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A New Approach to Natural Lift Curves of The Spherical Indicatrices of Timelike Bertrand Mate of a Spacelike Curve in Minkowski 3-Space

Mustafa Bilici

(Department of Mathematics, Educational Faculty of Ondokuz, Mayıs University, Atakum 55200, Samsun, Turkey)

Evren Ergün

(Çarşamba Chamber of Commerce Vocational School of Ondokuz, Mayıs University, Çarşamba 55500, Samsun, Turkey)

Mustafa Çalışkan

(Department of Mathematics, Faculty of Sciences of Gazi University, Teknik Okullar 06500, Ankara, Turkey)

E-mail: mbilici@omu.edu.tr, eergun@omu.edu.tr, mustafacalikan@gazi.edu.tr

Abstract: In this study, we present a new approach the natural lift curves for the spherical indicatrices of the timelike Bertrand mate of a spacelike curve on the tangent bundle $T\left(S_1^2\right)$ or $T\left(H_0^2\right)$ in Minkowski 3-space and we give some new characterizations for these curves. Additionally we illustrate an example of our main results.

Key Words: Bertrand curve, natural lift curve, geodesic spray, spherical indicatrix.

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§1. Introduction

Bertrand curves are one of the associated curve pairs for which at the corresponding points of the curves one of the Frenet vectors of a curve coincides with the one of the Frenet vectors of the other curve. These special curves are very interesting and characterized as a kind of corresponding relation between two curves such that the curves have the common principal normal i.e., the Bertrand curve is a curve which shares the normal line with another curve. It is proved in most texts on the subject that the characteristic property of such a curve is the existence of a linear relation between the curvature and the torsion; the discussion appears as an application of the Frenet-Serret formulas. So, a circular helix is a Bertrand curve. Bertrand mates represent particular examples of offset curves [11] which are used in computer-aided design (CAD) and computer-aided manufacturing (CAM). For classical and basic treatments of Bertrand curves, we refer to [3], [6] and [12].

There are recent works about the Bertrand curves. Ekmekçi and İlarslan studied Nonnull Bertrand curves in the n-dimensional Lorentzian space. Straightforward modication of classical theory to spacelike or timelike curves in Minkowski 3-space is easily obtained, (see [1]). Izumiya

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and Takeuchi [16] have shown that cylindrical helices can be constructed from plane curves and Bertrand curves can be constructed from spherical curves. Also, the representation formulae for Bertrand curves were given by [8].

In differential geometry, especially the theory of space curves, the Darboux vector is the areal velocity vector of the Frenet frame of a space curve. It is named after Gaston Darboux who discovered it. In terms of the Frenet-Serret apparatus, the Darboux vector can be expressed as $w = \tau t + \kappa b$. In addition, the concepts of the natural lift and the geodesic sprays have first been given by Thorpe (1979). On the other hand, Çalışkan et al. [4] have studied the natural lift curves and the geodesic sprays in Euclidean 3-space $\mathbb{R}^3$. Bilici et al. [7] have proposed the natural lift curves and the geodesic sprays for the spherical indicatrices of the involute-evolute curve couple in $\mathbb{R}^3$. Recently, Bilici [9] adapted this problem for the spherical indicatrices of the involutes of a timelike curve in Minkowski 3-space.

Kula and Yaylı [17] have studied spherical images of the tangent indicatrix and binormal indicatrix of a slant helix and they have shown that the spherical images are spherical helices. In [19] Süha et. all investigated tangent and trinormal spherical images of timelike curve lying on the pseudo hyperbolic space H_0^3 in Minkowski space-time. İyigün [20] defined the tangent spherical image of a unit speed timelike curve lying on the on the pseudo hyperbolic space H_0^2 in $\mathbb{R}_1^3$.

Şenyurt and Çalışkan [22] obtained arc-lengths and geodesic curvatures of the spherical indicatrices (T^*) , (N^*), (B^*) and the fixed pole curve (C^*) which are generated by Frenet trihedron and the unit Darboux vector of the timelike Bertrand mate of a spacelike curve with respect to Minkowski space $\mathbb{R}_1^3$ and Lorentzian sphere S_1^2 or hyperbolic sphere H_0^2. Furthermore, they give some criteria of being integral curve for the geodesic spray of the natural lift curves of this spherical indicatrices.

In this study, the conditions of being integral curve for the geodesic spray of the the natural lift curves of the the spherical indicatrices (T^*) , (N^*), (B^*) are investigated according to the relations given by [8] on the tangent bundle $T\left(S_1^2\right)$ or $T\left(H_0^2\right)$ in Minkowski 3-space. Also, we present an example which illustrates these spherical indicatrices (Figs. 1-4). It is seen that the principal normal indicatrix (N^*) is geodesic on S_1^2 and its natural lift curve is an integral curve for the geodesic spray on $T\left(S_1^2\right)$.

