

Inconsistency of \mathbb{N} from a not-finitist point of view

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Abstract

Considering the set of natural numbers \mathbb{N} , then in the context of Peano axioms, we find a fundamental contradiction from a not-finitist point of view.

1 Introduction

A formal system together an interpretation, constituted of an alphabet, grammar, inference rules, axioms, and a reference set, can produce formalized propositions and deductions (theorems) through with a finite number of steps, that is a finitist approach [5] [10].

A system is consistent whether a proposition and its negation are not deduced. Gödel's incompleteness theorems [4], developed on the basis of the system of Principia Mathematica including the axiom of infinity, represent a fortress of logic and consistency against inconsistency. But at the same time they represent a prelude of inconsistency. They give us necessary conditions of consistency, not sufficient ones (undecidable propositions and not demonstrable internal consistency are these necessary conditions).

Considering the successor function $S(x)$ and the existence of all natural numbers in concordance with Peano axiom and the axiom of infinity, we show a contradiction in \mathbb{N} , in a not-finitist way, that is thinking to take simultaneously all natural numbers.

2 Natural numbers set

The existence of \mathbb{N} is granted by the axiom of infinity [11] [8] [9]. This existence imply that one of each element of the set, in an actual sense, not time dependent, so taken all together. A finite set wouldn't admit Peano axiom $\forall x(S(x))$ with $S(x) \in \mathbb{N}$, because the greatest number doesn't have a successor into the finite set. All numbers of \mathbb{N} are defined by Peano axioms [7] [6] [3], together their proprieties thanks to the axiom of induction.

Nevertheless the axiom of infinity, natural numbers are almost always thought one by one over time, specially implicitly. A list of infinite elements is not contemplated by a finitist approach to obtain a deduction. In any case we will strive to think of all numbers simultaneously.

3 A fundamental contradiction

We show there is a greater natural number than all natural numbers, or equivalently there are fewer natural numbers than natural numbers. Two proofs are given.

1) We know, as it is demonstrable, that: $(x \in \mathbb{N})(\forall x(x < S(x)))$ with $S(x) \in \mathbb{N}$. So, in concordance with $\forall x$ (that defines each/all natural numbers) and considering all x together we have a greater natural number $S(n)$ than all natural numbers (all x). We can consider an oriented line with x marked points, from 1 to n; transposing $\forall x(x < S(x))$ on the line we see, imagining $n \rightarrow \infty$, $S(n)$ is greater than all natural numbers.

-- 1 -- 2 -- 3 -- 4 -- ----- n -- S(n) -- >
| ----- all x together ----- |

It could be said that it is not possible to consider all x together, but it is equivalent to not being able to consider \mathbb{N} , as defined by axiom of infinity. Then \mathbb{N} wouldn't be part of a formal system and its interpretation.

2) The two sets $\{0, \{S(x)|x \in \mathbb{N}\}\}$ and \mathbb{N} are the same, $\{0, \{S(x)|x \in \mathbb{N}\}\} = \mathbb{N}$, but we also have:

$$\sum_{x=0}^y x < \sum_{x=0}^y S(x) \quad \forall y \quad (\text{with} \quad \sum_{x=0}^y S(x) = 0 + \sum_{x=0}^y S(x)).$$

Although there are infinite sums (one for each y), **there is the sum of all natural numbers**, otherwise we wouldn't have considered all y, in agreement

with $\forall y$; the last term in $\sum x$ is exactly y , if in $\sum x$ there weren't all x (then all numbers), we could take others longer $\sum x$ then others greater y (necessary condition is: all y if all x , then all $y \longrightarrow$ all x). So we find that the sum of all elements of a set is less than the other; in a set there are elements that are not in the other. But the set is the same, \mathbb{N} . Then there are fewer natural numbers than natural numbers, that is a contradiction.

Here all x together are "automatically" considered in the sum.

These proofs seems to look like to Burali Forti antinomy [2] [1]. But we have only considered the \mathbb{N} set with its elements, while that one is centered on the "set" (actually named a class) of all ordinal numbers.

4 Conclusion

These proofs of inconsistency are not finitist because they need infinite totalities. But can we really summarize the finitist point of view as follows? "A list of infinity elements cannot be considered for deducing, although it has to exist in agreement with the axiom of infinity". Is this not an acceptable point of view and then are these not finitist proofs of inconsistency valid in a general way? The last consideration in "1)" seems to go in the direction of validity in a general way.

Anyway refusing the axiom of infinity could be a view to avoid this inconsistency.

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