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Two Fractional Integrals Involving Fractional Tangent Function

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***Abstract*: In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions, we study two fractional integrals involving fractional tangent function. We can obtain the exact solutions of these two fractional integrals by using some techniques. Moreover, our results are generalizations of the results of ordinary calculus.**

***Keywords*: Jumarie type of R-L fractional calculus, new multiplication, fractional analytic functions, fractional integrals, fractional tangent function.**

I. INTRODUCTION

Fractional calculus is a research hotspot in recent years. The application of fractional calculus in many fields such as numerical analysis, physics and engineering has aroused great interest [1-16]. In the first half of the 19th century, Abel, Liouville and Riemann correctly introduced fractional integral and derivative in the analysis. However, the use of generalized differential and integral operators became more familiar in the last decades of the 19th century because of the symbolic calculus of Heaviside and the work of mathematicians such as Hadamard, Hardy and Littlewood, M. Riesz, and H. Weyl.

Fractional calculus is different from traditional calculus. The definition of fractional derivative is not unique. Common definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [17-22]. Because Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions, we study the following two fractional integrals involving fractional tangent function:

$$\left({}_{p}I_{x}^{\alpha}\right)\left[[tan_{\alpha}(x^{\alpha})]^{\otimes_{\alpha}\left(\frac{1}{2}\right)}\right],$$

and

$$\left({}_{p}I_{x}^{\alpha}\right)\left[[tan_{\alpha}(x^{\alpha})]^{\otimes_{\alpha}\left(-\frac{1}{2}\right)}\right].$$

Where $0<\alpha\leq 1$, and p is a real number. Using some methods, the exact solutions of these two fractional integrals can be obtained. On the other hand, our results are generalizations of the results of traditional calculus.

II. PRELIMINARIES

Firstly, we introduce the fractional calculus used in this paper.

Definition 2.1 ([23])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ , \tag{1}$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \tag{2}$$

where $\Gamma(\ \ )$ is the gamma function.

In the following, some properties of Jumarie type of R-L fractional derivative are introduced.

Proposition 2.2 ([24])**:** *If* α,β,x_0,c *are real numbers and* $\beta\geq\alpha>0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right]=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[c]=0. \tag{4}$$

We introduce the definition of fractional analytic function below.

Definition 2.3 ([25])**:** If x,x_0, and a_k are real numbers for all k, $x_0\in(a,b)$, and $0<\alpha\leq 1$. If the function $f_\alpha:[a,b]\to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha:[a,b]\to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

Next, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([26])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{5}$$

$$g_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\ . \tag{6}$$

Then we define

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)\\&=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\otimes_\alpha\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\\&=\sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)(x-x_0)^{n\alpha}.\end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)\\&=\sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}\otimes_\alpha\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}\\&=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}.\end{aligned} \tag{8}$$

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Definition 2.5 ([27])**:** If $0<\alpha\leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}=\sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes_\alpha n}, \tag{9}$$

$$g_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}=\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes_\alpha n}. \tag{10}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha\circ g_\alpha)(x^\alpha)=f_\alpha(g_\alpha(x^\alpha))=\sum_{n=0}^{\infty}\frac{a_n}{n!}(g_\alpha(x^\alpha))^{\otimes_\alpha n}, \tag{11}$$

and

$$(g_\alpha\circ f_\alpha)(x^\alpha)=g_\alpha(f_\alpha(x^\alpha))=\sum_{n=0}^{\infty}\frac{b_n}{n!}(f_\alpha(x^\alpha))^{\otimes_\alpha n}. \tag{12}$$

Definition 2.6 ([28])**:** If $0<\alpha\leq 1$, and x is a real variable. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha n}. \tag{13}$$

And the α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$. On the other hand, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{(-1)^k x^{2n\alpha}}{\Gamma(2n\alpha+1)}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \tag{14}$$

and

$$sin_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{15}$$

Definition 2.7 ([29])**:** Let $0<\alpha\leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions. Then $(f_\alpha(x^\alpha))^{\otimes_\alpha n}=f_\alpha(x^\alpha)\otimes_\alpha\cdots\otimes_\alpha f_\alpha(x^\alpha)$ is called the nth power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)=1$, then $g_\alpha(x^\alpha)$ is called the $\otimes_\alpha$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $(f_\alpha(x^\alpha))^{\otimes_\alpha(-1)}$.

