World Journal of Advanced Research and Reviews
eISSN: 2581-9615 CODEN (USA): WJARAI
Cross Ref DOI: 10.30574/wjarr
Journal homepage: https://wjarr.com/

(Review Article)

The number of smallest parts of *Partitions* of *n*

A Manjusree [1,*] and Panuganti Srinivasa Sai [2]

[1] *Department of Mathematics, Hindu college, Acharya Nagarjuna University, Guntur, A.P-522002, India.*
[2] *Department of Mathematics, S. S and N College, Narasaraopet, Palnadu Dist, AP, India.*

World Journal of Advanced Research and Reviews, 2023, 17(01), 119–125

Publication history: Received on 06 November 2022; revised on 21 December 2022; accepted on 24 December 2022

Article DOI: https://doi.org/10.30574/wjarr.2023.17.1.1390

Abstract

George E Andrews derived formula for the number of smallest parts of partitions of a positive integer *n*. In this paper we derived the generating function for the number of smallest parts of all partitions of *n* utilizing r-partitions of *n*. We also derive the generating function for Ac(*n*) , the number of smallest parts of the partitions of *n* which are multiples of c and also to evaluate the sum of smallest parts of partitions of *n* by applying the concept of r-partitions of *n*.

Keywords: Partition; r-partition; Smallest parts of the partition; r-partition of positive integer *n*

1. Introduction

We adopt mostly the common notation on *partitions* used by Andrews [1] as given below.

A *partition* of a positive integer n is a finite non-increasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$ and is denoted by $n = (\lambda_1, \lambda_2, \ldots, \lambda_r), n = \lambda_1 + \lambda_2 + \lambda_3 + \ldots \lambda_r$ or $\lambda = (\lambda_1^{f_1}, \lambda_2^{f_2}, \lambda_3^{f_3}, \ldots)$ when λ_1 repeats f_1 times, λ_2 repeats f_2 times and so on. The λ_i are called the parts of the *partition*. In what follows λ stands for a *partition* of n , $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r), \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$.

$$p(n) = \begin{cases} 1 & \text{if } n = 0, \\ \text{number of } \textit{partitions} \text{ of } n & \text{if } n \in N, \\ 0 & \text{if } n \text{ is negative.} \end{cases}$$

1.1. Ex: partitions of 7.

7	6+1	5+1+1	4+1+1+1	3+1+1+1+1	2+1+1+1+1+1	2+1+1+1+1+1+1
	5+2	4+2+1	3+2+1+1	2+2+1+1+1		
	4+3	3+3+1	2+2+2+1			
		3+2+2				

*Corresponding author: Dr A Manjusree
Acharya Nagarjuna university , Department of Mathematics,Hindu College,Guntur, Andhra Pradesh INDIA.

If $1 \le r \le n$ then $\xi_r(n)$ is the set of partitions of *n* with *r* parts and its cardinality is denoted by $p_r(n)$. A *partition* of *n* with exactly *r* parts is called *r – partition* of *n*. We define

$$p_r(n) = \begin{cases} 0 & \text{if} \quad r = 0 \quad or \quad r > n \\ \text{number of } r - partitions \text{ of } n & \text{if} \quad 0 < r \le n \end{cases}$$

If $1 \le r \le n$ then $\xi_r(n)$ is the set of *partitions* of *n* with *r* parts and its cardinality is denoted by $p_r(n)$. A *partition* of *n* with exactly *r* parts is called *r – partition* of *n*. We define

$$p_r(n) = \begin{cases} 0 & \text{if} \quad r = 0 \quad or \quad r > n \\ \text{number of } r - partitions \text{ of } n & \text{if} \quad 0 < r \le n \end{cases}$$

1.2. Ex: 3-partitions of 8

$$6+1+1, \quad 5+2+1, \quad 4+3+1, \quad 4+2+2, \quad 3+3+2.$$

- $spt(n)$ denotes the number of smallest parts including repetitions in all *partitions* of n .
- $r - spt_i(n)$ denotes the number of i^{th} smallest parts in all $r - partitions$ of *n.*

Verification of the above illustration by $3-$ *partitions* of 10 having second smallest parts.

$$\underline{8}+1+1, \quad 7+\underline{2}+1, \quad 6+\underline{3}+1, \quad \underline{6}+2+2, \quad 5+\underline{4}+1, \quad 5+\underline{3}+2, \quad \underline{4}+\underline{4}+2, \quad \underline{4}+3+3.$$

