Goldbach's Conjecture — A Route to the Inconsistency of Arithmetic

Ralf Wüsthofen

Abstract. This paper proves an inconsistency in Peano arithmetic (PA). We express two propositions — a strengthened form of the strong Goldbach conjecture and its negation — using a specific set that varies depending on these propositions. On the other hand, we show that this set remains unchanged under the assumptions of the two statements. This causes a contradiction.

Notations. Let $\mathbb{N}$ denote the natural numbers starting from 1, let $\mathbb{N}_n$ denote the natural numbers starting from n > 1 and let $\mathbb{P}_3$ denote the prime numbers starting from 3.

Strengthened strong Goldbach conjecture (SSGB): *Every even integer greater than 6 can be expressed as the sum of two different primes.*

Theorem. *PA is contradictory, i.e. the statement FALSE can be derived.*

Proof. We define the set $S_g := \{ (pk, mk, qk) \mid k, m \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q; m = (p + q) / 2 \}$.

SSGB is equivalent to saying that every integer n ≥ 4 is the arithmetic mean of two different odd primes and so it is equivalent to saying that all integers n ≥ 4 appear as m in a middle component mk of S_g. So, by the definition of S_g we have

$$\text{SSGB} \iff \forall n \in \mathbb{N}_4 \;\; \exists (pk, mk, qk) \in S_g \quad n = m$$

$$\neg\text{SSGB} \iff \exists n \in \mathbb{N}_4 \;\; \forall (pk, mk, qk) \in S_g \quad n \neq m.$$

The set S_g has the following two properties.

First, the whole range of $\mathbb{N}_3$ can be expressed by the triple components of S_g ("*covering*"), because every integer x ≥ 3 can be written as some pk with k = 1 when x is prime, as some pk with k ≠ 1 when x is composite and not a power of 2, or as $(3 + 5)k / 2$ when x is a power of 2; $p \in \mathbb{P}_3$, $k \in \mathbb{N}$. So we have

(C) $\forall x \in \mathbb{N}_3 \;\; \exists (pk, mk, qk) \in S_g \quad x = pk \;\vee\; x = mk.$

A few examples of the covering:

x = 19: (**19·1**, 21·1, 23·1), (**19·1**, 60·1, 101·1)

x = 27: (**3·9**, 7·9, 11·9)

x = 38: (**19·2**, 21·2, 23·2)

x = 42: (**3·14**, 5·14, 7·14), (**7·6**, 9·6, 11·6)

x = 4096: (3·1024, **4·1024**, 5·1024)

x = 10000: (**5·2000**, 6·2000, 7·2000).

Second, according to the statement SSGB, all pairs (p, q) of distinct odd primes are used in the definition of the set S_g ("*maximality*"). So we have

(M) $\forall$ p, q $\in \mathbb{P}_3$, p < q $\forall$ k $\in \mathbb{N}$ (pk, mk, qk) $\in S_g$, where m = (p + q) / 2.

The property (C) excludes the possibility that there is an n ≥ 4 different from all m, i.e. ¬SSGB, for the reason that n is different from all S_g triple components pk and mk.

The property (M) excludes the possibility that there is an n ≥ 4 different from all m, i.e. ¬SSGB, for the reason that n is the arithmetic mean of a pair of primes not used in S_g. Thus (M) excludes the possibility that ¬SSGB applies due to a missing prime number pair. The proof would no longer be possible if we left out any prime number pair in the formulation of SSGB and S_g.

Therefore, in the case of ¬SSGB, neither ¬(C) nor ¬(M) applies.

The basic idea is now the following.

There are two possibilities for S_g, exactly one of which must occur: Either there is an n $\in \mathbb{N}_4$ in addition to all the numbers m defined in S_g or there is not. The latter is equivalent to SSGB and the former is equivalent to ¬SSGB.