Definition 2.8 ([30])**:** Let $0<\alpha\leq 1$ and r be a real number. The r-th power of the α-fractional analytic function $f_\alpha(x^\alpha)$ is defined by

$$[f_\alpha(x^\alpha)]^{\otimes_\alpha r}=E_\alpha\left(r\cdot Ln_\alpha(f_\alpha(x^\alpha))\right). \tag{16}$$

III. MAIN RESULTS

In this section, we will solve two fractional integrals involving fractional tangent function.

Theorem 3.1: *Let* $0<\alpha\leq 1$, *and* p *be a real number, then the* α*-fractional integrals*

$$\left({}_pI_x^\alpha\right)\left[[tan_\alpha(x^\alpha)]^{\otimes_\alpha\left(\frac{1}{2}\right)}\right]$$

$$=\frac{1}{\sqrt{2}}\left\{\begin{matrix} arcsin_\alpha\left(sin_\alpha(x^\alpha)-cos_\alpha(x^\alpha)\right)-Ln_\alpha\left(\left|sin_\alpha(x^\alpha)+cos_\alpha(x^\alpha)+[sin_\alpha(2x^\alpha)]^{\otimes_\alpha\left(\frac{1}{2}\right)}\right|\right)\\ -arcsin_\alpha\left(sin_\alpha(p^\alpha)-cos_\alpha(p^\alpha)\right)+Ln_\alpha\left(\left|sin_\alpha(p^\alpha)+cos_\alpha(p^\alpha)+[sin_\alpha(2p^\alpha)]^{\otimes_\alpha\left(\frac{1}{2}\right)}\right|\right)\end{matrix}\right\}. \tag{17}$$

$$\left({}_pI_x^\alpha\right)\left[[tan_\alpha(x^\alpha)]^{\otimes_\alpha\left(-\frac{1}{2}\right)}\right]$$

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$$=\frac{1}{\sqrt{2}}\begin{Bmatrix} arcsin_\alpha\left(sin_\alpha(x^\alpha)-cos_\alpha(x^\alpha)\right)+Ln_\alpha\left(\left|sin_\alpha(x^\alpha)+cos_\alpha(x^\alpha)+[sin_\alpha(2x^\alpha)]^{\otimes_\alpha\left(\frac{1}{2}\right)}\right|\right) \\ -arcsin_\alpha\left(sin_\alpha(p^\alpha)-cos_\alpha(p^\alpha)\right)-Ln_\alpha\left(\left|sin_\alpha(p^\alpha)+cos_\alpha(p^\alpha)+[sin_\alpha(2p^\alpha)]^{\otimes_\alpha\left(\frac{1}{2}\right)}\right|\right)\end{Bmatrix}. \tag{18}$$

Proof Since

$$\left({}_pI_x^\alpha\right)\left[[tan_\alpha(x^\alpha)]^{\otimes_\alpha\left(\frac{1}{2}\right)}\right]+\left({}_pI_x^\alpha\right)\left[[tan_\alpha(x^\alpha)]^{\otimes_\alpha\left(-\frac{1}{2}\right)}\right]$$

$$=\left({}_pI_x^\alpha\right)\left[[tan_\alpha(x^\alpha)]^{\otimes_\alpha\left(\frac{1}{2}\right)}+[tan_\alpha(x^\alpha)]^{\otimes_\alpha\left(-\frac{1}{2}\right)}\right]$$

$$=\left({}_pI_x^\alpha\right)\left[[sin_\alpha(x^\alpha)]^{\otimes_\alpha\left(\frac{1}{2}\right)}\otimes_\alpha[cos_\alpha(x^\alpha)]^{\otimes_\alpha\left(-\frac{1}{2}\right)}+[cos_\alpha(x^\alpha)]^{\otimes_\alpha\left(\frac{1}{2}\right)}\otimes_\alpha[sin_\alpha(x^\alpha)]^{\otimes_\alpha\left(-\frac{1}{2}\right)}\right]$$

