

# Inconsistency of $\mathbb{N}$ from a not-finitist point of view

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## Abstract

Considering the set of natural numbers  $\mathbb{N}$ , then in the context of Peano axioms, we find a fundamental contradiction from a not-finitist point of view.

## 1 Introduction

A formal system, constituted of a grammar, inference rules, axioms, and a reference set, can produce formalized propositions and deductions (theorems) through with a finite number of steps, that is a finitist approach [5] [10].

A system is consistent whether a proposition and its negation are not deduced. Gödel's incompleteness theorems [4], developed on the basis of the system of Principia Mathematica including the axiom of infinity, represent a fortress of logic and consistency against inconsistency. But at the same time they represent a prelude of inconsistency. They give us necessary conditions of consistency, not sufficient ones (undecidable propositions and not demonstrable internal consistency are these necessary conditions).

Considering the successor function  $S(x)$  and the existence and reachability of all natural numbers in concordance with Peano axiom and the axiom of infinity, we show a contradiction in  $\mathbb{N}$ , in a not-finitist and not completely formalized way, that is thinking to take all infinite natural numbers.

## 2 Natural numbers set

The existence of  $\mathbb{N}$  is granted by the axiom of infinity [11] [8] [9]. This existence imply that one of each element of the set, in an actual sense, not time dependent, so taken all together. Then the problem about the existence of natural numbers not yet effectively obtained (very large numbers) is avoided; in fact we could consider only effectively obtained numbers as natural numbers, but because the successor function  $S(x)$  exists, also others not-obtained numbers have to exist. All numbers of the set are defined by Peano axioms [7] [6] [3], together their proprieties thanks to the axiom of induction. In particular we focus attention on successor function  $S(x)$ .

Nevertheless the axiom of infinity, usually natural numbers are thought one by one over time, specially implicitly. But a list of infinite elements is not contemplated (nevertheless its existence?) by a finitist approach to obtain a deduction. In any case we will strive to think of all numbers simultaneously.

## 3 A fundamental contradiction

We show there is a greater natural number than all natural numbers, or equivalently there are fewer natural numbers than natural numbers. Three proofs are given.

1) We know that:  $(x \in \mathbb{N})(\forall x(x < S(x)))$  with  $S(x) \in \mathbb{N}$ , that is, in increasing value:

$$0 < S(0)$$

$$1 < S(1)$$

$$2 < S(2)$$

$$3 < S(3)$$

$$4 < S(4)$$

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**We imagine all natural numbers  $x$  in the left column of previous scheme**, so in the right one there is a greater number ( $S(x)$ ) than all natural numbers, that is  $(x, y \in \mathbb{N})(\exists y \forall x(x < S(y)))$ , in contradiction with  $S(x) \in \mathbb{N}$

(included in Peano axioms).  $\forall y(\forall x((x \leq y) \rightarrow (x < S(y))) \rightarrow \exists y\forall x(x < S(y)))$  represents the previous scheme together the implication in symbols.

**2)** Since the two sets  $\{0, \{S(x)|x \in \mathbb{N}\}\}$  and  $\mathbb{N}$  are the same:  $\{0, \{S(x)|x \in \mathbb{N}\}\} = \mathbb{N}$ , but because

$$\sum_{x=0}^y x < \sum_{x=0}^y S(x) \quad \forall y \quad (\text{with} \quad \sum_{x=0}^y S(x) = 0 + \sum_{x=0}^y S(x)),$$

although there are infinite sums, there is the sum of all natural numbers (otherwise we wouldn't have all x and then all y, in agreement with  $\forall y$ ) and we find that the sum of all elements of a set is less than the other; in a set there are elements that are not in the other. But the set is the same,  $\mathbb{N}$ . Then there are fewer natural numbers than natural numbers, that is a contradiction.

**3)** We know that  $\nexists y\forall x(x \leq y)$  is true. From this, considering all x and all y and the principle of the excluded third, we obtain  $\exists x\forall y(y < x)$ . It is very simple to understand this considering an oriented line with x,y marked points; transposing  $\nexists y\forall x(x \leq y)$  on the line we see  $\exists x\forall y(y < x)$ .

— — — — —  $x_1y_1$  — — — — —  $x_2y_2$  — — — — —  $x_my_m$  — — — — —  $x_n$  — — — — —  $>$

So  $\exists x\forall y(y < x)$ , but because  $x, y \in \mathbb{N}$ , a natural number x cannot be greater than all natural numbers. So we have a contradiction.

These proofs seems to look like to Burali Forti antinomy [2] [1]. But we have only considered the  $\mathbb{N}$  set with its elements, while that one is centered on the "set" (actually named a class) of all ordinal numbers.

## 4 Conclusion

These proofs of inconsistency are not finitist because they need infinite totalities. But can we really summarize the finitist point of view as follows? "A list of infinity elements cannot be considered for deducing, although it has to exist in agreement with the axiom of infinity". Is this not an acceptable point of view and then are these not finitist proofs of inconsistency valid in a general way?

Anyway a view to avoid this inconsistency could be to consider finite sets or not extending  $S(x)$  to all x (so, does infinity exist?), then revisiting Peano axioms.

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