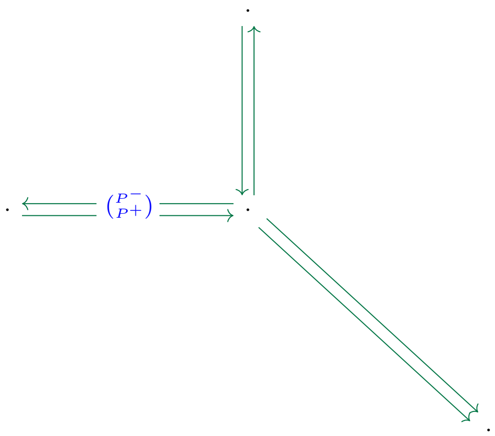
1-cells



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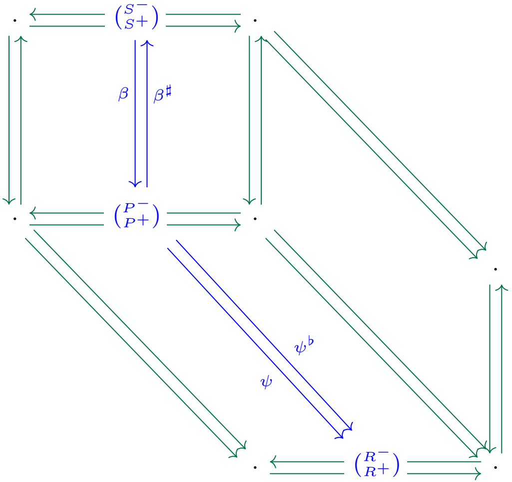

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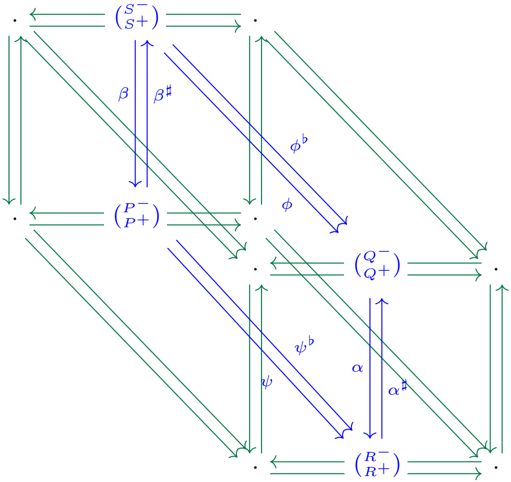
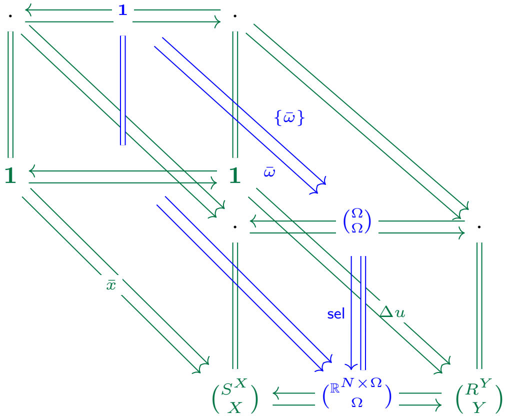


Example: fixpoints of games

When constructed suitably (i.e. as described in Capucci 2023), an open game is a basic 2-cell in $\mathbb{P}\mathbf{ara}(\mathbb{A}\mathbf{rena})$ and maps from the trivial basic 2-cell fix correspond to Nash equilibria:



Here $u : Y \to \mathbb{R}^N$ is a payoff function, $\bar{x} \in X$ an initial state and $\bar{\omega} \in \Omega$ a strategy profile.

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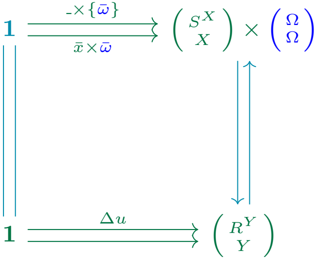

$$\iff \underbrace{\bar{\omega} \in \mathsf{sel}(\lambda\omega\,.\,\mathsf{coplay}(\bar{x}, \omega, \Delta u(\mathsf{play}(\bar{x}, \omega))))}_{\textbf{Nash equilibrium}}$$

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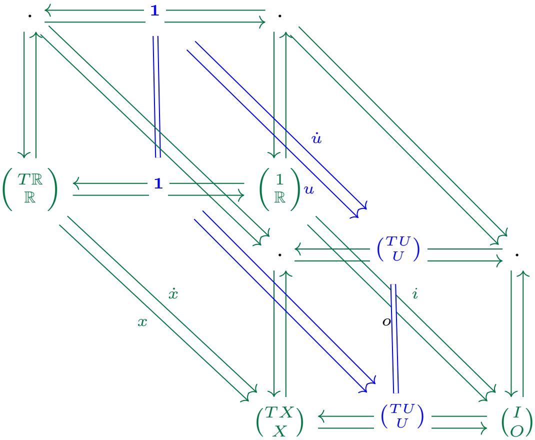
Example: trajectories of open controlled ODEs