§2. Preliminaries

To meet the requirements in the next sections, the basic elements of the theory of curves and hypersurfaces in the Minkowski 3-space are briefly presented in this section. A more detailed information can be found in [10].

The Minkowski 3-space $\mathbb{R}_1^3$ is the real vector space $\mathbb{R}^3$endowed with standard flat Lorentzian metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{R}_1^3$. A vector $V = (v_1, v_2, v_3) \in \mathbb{R}_1^3$ is said to be timelike if $g(V,V) < 0$, spacelike if $g(V,V) > 0$ or $V = 0$ and null (lightlike) if

$g(V,V)=0$ and $V\neq 0$ .Similarly, an arbitrary $\Gamma=\Gamma(s)$ curve in $\mathbb{R}_1^3$ can locally be timelike, spacelike or null (lightlike), if all of its velocity vectors Γ' are respectively timelike, spacelike or null (lightlike), for every t $\in$ I $\subset \mathbb{R}$. The pseudo-norm of an arbitrary vector $V\in\mathbb{R}_1^3$ is given by $\|V\|=\sqrt{|g(V,V)|}$. Γ is called a unit speed curve if the velocity vector V of Γ satisfies $\|V\|=1$. A timelike vector V is said to be positive (resp. negative) if and only if $v_1>0$ (resp.$v_1<0$) .

Let Γ be a unit speed spacelike curve with curvature κ and torsion τ. Denote by $\{t(s),n(s),b(s)\}$ the moving Frenet frame along the curve Γ in the space $\mathbb{R}_1^3$. Then t,n and b are the tangent, the principal normal and the binormal vector of the curve Γ, respectively.

The angle between two vectors in Minkowski 3-space is defined by [21]

Definition 2.1 *Let X and Y be spacelike vectors in $\mathbb{R}_1^3$ that span a spacelike vector subspace, then we have $|g(X,Y)|\leq\|X\|\|Y\|$ and hence, there is a unique positive real number φ such that*

$$|g(X,Y)|=\|X\|\|Y\|\cos\varphi.$$

The real numberφ is called the Lorentzian spacelike angle between X and Y.

Definition 2.2 *Let X and Y be spacelike vectors in $\mathbb{R}_1^3$ that span a timelike vector subspace, then we have $|g(X,Y)|>\|X\|\|Y\|$ and hence, there is a unique positive real number φ such that*

$$|g(X,Y)|=\|X\|\|Y\|\cosh\varphi.$$

The real number φ is called the Lorentzian timelike angle between X and Y.

Definition 2.3 *Let X be a spacelike vector and Y a positive timelike vector in $\mathbb{R}_1^3$, then there is a unique non-negative real number φ such that*

$$|g(X,Y)|=\|X\|\|Y\|\sinh\varphi.$$

The real number φ is called the Lorentzian timelike angle between X and Y.

Definition 2.4 *Let X and Y be positive (negative) timelike vectors in $\mathbb{R}_1^3$, then there is a unique non-negative real number φ such that*

$$g(X,Y)=\|X\|\|Y\|\cosh\varphi.$$

The real number φ is called the Lorentzian timelike angle between X and Y.

Case I. Let Γ be a unit speed spacelike curve with a spacelike binormal. For these Frenet vectors, we can write

$$T\times N=-B,\ N\times B=-T,\ B\times T=N$$

where "$\times$" is the Lorentzian cross product in space $\mathbb{R}_1^3$. Depending on the causal character of the curve Γ, the following Frenet formulae are given in [5].

$$\dot{T}=\kappa N,\ \dot{N}=\kappa T+\tau B,\dot{B}=\tau N$$

The Darboux vector for the spacelike curve with a spacelike binormal is defined by [11]:

$$w = -\tau T + \kappa B$$

If b and w spacelike vectors that span a spacelike vector subspace then by the Definition1. we can write

$$\begin{aligned} \kappa &= \|w\| \cosh \varphi \\ \tau &= \|w\| \sinh \varphi, \end{aligned}$$

where $\|w\|^2 = g(w, w) = \tau^2 + \kappa^2$.

Case II. Let Γ be a unit speed spacelike curve with a timelike binormal. For these Frenet vectors, we can write

$$T \times N = B, \; N \times B = -T, \; B \times T = -N$$

Depending on the causal character of the curve Γ , the following Frenet formulae are given in [5].

$$\dot{T} = \kappa N, \; \dot{N} = -\kappa T + \tau B, \dot{B} = \tau N$$

The Darboux vector for the spacelike curve with a timelike binormal is defined by [11]:

$$w = \tau T - \kappa B$$

There are two cases corresponding to the causal characteristic of Darboux vector w.

