

Goldbach's Conjecture — Towards the Inconsistency of Arithmetic

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Abstract. This paper proves an inconsistency in Peano arithmetic (PA). We express a strengthened form of the strong Goldbach conjecture and its negation by using a specific set that varies according to whether the conjecture or the negation is assumed. We show that, on the other hand, this set remains unchanged under these assumptions. This causes a contradiction.

Notations. Let \mathbb{N} denote the natural numbers starting from 1, let \mathbb{N}_n denote the natural numbers starting from $n > 1$ and let \mathbb{P}_3 denote the prime numbers starting from 3.

Theorem. *PA is contradictory, i.e. the statement FALSE can be derived.*

Proof. Let T be a non-empty subset of $\mathbb{P}_3 \times \mathbb{P}_3$ with $p < q$ for all $(p, q) \in T$. Then, we define $S_T := \{ (pk, mk, qk) \mid k, m \in \mathbb{N}; (p, q) \in T; m = (p + q) / 2 \}$.

Let P_T denote the following proposition.

Every even integer greater than 6 can be expressed as $p + q$ for some $(p, q) \in T$.

Then, P_T is equivalent to saying that every integer $n \geq 4$ is the arithmetic mean of p and q for some $(p, q) \in T$ and so it is equivalent to saying that all integers $n \geq 4$ appear as m in a middle component mk of S_T . So, by the definition of S_T we have

$$P_T \iff \forall n \in \mathbb{N}_4 \exists (pk, mk, qk) \in S_T \quad n = m$$
$$\neg P_T \iff \exists n \in \mathbb{N}_4 \forall (pk, mk, qk) \in S_T \quad n \neq m.$$

We consider the following two conditions.

First, the whole range of \mathbb{N}_3 can be expressed by the triple components of S_T ("covering"), that is,

$$(C) \quad \forall x \in \mathbb{N}_3 \quad \exists (pk, mk, qk) \in S_T \quad x = pk \vee x = mk \vee x = qk.$$

Second, all pairs (p, q) of distinct odd primes are used in the definition of the set S_T ("maximality"), that is,

$$(M) \quad \forall p, q \in \mathbb{P}_3, p < q \quad \forall k \in \mathbb{N} \quad (pk, mk, qk) \in S_T, \text{ where } m = (p + q) / 2.$$

$\neg(C)$ immediately implies $\neg P_T$ since an $n \geq 4$ that is different from all S_T triple components pk, mk and qk is in particular different from all m in S_T . So the condition (C) excludes the possibility that $\neg P_T$ holds because of $\neg(C)$.

The condition (M) excludes the possibility that an $n \geq 4$ different from all m is the arithmetic mean of a pair of primes not used in S_T . So (M) excludes the possibility that $\neg P_T$ holds due to a missing prime number pair.

Now, we define $S_g := S_T$ with $T = \{ (p, q) \in \mathbb{P}_3 \times \mathbb{P}_3 \mid p < q \}$, i.e.,

$$S_g := \{ (pk, mk, qk) \mid k, m \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q; m = (p + q) / 2 \}.$$

Then, the corresponding P_T is the following strengthened form of the strong Goldbach conjecture which we denote by SSGB.

Every even integer greater than 6 can be expressed as the sum of two different primes.

In analogy to P_T and S_T , SSGB is equivalent to saying that all integers $n \geq 4$ appear as m in a middle component mk of S_g .

The set S_g satisfies the conditions (C) and (M). (M) holds by the definition of S_g , and (C) holds because every integer $x \geq 3$ can be written as some pk with $k = 1$ when x is prime, as some pk with $k \neq 1$ when x is composite and not a power of 2, or as $(3 + 5)k / 2$ when x is a power of 2; $p \in \mathbb{P}_3, k \in \mathbb{N}$.

In particular, from the above it follows that $(M) \Rightarrow (C)$.

A few examples of the covering:

$$x = 19: (\mathbf{19\cdot1}, 21\cdot1, 23\cdot1), (\mathbf{19\cdot1}, 60\cdot1, 101\cdot1)$$

$$x = 27: (\mathbf{3\cdot9}, 7\cdot9, 11\cdot9)$$

$$x = 38: (\mathbf{19\cdot2}, 21\cdot2, 23\cdot2)$$

$$x = 42: (\mathbf{3\cdot14}, 5\cdot14, 7\cdot14), (\mathbf{7\cdot6}, 9\cdot6, 11\cdot6)$$

$$x = 4096: (3\cdot1024, \mathbf{4\cdot1024}, 5\cdot1024)$$

$$x = 10000: (\mathbf{5\cdot2000}, 6\cdot2000, 7\cdot2000).$$

The basic idea is now the following.