2. Generating function for *spt(n)*

The generating function for the number of smallest parts of all *partitions* of positive integer *n* is derived by Andrews [2]. By utilizing $r - partitions$ of *n*, we present a formula for finding the number of smallest parts of *n.*

2.1. Theorem

$$spt(n) = \sum_{k=1}^{\infty}\sum_{t=1}^{\infty} p(k, n-tk) + d(n)$$

Where $d(n)$ is the number of positive divisors of n

2.1.1. Proof

Let $n = (\lambda_1, \lambda_2, \ldots, \lambda_r) = \left(\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, \ldots, \mu_{l-1}^{\alpha_{l-1}}, k^{\alpha_l}\right)$ be any $r - partition$ of n with *l* distinct parts.(1)

Case 1

Let $r > \alpha_l = t$ which implies $\lambda_{r-t} > k$

Subtract all $k's$, we get $n - tk = \left(\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, \ldots, \mu_{l-1}^{\alpha_{l-1}}\right)$

Hence $n - tk = \left(\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, \ldots, \mu_{l-1}^{\alpha_{l-1}}\right)$ is a $(r-t) - partition$ of $n - tk$ with l - 1 distinct parts and each part is greater than or equal to $k+1$.

Therefore the number of $r-partitions$ with smallest part *k* that occurs exactly *t* times among all $r-partitions$ of *n* is $p_{r-t}(k+1, n-tk)$

Case 2

Let $r > \alpha_l > t$ which implies $\lambda_{r-t} = k$

Omit $k's$ from last t places, we get $n-tk = \left(\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, \ldots, \mu_{l-1}^{\alpha_{l-1}}, k^{\alpha_l - t}\right)$

Hence $n-tk = \left(\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, \ldots, \mu_{l-1}^{\alpha_{l-1}}, k^{\alpha_l - t}\right)$ is a $(r-t)-partition$ of $n-tk$ with l distinct parts and the least part is k .

Now we get the number of $r-partitions$ with smallest part *k* that occurs more than *t* times among all $r-partitions$ of *n* is $f_{r-t}(k, n-tk)$.

Case 3

Let $r = \alpha_l = t$ which implies all parts in the *partition* are equal.

The number of *partitions* of *n* with equal parts is β which is equal to thenumber of positive divisors of *n*. Since the positive number of divisors of *n* is $d(n)$, the number of $partitions$ of *n* with all parts are equal is $d(n)$.

$$\text{where} \quad \beta = \begin{cases} 1 & \text{if } r \mid n \\ 0 & \text{otherwise} \end{cases}$$

From cases (1), (2) and (3) we get $r-partitions$ of *n* with smallest part *k* that occurs atleast *t* times is

$$f_{r-t}(k, n-tk) + p_{r-t}(k+1, n-tk) + \beta$$

$$= p_{r-t}(k, n-tk) + \beta \tag{2}$$

From [2], the number of smallest parts in $partitions$ of *n* is

$$spt(n) = \sum_{k=1}^{\infty} \sum_{t=1}^{\infty} p(k, n-tk) + d(n). \quad \blacksquare$$

The above illustration is verified from the following *partitions* of 8.

$\underline{8},\ 7+\underline{1},\ 6+\underline{2},\ 5+\underline{3},\ \underline{4}+\underline{4},\ 6+\underline{1}+\underline{1},\ 5+2+\underline{1},\ 4+3+\underline{1},\ 4+\underline{2}+\underline{2},\ 3+3+\underline{2},\ 5+\underline{1}+\underline{1}+\underline{1},$

$4+2+\underline{1}+\underline{1},\ 3+3+\underline{1}+\underline{1},\ 3+2+2+\underline{1},\ \underline{2}+\underline{2}+\underline{2}+\underline{2},\ 4+\underline{1}+\underline{1}+\underline{1}+\underline{1},\ 3+2+\underline{1}+\underline{1}+\underline{1},$

$2+2+2+\underline{1}+\underline{1},\ 3+\underline{1}+\underline{1}+\underline{1}+\underline{1}+\underline{1},\ 2+2+\underline{1}+\underline{1}+\underline{1}+\underline{1},\ 2+\underline{1}+\underline{1}+\underline{1}+\underline{1}+\underline{1}+\underline{1},$

$\underline{1}+\underline{1}+\underline{1}+\underline{1}+\underline{1}+\underline{1}+\underline{1}+\underline{1}.$

2.2. Theorem

$$p_r(k+1,n) = p_r(n-kr) \quad (3)$$

2.2.1. Proof

Let $n=(\lambda_1,\lambda_2,\ldots,\lambda_r),\lambda_i>k \quad \forall i$ be any $r-partition$ of n .