(i) If $|\kappa| < |\tau|$, then w is a timelike vector. In this situation, we have

$$\begin{aligned} \kappa &= \|w\| \sinh \varphi \\ \tau &= \|w\| \cosh \varphi, \end{aligned}$$

where $\|w\|^2 = -g(w, w) = \tau^2 - \kappa^2$. So the unit vector c of direction w is

$$c = \frac{1}{\|w\|} w = \sinh \varphi T - \cosh \varphi B.$$

(ii) If $|\kappa| > |\tau|$, then w is a spacelike vector. In this situation, we can write

$$\begin{aligned} \kappa &= \|w\| \cosh \varphi \\ \tau &= \|w\| \sinh \varphi, \end{aligned}$$

where $\|w\|^2 = g(w, w) = \kappa^2 - \tau^2$. So the unit vector c of direction w is

$$c = \frac{1}{\|w\|} w = \sinh \varphi T - \cosh \varphi B.$$

Proposition 2.5([13]) *Let α be a timelike (or spacelike) curve with curvatures κ and τ. The* *curve is a general helix if and only if $\frac{\tau}{\kappa}$ is constant.*

Remark 2.6 We can easily see fromequations of the section Case I and Case II that: $\frac{\tau}{\kappa} = \tan\varphi$, $\frac{\tau}{\kappa} = \tanh\varphi$ $\left(or\ \frac{\tau}{\kappa} = \coth\varphi\right)$, if φ =constant then α is a general helix.

Lemma 2.7([9]) *The natural lift $\overline{\alpha}$ of the curve α is an integral curve of the geodesic spray X if and only if α is a geodesic on M.*

Definition 2.8 *Let $\alpha = (\alpha(s); T(s), N(s), B(s))$ and $\beta = (\beta(s^*); T^*(s^*), N^*(s^*), B^*(s^*))$ be two regular non-null curves in $\mathbb{R}^3_1$. $\alpha(s)$ and $\beta(s^*)$ are called Bertrand curves if $N(s)$ and $N^*(s^*)$ are linearly dependent. In this situation,(α, β) is called a Bertrand couple in $\mathbb{R}^3_1$.* (See [1] for the more details in the n-dimensional space).

Lemma 2.9 *Let α be a spacelike curve with a timelike binormal. In this situation, β is a timelike Bertrand mate of α. The relations between the Frenet vectors of the (α, β) is as follow*

$$\begin{bmatrix} T^* \\ N^* \\ B^* \end{bmatrix} = \begin{bmatrix} \sinh\theta & 0 & \cosh\theta \\ 0 & 1 & 0 \\ \cosh\theta & 0 & \sinh\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, g(T, T^*) = \sinh\theta = constant, [8].$$

Definition 2.11([10]) *Let S^2_1 and H^2_0 be hypersphere in $\mathbb{R}^3_1$. The Lorentzian sphere and hyperbolic sphere of radius 1 in are given by*

$$S^2_1 = \left\{V = (v_1, v_2, v_3) \in \mathbb{R}^3_1 \ : \ g(V, V) = 1\right\}$$

and

$$H^2_0 = \left\{V = (v_1, v_2, v_3) \in \mathbb{R}^3_1 \ : \ g(V, V) = -1\right\}$$

respectively.

Definition 2.12([9]) *Let M be a hypersurface in $\mathbb{R}^3_1$ equipped with a metric g. Let TM be the set $\cup\{T_p(M) : p \in M\}$ of all tangent vectors to M. Then each $v \in TM$ is in a unique $T_p(M)$, and the projection $\pi : TM \to M$ sends v to p. Thus $\pi^{-1}(p) = T_p(M)$. There is a natural way to makeTM a manifold, called the tangent bundle of M.*

A vector field $X \in \chi(M)$ is exactly a smooth section of TM, that is, a smooth function $X : M \to TM$ such that $\pi \circ X = id_M$.

Definition 2.13([9]) *Let M be a hypersurface in $\mathbb{R}^3_1$. A curve $\alpha : I \to TM$ is an integral curve of $X \in \chi(M)$ provided $\dot{\alpha} = X_\alpha$; that is*

$$\frac{d}{ds}(\alpha(s)) = X(\alpha(s)) \ \ for\ all\ s \in I, [10]. \tag{1}$$

Definition 2.14 *For any parametrized curve $\alpha : I \to TM$, the parametrized curve given by*

$\overline{\alpha}: I \to TM$

$$s \to \overline{\alpha}(s) = (\alpha(s), \dot{\alpha}(s)) = \dot{\alpha}(s)\,|_{\alpha(s)} \tag{2}$$

is called the natural lift of α on TM. Thus, we can write

$$\frac{d\overline{\alpha}}{ds} = \frac{d}{ds}\left(\alpha'(s)\,|_{\alpha(s)}\right) = D_{\dot{\alpha}'(s)}\dot{\alpha}(s), \tag{3}$$

where D is the standard connection on $\mathbb{R}^3_1$.