There are two possibilities for S_g , exactly one of which must occur: Either there is an $n \in \mathbb{N}_4$ in addition to all the numbers m defined in S_g or there is not. The latter is equivalent to SSGB and the former is equivalent to \neg SSGB.

Since, due to (M), an $n \geq 4$ different from all m cannot be the arithmetic mean of a pair of primes not used in S_g and since, due to (C), this n equals a component of some S_g triple that exists by definition, the covering of \mathbb{N}_3 by the S_g triples in the case n exists (\neg SSGB) is equal to that in the case n does not exist (SSGB). This causes a contradiction because in the case SSGB the numbers m defined in S_g take all integer values $x \geq 4$ whereas in the case \neg SSGB they don't.

The following steps are independent of the choice of n if, in the case of \neg SSGB, there is more than one that is different from all m . For example, the minimal such n works.

We split S_g into two complementary subsets in the following way. For any $y \in \mathbb{N}_3$, we write

$$S_g = S_{g+(y)} \cup S_{g-(y)}, \text{ with}$$

$$S_{g+(y)} := \{ (pk, mk, qk) \in S_g \mid \exists k' \in \mathbb{N} \quad pk = yk' \vee mk = yk' \vee qk = yk' \}$$

$$S_{g-(y)} := \{ (pk, mk, qk) \in S_g \mid \forall k' \in \mathbb{N} \quad pk \neq yk' \wedge mk \neq yk' \wedge qk \neq yk' \}.$$

We define

$$S_1 := \{ (pk, mk, qk) \in S_g \mid SSGB \wedge ((C) \wedge (M)) \}$$

$$S_2 := \{ (pk, mk, qk) \in S_g \mid \neg SSGB \wedge ((C) \wedge (M)) \}.$$

Under the assumption $\neg SSGB$ there is an $n \in \mathbb{N}_4$ as described above and under the assumption $SSGB$ there is no such n . Then,

$$\left(\left(\forall y \in \mathbb{N}_3 \quad SSGB \Rightarrow S_1 = S_{g+(y)} \cup S_{g-(y)} \right) \wedge \left(\neg SSGB \Rightarrow S_2 = S_{g+(n)} \cup S_{g-(n)} \right) \right)$$

(1) \vee

$$\left(\neg(C) \vee \neg(M) \right).$$

Since S_g satisfies both (C) and (M), and since $S_{g+(n)} \cup S_{g-(n)}$ is independent of n , we get

$$(1.1) \quad \forall y \in \mathbb{N}_3 \quad SSGB \Rightarrow S_1 = S_{g+(y)} \cup S_{g-(y)}$$

\wedge

$$(1.2) \quad \forall y \in \mathbb{N}_3 \quad \neg SSGB \Rightarrow S_2 = S_{g+(y)} \cup S_{g-(y)}.$$

Now, we will make use of the following principle.

If two sets of (possibly infinitely many) x -tuples are equal, then the sets of their corresponding i -th components are equal; $1 \leq i \leq x$.

To this end, for each $k \in \mathbb{N}$ we define

$$M_1(k) := \{ mk \mid (pk, mk, qk) \in S_1 \}$$

$$M_2(k) := \{ mk \mid (pk, mk, qk) \in S_2 \}.$$

Then, applying the principle above to the middle component of the triples (pk, mk, qk) , $((1.1) \wedge (1.2))$ implies

$$(2.1) \quad \forall k \in \mathbb{N} \quad \forall y \in \mathbb{N}_3 \quad \text{SSGB} \Rightarrow M_1(k) = \{ mk \mid (pk, mk, qk) \in S_{g^+}(y) \cup S_{g^-}(y) \}$$

\wedge

$$(2.2) \quad \forall k \in \mathbb{N} \quad \forall y \in \mathbb{N}_3 \quad \neg \text{SSGB} \Rightarrow M_2(k) = \{ mk \mid (pk, mk, qk) \in S_{g^+}(y) \cup S_{g^-}(y) \}.$$

Setting $M_1 := M_1(1)$ and $M_2 := M_2(1)$, we get

$$(2.1') \quad \forall y \in \mathbb{N}_3 \quad \text{SSGB} \Rightarrow M_1 = \{ m \mid (p, m, q) \in S_{g^+}(y) \cup S_{g^-}(y) \}$$