Since, due to (M), an n ≥ 4 different from all m cannot be the arithmetic mean of a pair of primes not used in S_g and since, due to (C), this n equals a component of some S_g triple that exists by definition, the covering of $\mathbb{N}_3$ by the S_g triples in the case n exists (¬SSGB) is equal to that in the case n does not exist (SSGB). This causes a contradiction because in the case SSGB the numbers m defined in S_g take all integer values x ≥ 4 whereas in the case ¬SSGB they don't.

The following steps are independent of the choice of n if, in the case of $\neg$SSGB, there is more than one that is different from all m. For example, the minimal such n works.

We split S_g into two complementary subsets in the following way. For any $y \in \mathbb{N}_3$, we write

$S_g = S_g+(y) \cup S_g-(y)$, with

$S_g+(y) := \{ (pk, mk, qk) \in S_g \mid \exists\, k' \in \mathbb{N} \;\; pk = yk' \;\vee\; mk = yk' \;\vee\; qk = yk' \}$

$S_g-(y) := \{ (pk, mk, qk) \in S_g \mid \forall\, k' \in \mathbb{N} \;\; pk \neq yk' \;\wedge\; mk \neq yk' \;\wedge\; qk \neq yk' \}$.

We define

$S_1 := \{ (pk, mk, qk) \in S_g \mid \;\; \text{SSGB} \}$

$S_2 := \{ (pk, mk, qk) \in S_g \mid \neg\text{SSGB} \;\wedge\; ((C) \wedge (M)) \}$.

Under the assumption SSGB there is no $n \in \mathbb{N}_4$ in addition to all m defined in S_g. Then, under this assumption, we can choose any $y \in \mathbb{N}_3$ such that $S_g = S_g+(y) \cup S_g-(y)$. So,

(1.1) $\forall\, y \in \mathbb{N}_3 \qquad \text{SSGB} \Rightarrow S_1 = S_g+(y) \cup S_g-(y)$.

Under the assumption $\neg$SSGB there is an $n \in \mathbb{N}_4$ as described above. Then, since the possibilities $\neg$(C) and $\neg$(M) for the existence of n are ruled out,

(1.2) $\neg\text{SSGB} \Rightarrow S_2 = S_g+(n) \cup S_g-(n)$.

$S_g+(n) \cup S_g-(n)$ is independent of n. Hence, from (1.2) we get

(1.2') $\forall\, y \in \mathbb{N}_3 \qquad \neg\text{SSGB} \Rightarrow S_2 = S_g+(y) \cup S_g-(y)$.

Now, we will make use of the following principle.

If two sets of (possibly infinitely many) x-tuples are equal, then the sets of their corresponding i-th components are equal; $1 \le i \le x$.

To this end, for each $k \in \mathbb{N}$ we define

$M_1(k) := \{ mk \mid (pk, mk, qk) \in S_1 \}$

$M_2(k) := \{ mk \mid (pk, mk, qk) \in S_2 \}$.

Then, applying the principle above to the middle component of the triples (pk, mk, qk), ((1.1) $\wedge$ (1.2')) implies

(2.1) $\forall k \in \mathbb{N} \quad \forall y \in \mathbb{N}_3 \qquad \text{SSGB} \Rightarrow M_1(k) = \{ mk \mid (pk, mk, qk) \in S_g+(y) \cup S_g-(y) \}$

$\wedge$

(2.2) $\forall k \in \mathbb{N} \quad \forall y \in \mathbb{N}_3 \qquad \neg\text{SSGB} \Rightarrow M_2(k) = \{ mk \mid (pk, mk, qk) \in S_g+(y) \cup S_g-(y) \}$.

Setting $M_1 := M_1(1)$ and $M_2 := M_2(1)$, we get

(2.1') $\forall y \in \mathbb{N}_3 \qquad \text{SSGB} \Rightarrow M_1 = \{ m \mid (p, m, q) \in S_g+(y) \cup S_g-(y) \}$

$\wedge$

(2.2') $\forall y \in \mathbb{N}_3 \qquad \neg\text{SSGB} \Rightarrow M_2 = \{ m \mid (p, m, q) \in S_g+(y) \cup S_g-(y) \}$.