Definition 2.15([9]) *For $v \in TM$, the smooth vector field $X \in \chi(TM)$ defined by*

$$X(v) = \varepsilon g(v, S(v))\,\xi\,|_{\alpha(s)},\ \ \varepsilon = g(\xi, \xi) \tag{4}$$

is called the geodesic spray on the manifold TM, where ξ is the unit normal vector field of M and S is the shape operator of M.

§3. Natural Lift Curves for the Spherical Indicatrices of Spacelike-Timelike Bertrand Couple in Minkowski 3-Space

In this section we investigate the natural lift curves of the spherical indicatrices of Bertrand curves (α, β) as in Lemma 2.9. Furthermore, some interesting theorems about the original curve were obtained depending on the assumption that the natural lift curves should be the integral curve of the geodesic spray on the tangent bundle $T\left(S^2_1\right)$ or $T\left(H^2_0\right)$.

Note that $\bar{D}$ and $\bar{\bar{D}}$ are Levi-Civita connections on S^2_1 and H^2_0 , respectively. Then Gauss equations are given by the followings

Let D, $\bar{D}$ and $\bar{\bar{D}}$ be connections in $\mathbb{R}^3_1$, S^2_1 and H^2_0 respectively and ξ be a unit normal vector field of S^2_1 and H^2_0. Then Gauss Equations are given by the followings

$$D_XY = \bar{D}_XY + \varepsilon g(S(X), Y)\,\xi,\ \ D_XY = \bar{\bar{D}}_XY + \varepsilon g(S(X), Y)\,\xi, \varepsilon = g(\xi, \xi)$$

where ξ is a unit normal vector field and S is the shape operator of S^2_1 (or H^2_0).

3.1 The natural lift of the spherical indicatrix of the tangent vector of β

Let (α, β) be Bertrand curves as in Lemma 2.9. We will investigate the curve α to satisfy the condition that the natural lift curve of $\bar{\beta}_{T^*}$ is an integral curve of geodesic spray, where β_{T^*} is the tangent indicatrix of β. If the natural lift curve $\bar{\beta}_{T^*}$ is an integral curve of the geodesic spray, then by means of Lemma 2.9. we get,

$$\bar{\bar{D}}_{\dot{\beta}_{T^*}}\dot{\beta}_{T^*} = 0, \tag{5}$$

where $\bar{\bar{D}}$ is the connection on the hyperbolic unit sphere H^2_0 and the equation of tangent indicatrix is $\beta_{T^*} = T^*$. Thus from the Gauss equation we can write

$$D_{\dot{\beta}_{T^*}} \dot{\beta}_{T^*} = \overline{\overline{D}}_{\dot{\beta}_{T^*}} \dot{\beta}_{T^*} + \varepsilon g\left(S\left(\dot{\beta}_{T^*}\right), \dot{\beta}_{T^*}\right) T^*, \varepsilon = g\left(T^*, T^*\right) = -1$$

On the other hand, from the Lemma 2.9. straightforward computation gives

$$\dot{\beta}_{T^*} = t_{T^*} = \frac{d\beta_{T^*}}{ds}\frac{ds}{ds_{T^*}} = (\kappa \sinh\theta + \tau\cosh\theta)\, N \frac{ds}{ds_{T^*}}$$

Moreover, we get

$$\frac{ds}{ds_{T^*}} = \frac{1}{\kappa\sinh\theta + \tau\cosh\theta}, t_{T^*} = N,$$

$$D_{t_{T^*}} t_{T^*} = -\frac{\kappa}{\kappa\sinh\theta + \tau\cosh\theta} T + \frac{\tau}{\kappa\sinh\theta + \tau\cosh\theta} B$$

and $g\left(S\left(t_{T^*}\right), t_{T^*}\right) = -1$.

Using these in the Gauss equation, we immediately have

$$\overline{\overline{D}}_{t_{T^*}} t_{T^*} = -\frac{\kappa}{\kappa\sinh\theta + \tau\cosh\theta} T + \frac{\tau}{\kappa\sinh\theta + \tau\cosh\theta} B - T^*.$$

From the Eq. (5) and Lemma 2.9.ii) we get

$$\left(-\frac{\kappa}{\kappa\sinh\theta + \tau\cosh\theta} - \sinh\theta\right) T + \left(\frac{\tau}{\kappa\sinh\theta + \tau\cosh\theta} - \cosh\theta\right) B$$

Since T, N, B are linearly independent, we have

$$-\frac{\kappa}{\kappa\sinh\theta + \tau\cosh\theta} - \sinh\theta = 0, \qquad \frac{\tau}{\kappa\sinh\theta + \tau\cosh\theta} - \cosh\theta = 0.$$

It follows that,

$$\kappa\cosh\theta + \tau\sinh\theta = 0 \tag{6}$$

$$\frac{\tau}{\kappa} = -\coth\theta \tag{7}$$

So from the Eq. (7) and Remark 2.6. we can give the following proposition.