\wedge

$$(2.2') \quad \forall y \in \mathbb{N}_3 \quad \neg \text{SSGB} \Rightarrow M_2 = \{ m \mid (p, m, q) \in S_{g^+}(y) \cup S_{g^-}(y) \}.$$

Since for every $y \in \mathbb{N}_3$ $S_{g^+}(y) \cup S_{g^-}(y)$ equals S_g by definition, for every $y \in \mathbb{N}_3$ $\{ m \mid (p, m, q) \in S_{g^+}(y) \cup S_{g^-}(y) \}$ equals the set $X := \{ m \mid (p, m, q) \in S_g \}$. So, from $((2.1') \wedge (2.2'))$ we obtain

$$(3) \quad (\text{SSGB} \Rightarrow M_1 = X) \quad \wedge \quad (\neg \text{SSGB} \Rightarrow M_2 = X).$$

The set X is a free variable in (3) that is either equal to \mathbb{N}_4 or to some non-empty proper subset Y of \mathbb{N}_4 .

Now, we make use of the following rule.

Let $P = P(A)$ be a proposition that depends on a set A . Then, for any set B ,

$(\text{we have a proof of } P(A) \wedge \text{we have a proof of } A = B) \Rightarrow \text{we have a proof of } P(B)$.

In the special case that A is a free variable that is replaced by the value B , the above conjunct $(\text{we have a proof of } A = B)$ is trivially true.

Since the set X is a free variable in (3) and since we have a proof of (3), we can apply the above rule with $P = (3)$. If $X = \mathbf{N}_4$ we use the rule with $A = X$ and $B = \mathbf{N}_4$, and if $X = Y$ we use it with $A = X$ and $B = Y$. Then, since either $X = \mathbf{N}_4$ or $X = Y$, from (3) we obtain

(3.1) we have a proof of $(\text{SSGB} \Rightarrow M_1 = \mathbf{N}_4 \quad \wedge \quad \neg\text{SSGB} \Rightarrow M_2 = \mathbf{N}_4)$

\vee

(3.2) we have a proof of $(\text{SSGB} \Rightarrow M_1 = Y \neq \mathbf{N}_4 \quad \wedge \quad \neg\text{SSGB} \Rightarrow M_2 = Y \neq \mathbf{N}_4)$.

This implies

(3.1') (we have a proof of $(\text{SSGB} \Rightarrow M_1 = \mathbf{N}_4)$

\wedge
we have a proof of $(\neg\text{SSGB} \Rightarrow M_2 = \mathbf{N}_4))$

\vee

(3.2') (we have a proof of $(\text{SSGB} \Rightarrow M_1 = Y \neq \mathbf{N}_4)$

\wedge
we have a proof of $(\neg\text{SSGB} \Rightarrow M_2 = Y \neq \mathbf{N}_4))$.

Now, we will establish a contradiction to $((3.1') \vee (3.2'))$.

Under the assumption SSGB the set $X = \{ m \mid (p, m, q) \in S_g \}$ is equal to \mathbf{N}_4 and under $\neg\text{SSGB}$ it is equal to $Y \neq \mathbf{N}_4$. Therefore,

(4.1) we have a proof of $(\text{SSGB} \Rightarrow M_1 = \mathbf{N}_4)$

\wedge

(4.2) we have a proof of $(\neg\text{SSGB} \Rightarrow M_2 = Y \neq \mathbf{N}_4)$.

Then, $((3.1') \vee (3.2'))$ together with $((4.1) \wedge (4.2))$ implies

(5.1) we have a proof of $(\neg \text{SSGB} \Rightarrow M_2 = \mathbb{N}_4)$

\vee

(5.2) we have a proof of $(\text{SSGB} \Rightarrow M_1 = Y \neq \mathbb{N}_4)$.

Because of $((4.1) \wedge (4.2))$ and because

$\text{SSGB} \Rightarrow M_2 = \{\} \neq \mathbb{N}_4$

and

$\neg \text{SSGB} \Rightarrow M_1 = \{\} \neq Y$,

we have a proof that $(M_2 = \mathbb{N}_4)$ is false and we have a proof that $(M_1 = Y \neq \mathbb{N}_4)$ is false.

So, $((5.1) \vee (5.2))$ yields

(6.1) we have a proof of SSGB

\vee

(6.2) we have a proof of $\neg \text{SSGB}$.

Since we have neither a proof of SSGB nor of $\neg \text{SSGB}$, both (6.1) and (6.2) are false.

Therefore, we obtain $(\text{FALSE} \vee \text{FALSE})$ and thus **FALSE**.

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