Subtracting k from each part, we get $n-kr=(\lambda_1-k,\lambda_2-k,\ldots,\lambda_r-k)$

Hence $n-kr=(\lambda_1-k,\lambda_2-k,\ldots,\lambda_r-k)$ is a $r-partition$ of $n-kr$.

Therefore the number of $r-partitions$ of n with parts greater than or equal to $k+1$ is $p_r(n-kr)$.

2.3. Illustration

Let $n=9,\quad k=2$ and $r=3$.

$$\xi_r(k+1,n)=\xi_3(2+1,9)=\xi_3(3,9)=\{3+3+3\}$$

$$\text{Hence } p_3(3,9)=1$$

$$\xi_r(n-kr)=\xi_3(9-2.3)=\xi_3(3)=\{1+1+1\}$$

$$\text{Hence } p_3(3)=1$$

$$\text{Hence } p_r(k+1,n)=p_r(n-kr).$$

Further we also derive generating function for the number of smallest parts of all *partitions*of *n* utilizing $r-partitions$ of *n*.

2.4. Theorem

$$\sum_{n=1}^{\infty} spt(n)q^n = \frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{q^n(q)_{n-1}}{(1-q^n)}.$$

2.4.1. Proof

From theorem (1) we have

$$spt(n)=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}p(k,n-tk)+d(n)$$

where $d(n)$ is the number of positive divisors of n.

$$=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}p_r(k,n-tk)+d(n) \qquad \text{since } p(n)=\sum_{r=1}^{\infty}p_r(n)$$

First replace $k+1$ by k, then replace n by $n-tk$ in (3)

$$=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}p_r(n-tk-r(k-1))+d(n)$$

From (4)

$$\sum_{n=1}^{\infty} spt(n)q^n = \sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{q^{r+tk+r(k-1)}}{(q)_r} + \sum_{k=1}^{\infty}\frac{q^k}{1-q^k} \qquad \text{since } \frac{n}{r}=k$$

$$= \sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{q^{tk+rk}}{(q)_r} + \sum_{k=1}^{\infty}\frac{q^k}{1-q^k}$$

$$= \sum_{k=1}^{\infty}\sum_{t=1}^{\infty}q^{tk}\left[\sum_{r=1}^{\infty}\frac{\left(q^k\right)^r}{(q)_r}\right] + \sum_{k=1}^{\infty}\frac{q^k}{1-q^k}$$

$$= \sum_{k=1}^{\infty}\frac{q^k}{\left(1-q^k\right)}\left[\left(1+\sum_{r=1}^{\infty}\frac{\left(q^k\right)^r}{(q)_r}\right)-1\right] + \sum_{k=1}^{\infty}\frac{q^k}{1-q^k}$$

$$= \sum_{k=1}^{\infty}\frac{q^k}{\left(1-q^k\right)}\left(1+\sum_{r=1}^{\infty}\frac{\left(q^k\right)^r}{(q)_r}\right)$$

$$= \sum_{k=1}^{\infty}\frac{q^k}{\left(1-q^k\right)}\prod_{r=0}^{\infty}\left(\frac{1}{1-q^r q^k}\right) \qquad \text{from [1]}$$

$$= \sum_{k=1}^{\infty}\frac{q^k}{\left(1-q^k\right)}\prod_{r=0}^{\infty}\left(\frac{1}{1-q^{r+k}}\right)$$

$$= \frac{1}{(q)_\infty}\sum_{k=1}^{\infty}\frac{q^k (q)_{k-1}}{\left(1-q^k\right)}$$

$$= \frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{q^n (q)_{n-1}}{\left(1-q^n\right)}$$

3. Corollary

The generating function for $A_c(n)$, the number of smallest parts of the *partitions* of n which are multiples of c is

$$\sum_{n=1}^{\infty} A_c(n)q^n = \frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{q^{cn}(q)_{cn-1}}{\left(1-q^{cn}\right)}$$

To evaluate the sum of smallest parts of *partitions* of *n* by applying the concept of *r-partitions* of *n*, we propose the following theorem.