Proposition 3.1 *Let (α, β) be Bertrand curves as in Lemma 2.9. If α is a general helix, then the tangent indicatrix β_{T^*} of β is a geodesic on H_0^2.*

Moreover from Lemma 2.7. and Proposition 3.1 we can give the following theorem to characterize the natural lift of the tangent indicatrix of β without proof.

Theorem 3.2 *Let (α, β) be Bertrand curves as in Lemma 2.9. If α is a general helix, then the natural lift $\bar{\beta}_{T^*}$ of the tangent indicatrix β_{T^*} of β is an integral curve of the geodesic spray on the tangent bundle $T\left(H_0^2\right)$.*

3.2 The natural lift of the spherical indicatrix of the principal normal vectors of β

Let β_{N^*} be the spherical indicatrix of principal normal vectors of β and $\bar{\beta}_{N^*}$ be the natural lift of the curve . If $\bar{\beta}_{N^*}$ is an integral curve of the geodesic spray, then by means of Lemma 2.7. we get,

$$\bar{D}_{t_{N^*}} t_{N^*} = 0, \tag{8}$$

that is

$$D_{t_{N^*}} t_{N^*} = \bar{D}_{t_{N^*}} t_{N^*} + \varepsilon g\left(S\left(t_{N^*}\right), t_{N^*}\right) N^*, \varepsilon = g\left(N^*, N^*\right) = 1$$

On the other hand, from Lemma 2.9. and Case II. i) straightforward computation gives

$$\dot{\beta}_{N^*} = t_{N^*} = -\sinh\varphi T + \cosh\varphi B$$

Moreover we get

$$D_{t_{N^*}} t_{N^*} = -\frac{\dot{\varphi}\cosh\varphi}{\|W\|}T + \frac{-\kappa\sinh\varphi + \tau\cosh\varphi}{\|W\|}N + \frac{\dot{\varphi}\sinh\varphi}{\|W\|}B \text{ and } g\left(S\left(t_{N^*}\right), t_{N^*}\right) = 1$$

Using these in the Gauss equation, we immediately have

$$\bar{D}_{t_{N^*}} t_{N^*} = -\frac{\dot{\varphi}\cosh\varphi}{\|W\|}T + \frac{\dot{\varphi}\sinh\varphi}{\|W\|}B.$$

Since T, N, B are linearly independent, we have

$$-\frac{\dot{\varphi}\cosh\varphi}{\|W\|} = 0, \qquad \frac{\dot{\varphi}\sinh\varphi}{\|W\|} = 0.$$

It follows that,

$$\dot{\varphi} = 0, \tag{9}$$

$$\frac{\tau}{\kappa} = cons\tan t. \tag{10}$$

So from the Eq. (10) and Remark 2.6. we can give the following proposition.

Proposition 3.3 *Let (α, β) be Bertrand curves as in Lemma 2.9. If α is a general helix, then the principal normal indicatrix β_{N^*} of β is a geodesic on S_1^2.*

Moreover from Lemma 2.7. and Proposition 4.3. we can give the following theorem to characterize the natural lift of the principal normal indicatrix of β without proof.

Theorem 3.4 *Let (α, β) be Bertrand curves as in Lemma 2.9. If α is a general helix, then the natural lift $\bar{\beta}_{N^*}$ of the principal normal indicatrix of β_{N^*} is β an integral curve of the geodesic spray on the tangent bundle $T\left(S_1^2\right)$.*

3.3 The natural lift of the spherical indicatrix of the binormal vectors of β

Let β_{B^*} be the spherical indicatrix of binormal vectors of β and $\bar{\beta}_{B^*}$ be the natural lift of the curve β_{B^*} . If $\bar{\beta}_{B^*}$ is an integral curve of the geodesic spray, then by means of Lemma 2.7. we get

$$\bar{D}_{t_{B^*}} t_{B^*} = 0, \tag{11}$$

that is

$$D_{t_{B^*}} t_{B^*} = \bar{D}_{t_{B^*}} t_{B^*} + \varepsilon g\left(S\left(t_{B^*}\right), t_{B^*}\right) B^*, \varepsilon = g\left(B^*, B^*\right) = 1$$

On the other hand, from Lemma 2.9.ii) straightforward computation gives

$$t_{B^*} = \left(\kappa \cosh\theta + \tau \sinh\theta\right) N \frac{ds}{ds_{B^*}}$$

Moreover we get

$$\frac{ds}{ds_{B^*}} = \frac{1}{\kappa \cosh\theta + \tau \sinh\theta}, t_{B^*} = N,$$

$$D_{t_{B^*}} t_{B^*} = -\frac{\kappa}{\kappa \cosh\theta + \tau \sinh\theta} T + \frac{\tau}{\kappa \cosh\theta + \tau \sinh\theta} B$$

and $g\left(S\left(t_{B^*}\right), t_{B^*}\right) = -1$.