$$=\left({}_pI_x^\alpha\right)\left[[sin_\alpha(x^\alpha)+cos_\alpha(x^\alpha)]\otimes_\alpha\left[sin_\alpha(x^\alpha)\otimes_\alpha cos_\alpha(x^\alpha)\right]^{\otimes_\alpha\left(-\frac{1}{2}\right)}\right]$$

$$=\left({}_pI_x^\alpha\right)\left[\sqrt{2}\cdot\left[2sin_\alpha(x^\alpha)\otimes_\alpha cos_\alpha(x^\alpha)\right]^{\otimes_\alpha\left(-\frac{1}{2}\right)}\otimes_\alpha\left({}_pD_x^\alpha\right)[sin_\alpha(x^\alpha)-cos_\alpha(x^\alpha)]\right]$$

$$=\sqrt{2}\cdot\left({}_pI_x^\alpha\right)\left[\left[1-[sin_\alpha(x^\alpha)-cos_\alpha(x^\alpha)]^{\otimes_\alpha 2}\right]^{\otimes_\alpha\left(-\frac{1}{2}\right)}\otimes_\alpha\left({}_pD_x^\alpha\right)[sin_\alpha(x^\alpha)-cos_\alpha(x^\alpha)]\right]$$

$$=\sqrt{2}\cdot arcsin_\alpha\left(sin_\alpha(x^\alpha)-cos_\alpha(x^\alpha)\right)-\sqrt{2}\cdot arcsin_\alpha\left(sin_\alpha(p^\alpha)-cos_\alpha(p^\alpha)\right). \tag{19}$$

And

$$\left({}_pI_x^\alpha\right)\left[[tan_\alpha(x^\alpha)]^{\otimes_\alpha\left(\frac{1}{2}\right)}\right]-\left({}_pI_x^\alpha\right)\left[[tan_\alpha(x^\alpha)]^{\otimes_\alpha\left(-\frac{1}{2}\right)}\right]$$

$$=\left({}_pI_x^\alpha\right)\left[[tan_\alpha(x^\alpha)]^{\otimes_\alpha\left(\frac{1}{2}\right)}-[tan_\alpha(x^\alpha)]^{\otimes_\alpha\left(-\frac{1}{2}\right)}\right]$$

$$=\left({}_pI_x^\alpha\right)\left[[sin_\alpha(x^\alpha)]^{\otimes_\alpha\left(\frac{1}{2}\right)}\otimes_\alpha[cos_\alpha(x^\alpha)]^{\otimes_\alpha\left(-\frac{1}{2}\right)}-[cos_\alpha(x^\alpha)]^{\otimes_\alpha\left(\frac{1}{2}\right)}\otimes_\alpha[sin_\alpha(x^\alpha)]^{\otimes_\alpha\left(-\frac{1}{2}\right)}\right]$$

$$=\left({}_pI_x^\alpha\right)\left[[sin_\alpha(x^\alpha)-cos_\alpha(x^\alpha)]\otimes_\alpha\left[sin_\alpha(x^\alpha)\otimes_\alpha cos_\alpha(x^\alpha)\right]^{\otimes_\alpha\left(-\frac{1}{2}\right)}\right]$$

$$=-\sqrt{2}\cdot\left({}_pI_x^\alpha\right)\left[\left[[sin_\alpha(x^\alpha)+cos_\alpha(x^\alpha)]^{\otimes_\alpha 2}-1\right]^{\otimes_\alpha\left(-\frac{1}{2}\right)}\otimes_\alpha\left({}_pD_x^\alpha\right)[sin_\alpha(x^\alpha)+cos_\alpha(x^\alpha)]\right]$$

$$=-\sqrt{2}\cdot Ln_\alpha\left(\left|sin_\alpha(x^\alpha)+cos_\alpha(x^\alpha)+[sin_\alpha(2x^\alpha)]^{\otimes_\alpha\left(\frac{1}{2}\right)}\right|\right)$$

$$+\sqrt{2}\cdot Ln_\alpha\left(\left|sin_\alpha(p^\alpha)+cos_\alpha(p^\alpha)+[sin_\alpha(2p^\alpha)]^{\otimes_\alpha\left(\frac{1}{2}\right)}\right|\right). \tag{20}$$