3.1. Theorem

The generating function for the sum of smallest parts of the *partitions* of n is

$$\sum_{n=1}^{\infty} sum\,spt(n)q^n = \frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{nq^n(q)_{n-1}}{(1-q^n)}$$

3.1.1. Proof

From [3] we have the sum of smallest parts $sum\ spt(n)$ of the *partitions* of a positive integer n is

$$sum\ spt(n) = \sum_{k=1}^{\infty}\sum_{t=1}^{\infty} k \quad p(k, n-tk) + \sigma(n)$$

where $\sigma(n)$ is sum of positive divisors of n.

$$= \sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty} k \quad p_r(k, n-tk)$$

First replace $k+1$ by k, then replace n by $n-tk$ in (3)

$$= \sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty} k\, p_r\big(n-tk-r(k-1)\big) + \sigma(n)$$

Hence the generating function for the sum of smallest parts of the *partitions* of a positive integer n is

$$\sum_{n=1}^{\infty} sum\,spt(n)q^n = \sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{kq^{r+tk+r(k-1)}}{(q)_r} + \sum_{k=1}^{\infty}\frac{kq^k}{1-q^k} \text{ from [4]}$$

$$= \sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{kq^{tk+rk}}{(q)_r} + \sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}$$

$$= \sum_{k=1}^{\infty}\sum_{t=1}^{\infty} kq^{tk}\left[\sum_{r=1}^{\infty}\frac{\left(q^k\right)^r}{(q)_r}\right] + \sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}$$

$$= \sum_{k=1}^{\infty}\sum_{t=1}^{\infty} kq^{tk}\left[1+\sum_{r=1}^{\infty}\frac{\left(q^k\right)^r}{(q)_r}-1\right] + \sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}$$

$$= \sum_{k=1}^{\infty}\frac{kq^k}{\left(1-q^k\right)}\prod_{r=0}^{\infty}\left(\frac{1}{1-q^r q^k}\right)$$

$$= \sum_{k=1}^{\infty}\frac{kq^k}{\left(1-q^k\right)}\prod_{r=0}^{\infty}\left(\frac{1}{1-q^{r+k}}\right)$$

$$= \frac{1}{(q)_\infty} \sum_{k=1}^{\infty} \frac{kq^k (q)_{k-1}}{(1-q^k)}$$

$$= \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} \frac{nq^n (q)_{n-1}}{(1-q^n)}$$

$$\sum_{n=1}^{\infty} sum\, spt(n) q^n = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} \frac{nq^n (q)_{n-1}}{(1-q^n)} \quad \blacksquare$$

4. Conclusion

The formula for the number of smallest parts of partitions of a positive integer *n* was first derived by George E Andrews. In the present article we derived the generating function for the number of smallest parts of all partitions of *n* utilizing r-partitions of *n*. We also derived the generating function for Ac(n) , the number of smallest parts of the partitions of *n* which are multiples of c and also to evaluate the sum of smallest parts of partitions of *n* by applying the concept of r-partitions of *n*.

There are many such applications. A straightforward one is that partitions can be used in statistical mechanics to count available states to many-particle bosonic/fermionic systems and in the calculation of their "partition" functions.

Compliance with ethical standards

Acknowledgments

Authors are thankful to Dr. K. Hanuma Reddy for his suggestions during r-partitions

Disclosure of conflict of interest

We both assure the publishers that Dr. A. Manjusree will be the main author and she will followup all the correspondence. We also disclose that we are bound to the rules and regulations of World Journal of Advanced Research and Reviews.

References

[1] Andrews G. E (1976), The Theory of Partitions, Vol. 2, Encycl. of Math. and Its Appl, Addison-Wesley, Reading. (re printed: Cambridge University Press, 1998).

[2] Andrews, G. E. (2007) The number of smallest parts in the partitions of n, J. ReineAngew. Math., to appear.

[3] Hanuma Reddy K. (2010), A Note on r – partitions of n in which the least part is k, International Journal of Computational Mathematical Ideas, 2, 1, pp. 6-12.

[4] Hanuma Reddy K. (2010), A Study of r – partitions, submitted to Acharya Nagarjuna University, awarded of Ph.D. in Mathematics.