Using these in the Gauss equation, we immediately have

$$\bar{D}_{t_{B^*}} t_{B^*} = \frac{\kappa}{\kappa \cosh\theta + \tau \sinh\theta} T - \frac{\tau}{\kappa \cosh\theta + \tau \sinh\theta} B + B^*$$

From the Eq. (11) and Lemma 2.9.ii) we get

$$\left(-\frac{\kappa}{\kappa \cosh\theta + \tau \sinh\theta} + \cosh\theta\right) T + \left(\frac{\tau}{\kappa \cosh\theta + \tau \sinh\theta} + \sinh\theta\right) B = 0.$$

Since T, N, B are linearly independent, we have

$$\begin{aligned} -\frac{\kappa}{\kappa \cosh\theta + \tau \sinh\theta} + \cosh\theta &= 0 \\ \frac{\tau}{\kappa \cosh\theta + \tau \sinh\theta} + \sinh\theta &= 0 \end{aligned}$$

it follows that

$$\kappa \sinh\theta + \tau \cosh\theta = 0 \tag{12}$$

$$\frac{\tau}{\kappa} = -\tanh\theta \tag{13}$$

So from the Eq. (13) and Remark 2.6. we can give the following proposition.

Proposition 3.5 *Let (α, β) be Bertrand curves as in Lemma 2.9. If α is a general helix, then the binormal indicatrix β_{B^*} of β is a geodesic on S_1^2.*

Moreover from Lemma 2.7. and Proposition 4.5. we can give the following theorem to characterize the natural lift of the binormal indicatrix of β without proof.

Theorem 3.6 *Let (α, β) be Bertrand curves as in Lemma 2.9. If α is a general helix, then the natural lift $\bar{\beta}_{B^*}$ of the binormal indicatrix β_{B^*} of β is an integral curve of the geodesic spray on the tangent bundle $T\left(S_1^2\right)$.*

From the classification of all W-curves (i.e. a curves for which a curvature and a torsion are constants) in (Walrawe, 1995), we have following proposition with relation to curve.

Proposition 3.7 (1) *If the curve* α *with* κ *=constant* >0 , $\tau=0$ *then* α *is a part of a circle;*

(2) *If the curve* α *with* κ *=constant* >0 , τ *=constant* $\neq 0$, *and* $|\tau|>\kappa$ *then* α *is a part of a spacelike hyperbolic helix,*

$$\alpha(s)=\frac{1}{K}\left(\kappa\sinh\left(\sqrt{K}s\right),\sqrt{\tau^2K}s,\kappa\cosh\left(\sqrt{K}s\right)\right),\ K=\tau^2-\kappa^2;$$

(3) *If the curve* α *with* κ *=constant* >0, τ *=constant* $\neq 0$ *and* $|\tau|<\kappa$, *then* α *is a part of a spacelike circular helix,*

$$\alpha(s)=\frac{1}{K}\left(\sqrt{\tau^2K}s,\kappa\cos\left(\sqrt{K}s\right),\kappa\sin\left(\sqrt{K}s\right),\sqrt{\tau^2K}s,\right),\ K=\kappa^2-\tau^2;$$

From Lemma 3.1 in Choi et al 2012, we can write the following proposition.

Proposition 3.8 *There is no spacelike general helix of spacelike curve with a timelike binormal in Minkowski 3-space with condition* $|\tau|=|\kappa|$.

Example 3.9 Let $\alpha(s)=\frac{1}{3}\left(\sinh\left(\sqrt{3}s\right),2\sqrt{3}s,\cosh\left(\sqrt{3}s\right)\right)$be a unit speed spacelike hyperbolic helix with

$$\begin{aligned} T &= \frac{\sqrt{3}}{3}\left(\cosh\left(\sqrt{3}s\right),2,\sinh\left(\sqrt{3}s\right)\right) \\ N &= \left(\sinh\left(\sqrt{3}s\right),0,\cosh\left(\sqrt{3}s\right)\right), \qquad \kappa=1 \textit{ and } \tau=2 \\ B &= \frac{\sqrt{3}}{3}\left(2\cosh\left(\sqrt{3}s\right),1,2\sinh\left(\sqrt{3}s\right)\right) \end{aligned}$$

In this situation, spacelike with spacelike binormal Bertrand mate for can be given by the equation

$$\beta(s)=\left(\left(\lambda+\frac{1}{3}\right)\sinh\left(\sqrt{3}s\right),\frac{2\sqrt{3}}{3}s,\left(\lambda+\frac{1}{3}\right)\cosh\left(\sqrt{3}s\right)\right),\lambda\in\mathbb{R}$$

For $\lambda=\frac{\sqrt{7}-1}{3}$, we have

$$\beta(s)=\left(\frac{\sqrt{7}}{3}\sinh\left(\sqrt{3}s\right),\frac{2\sqrt{3}}{3}s,\frac{\sqrt{7}}{3}\cosh\left(\sqrt{3}s\right)\right).$$