Since for every $y \in \mathbb{N}_3$ $S_g+(y) \cup S_g-(y)$ equals S_g by definition, for every $y \in \mathbb{N}_3$ $\{ m \mid (p, m, q) \in S_g+(y) \cup S_g-(y) \}$ equals the set $X := \{ m \mid (p, m, q) \in S_g \}$. So, from ((2.1') $\wedge$ (2.2')) we obtain

(3) $(SSGB \Rightarrow M_1 = X) \wedge (\neg SSGB \Rightarrow M_2 = X)$.

The set X is a free variable in (3) that is either equal to $\mathbb{N}_4$ or to some non-empty proper subset Y of $\mathbb{N}_4$.

Now, we make use of the following rule.

Let P = P(A) be a proposition that depends on a set A. Then, for any set B,

(we have a proof of P(A) $\wedge$ we have a proof of A = B) => we have a proof of P(B).

In the special case that A is a free variable that is replaced by the value B, the above conjunct (we have a proof of A = B) is trivially true.

Since the set X is a free variable in (3) and since we have a proof of (3), we can apply the above rule with P = (3). If $X = \mathbb{N}_4$ we use the rule with A = X and $B = \mathbb{N}_4$, and if X = Y we use it with A = X and B = Y. Then, since either $X = \mathbb{N}_4$ or X = Y, from (3) we obtain

(3.1) we have a proof of $(SSGB \Rightarrow M_1 = \mathbb{N}_4 \wedge \neg SSGB \Rightarrow M_2 = \mathbb{N}_4)$

$\vee$

(3.2) we have a proof of $(SSGB \Rightarrow M_1 = Y \neq \mathbb{N}_4 \wedge \neg SSGB \Rightarrow M_2 = Y \neq \mathbb{N}_4)$.

This implies

(3.1') (we have a proof of $(SSGB \Rightarrow M_1 = \mathbb{N}_4)$
$\wedge$
we have a proof of $(\neg SSGB \Rightarrow M_2 = \mathbb{N}_4)$)

$\vee$

(3.2') (we have a proof of $(SSGB \Rightarrow M_1 = Y \neq \mathbb{N}_4)$
$\wedge$
we have a proof of $(\neg SSGB \Rightarrow M_2 = Y \neq \mathbb{N}_4)$).

Now, we will establish a contradiction to ((3.1') $\vee$ (3.2')).

Under the assumption SSGB the set $X = \{ m \mid (p, m, q) \in S_g \}$ is equal to $\mathbb{N}_4$ and under $\neg$SSGB it is equal to $Y \neq \mathbb{N}_4$. Therefore,

(4.1) we have a proof of (SSGB => $M_1 = \mathbb{N}_4$)

$\wedge$

(4.2) we have a proof of ($\neg$SSGB => $M_2 = Y \neq \mathbb{N}_4$).

Then, ((3.1') $\vee$ (3.2')) together with ((4.1) $\wedge$ (4.2)) implies

(5.1) we have a proof of ($\neg$SSGB => $M_2 = \mathbb{N}_4$)

$\vee$

(5.2) we have a proof of (SSGB => $M_1 = Y \neq \mathbb{N}_4$).

Because of ((4.1) $\wedge$ (4.2)) and because

SSGB => $M_2 = \{ \} \neq \mathbb{N}_4$

and

$\neg$SSGB => $M_1 = \{ \} \neq Y$,

we have a proof that ($M_2 = \mathbb{N}_4$) is false and we have a proof that ($M_1 = Y \neq \mathbb{N}_4$) is false. So, ((5.1) $\vee$ (5.2)) yields

(6.1) we have a proof of SSGB

$\vee$

(6.2) we have a proof of $\neg$SSGB.

Since we have neither a proof of SSGB nor of $\neg$SSGB, both (6.1) and (6.2) are false.

Therefore, we obtain (FALSE $\vee$ FALSE) and thus FALSE.

□