Let $\binom{f^\sharp}{f} : \binom{TX}{X} \otimes \binom{TU}{U} \leftrightarrows \binom{I}{O}$ be an open controlled ODE. Let clock be the 'walking trajectory' system,i.e. the uncontrolled ODE on $\mathbb{R}$ defined as $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$. Then maps from the latter into the first in $\mathbb{A}\mathbf{rena}(\mathrm{subm})$ correspond to solutions of the open controlled ODE:



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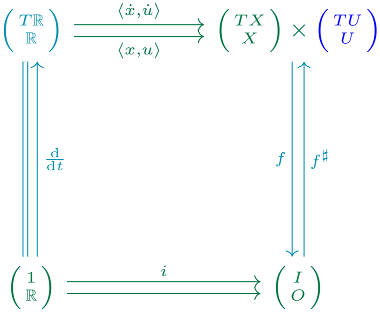


$$\iff$$

$$\underbrace{\begin{aligned} o(t) &= f(x(t), u(t)) \\ \langle \dot{x}(t), \dot{u}(t) \rangle &= f^\sharp(i(t), x(t), u(t)) \end{aligned}}_{\text{trajectory of the open controlled ODE}}$$

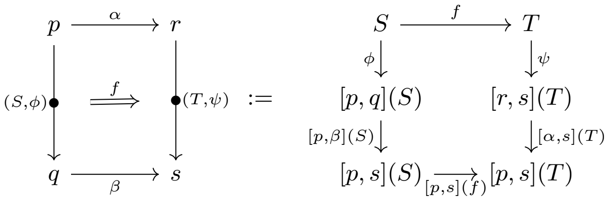
Bonus: $\mathbb{P}\mathrm{ara}(\mathbb{A}\mathrm{rena})$ and $\mathbb{O}\mathbf{rg}$

In (Shapiro and Spivak 2022) they define a double category $\mathbb{O}\mathbf{rg}$ where

- objects are *polynomial functors*, i.e. functors of the form $p = \sum_{i:p(1)} y^{p[i]}$
- loose arrows $(S, \phi) : p \to q$ are *polynomial coalgebras*, i.e. coalgebras of the form

$$S : \mathbf{Set}, \quad \phi : S \longrightarrow [p, q](S)$$

  where $[-,-]$ is the closed structure associated to the Hancock product,
- tight arrows $h : p \to r$ are morphisms of polynomial functors,
- squares are given by maps between the carriers of the coalgebras, plus a commutativity condition:



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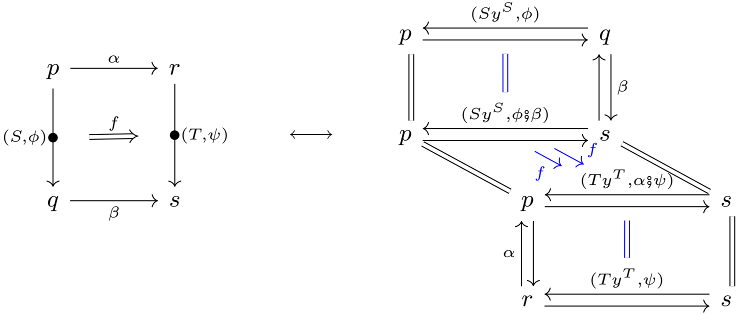
Recalling that $\mathbf{Poly} \cong \mathbf{Lens}(\mathrm{cod}_{\mathbf{Set}})$, and that polynomial coalgebras can equivalently be given as parametric maps $Sy^S \otimes p \to q$, and that coalgebra maps between them are *charts*, we see that $\mathbb{O}\mathbf{rg}$ embeds in $\mathbb{P}\mathrm{ara}(\mathbb{A}\mathbf{rena})$ ‘diagonally’:



Hence $\mathbb{O}\mathbf{rg}$ distills the structure of $\mathbb{P}\mathrm{ara}(\mathbb{A}\mathbf{rena})$ (or variants thereof) for the purposes of “dynamic enrichment”. We converge on the same structure!

Question: is enrichment in $\mathbb{P}\mathrm{ara}(\mathbb{A}\mathbf{rena})$ interesting?

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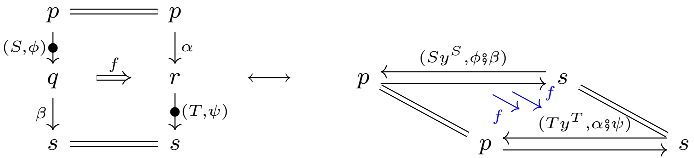

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What we left out:

- The gory categorical details of the generalised $\mathbb{P}\mathbf{ara}$ construction,
- How to actually get cybernetic **systems**, by running $\mathbb{P}\mathbf{ara}$ in $\mathbb{S}\mathbf{ys}\mathbb{T}\mathbf{h}$ $(= \mathbb{S}\mathbf{ym}\mathbb{M}\mathbf{on}\mathbb{D}\mathbf{bl}\mathbb{I}\mathbf{x}\mathbb{C}\mathbf{at}^{v})$

Thanks for your attention!

Questions?

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