The straight forward calculations give the following spherical indicatrices and natural lift curves of spherical indicatrices for β,

$$
\begin{aligned}
\beta_{T^*} &= \frac{\sqrt{3}}{3}\left(\sqrt{7}\cosh\left(\sqrt{3}s\right),2,\sqrt{7}\sinh\left(\sqrt{3}s\right)\right)\\
\beta_{N^*} &= \left(\sinh\left(\sqrt{3}s\right),0,\cosh\left(\sqrt{3}s\right)\right)\\
\beta_{B^*} &= \frac{\sqrt{3}}{3}\left(-2\cosh\left(\sqrt{3}s\right),\frac{\sqrt{7}}{3},-2\sinh\left(\sqrt{3}s\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
\bar{\beta}_{T^*} &= \frac{\sqrt{3}}{3}\left(\sqrt{7}\sinh\left(\sqrt{3}s\right),0,\sqrt{7}\cosh\left(\sqrt{3}s\right)\right)\\
\bar{\beta}_{N^*} &= \left(\cosh\left(\sqrt{3}s\right),0,\sinh\left(\sqrt{3}s\right)\right)\\
\bar{\beta}_{B^*} &= -2\left(\sinh\left(\sqrt{3}s\right),0,\cosh\left(\sqrt{3}s\right)\right)
\end{aligned}
$$

respectively, (Figs. 1-4).

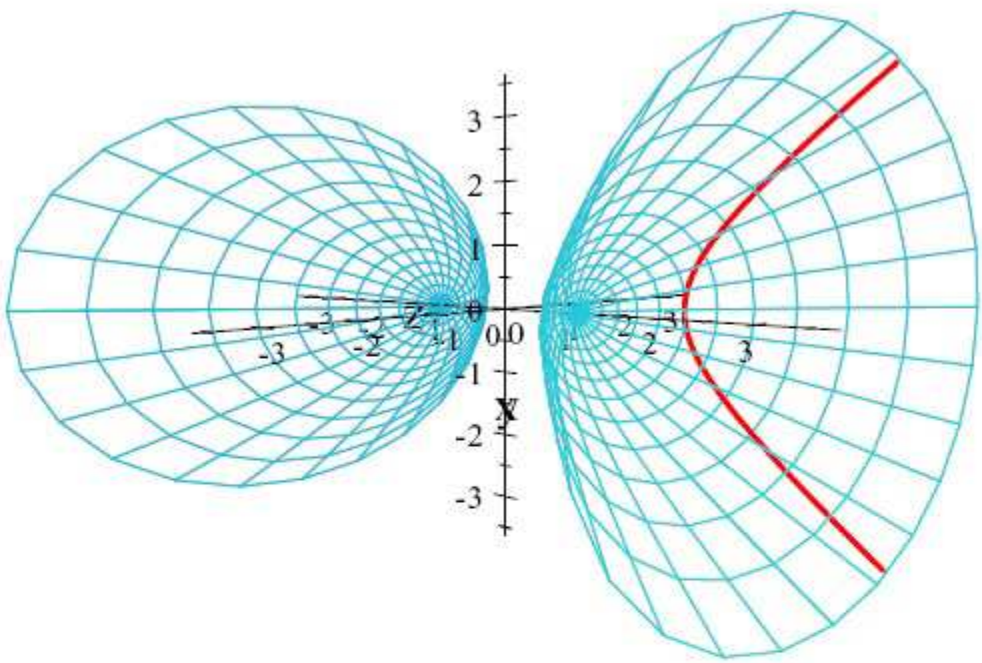

Figure 1. Tangent indicatrix β_{T^*} for Bertrand mate of α on H_0^2

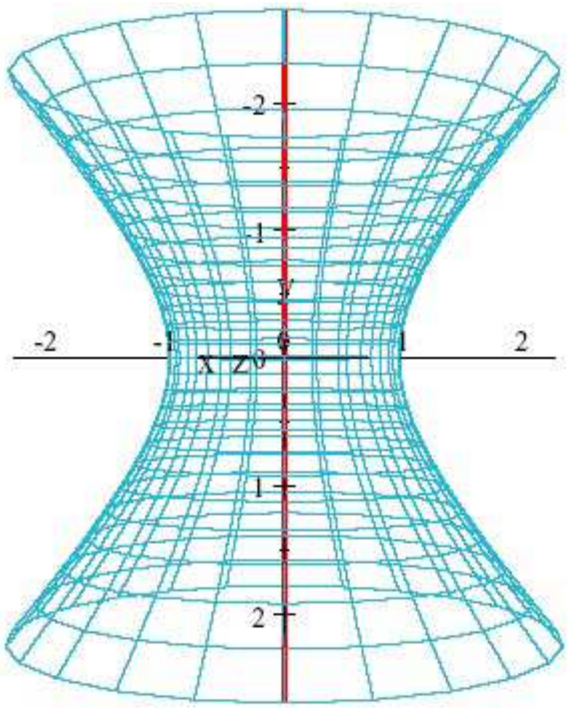

Figure 2. Principal norma indicatrix β_{N^*} for Bertrand mate of α on S_1^2

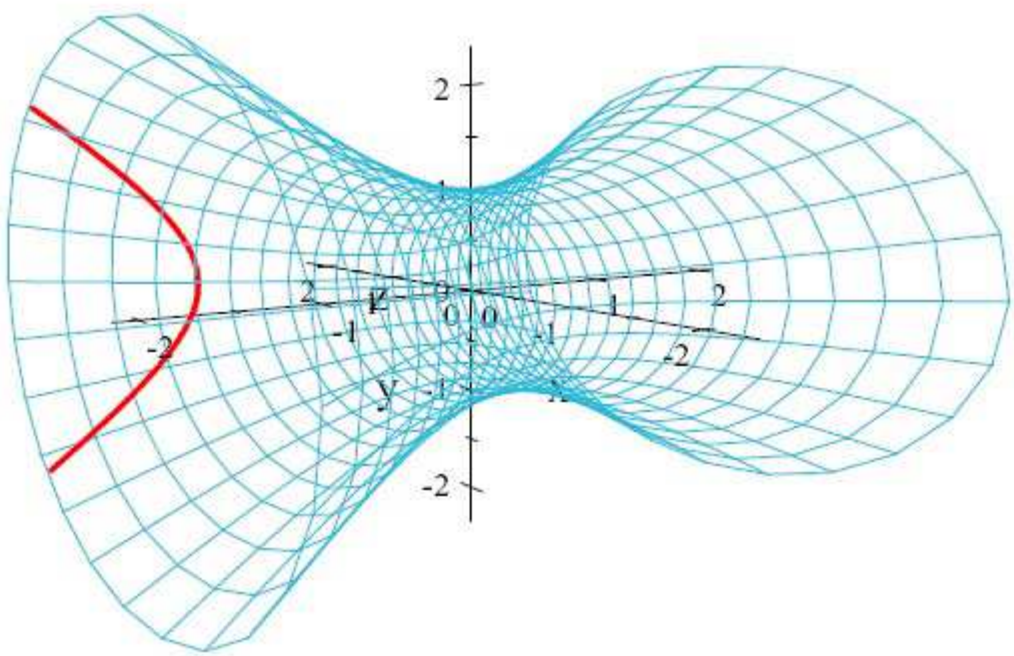

Figure 3. Binormal indicatrix β_{B^*} for Bertrand mate of α on S_1^2

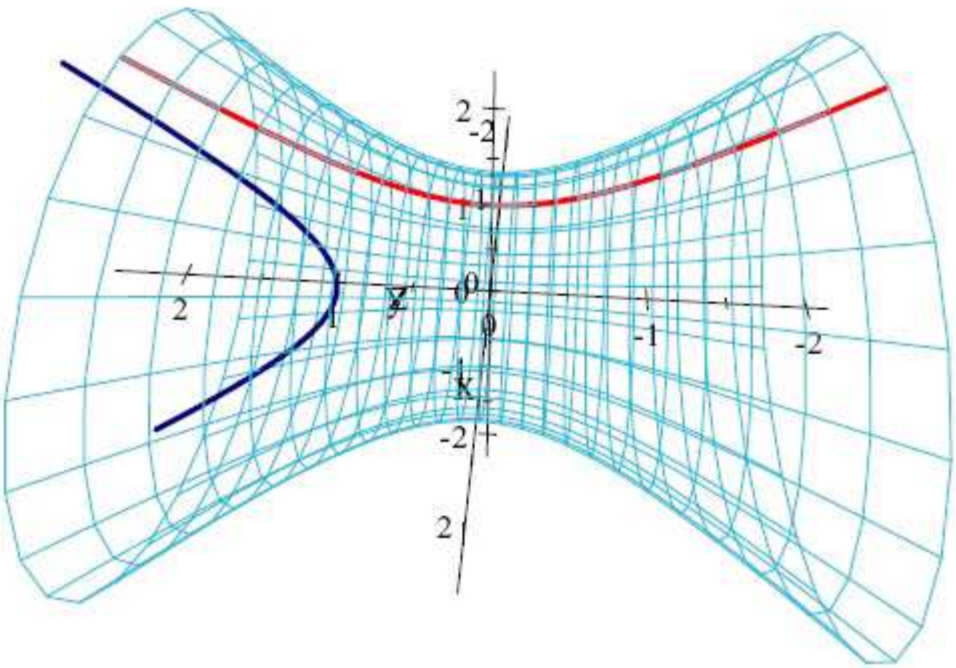

Figure 4. Principal norma indicatrix β_{N^*} and its natural lift curve $\bar{\beta}_{N^*}$ on S_1^2

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