It follows that

$$\left({}_pI_x^\alpha\right)\left[[tan_\alpha(x^\alpha)]^{\otimes_\alpha\left(\frac{1}{2}\right)}\right]$$

$$=\frac{1}{2}\begin{Bmatrix}\sqrt{2}\cdot arcsin_\alpha\left(sin_\alpha(x^\alpha)-cos_\alpha(x^\alpha)\right)-\sqrt{2}\cdot arcsin_\alpha\left(sin_\alpha(p^\alpha)-cos_\alpha(p^\alpha)\right) \\ -\sqrt{2}\cdot Ln_\alpha\left(\left|sin_\alpha(x^\alpha)+cos_\alpha(x^\alpha)+[sin_\alpha(2x^\alpha)]^{\otimes_\alpha\left(\frac{1}{2}\right)}\right|\right) \\ +\sqrt{2}\cdot Ln_\alpha\left(\left|sin_\alpha(p^\alpha)+cos_\alpha(p^\alpha)+[sin_\alpha(2p^\alpha)]^{\otimes_\alpha\left(\frac{1}{2}\right)}\right|\right)\end{Bmatrix}$$

$$=\frac{1}{\sqrt{2}}\left\{\begin{array}{l} arcsin_\alpha\left(sin_\alpha(x^\alpha)-cos_\alpha(x^\alpha)\right)-Ln_\alpha\left(\left|sin_\alpha(x^\alpha)+cos_\alpha(x^\alpha)+[sin_\alpha(2x^\alpha)]^{\otimes_\alpha\left(\frac{1}{2}\right)}\right|\right) \\ -arcsin_\alpha\left(sin_\alpha(p^\alpha)-cos_\alpha(p^\alpha)\right)+Ln_\alpha\left(\left|sin_\alpha(p^\alpha)+cos_\alpha(p^\alpha)+[sin_\alpha(2p^\alpha)]^{\otimes_\alpha\left(\frac{1}{2}\right)}\right|\right)\end{array}\right\}.$$

And

$$\left({}_pI_x^\alpha\right)\left[[tan_\alpha(x^\alpha)]^{\otimes_\alpha\left(-\frac{1}{2}\right)}\right]$$

$$=\frac{1}{2}\left\{\begin{array}{c}\sqrt{2}\cdot arcsin_\alpha\left(sin_\alpha(x^\alpha)-cos_\alpha(x^\alpha)\right)-\sqrt{2}\cdot arcsin_\alpha\left(sin_\alpha(p^\alpha)-cos_\alpha(p^\alpha)\right) \\ +\sqrt{2}\cdot Ln_\alpha\left(\left|sin_\alpha(x^\alpha)+cos_\alpha(x^\alpha)+[sin_\alpha(2x^\alpha)]^{\otimes_\alpha\left(\frac{1}{2}\right)}\right|\right) \\ -\sqrt{2}\cdot Ln_\alpha\left(\left|sin_\alpha(p^\alpha)+cos_\alpha(p^\alpha)+[sin_\alpha(2p^\alpha)]^{\otimes_\alpha\left(\frac{1}{2}\right)}\right|\right)\end{array}\right\}$$

$$=\frac{1}{\sqrt{2}}\left\{\begin{array}{l} arcsin_\alpha\left(sin_\alpha(x^\alpha)-cos_\alpha(x^\alpha)\right)+Ln_\alpha\left(\left|sin_\alpha(x^\alpha)+cos_\alpha(x^\alpha)+[sin_\alpha(2x^\alpha)]^{\otimes_\alpha\left(\frac{1}{2}\right)}\right|\right) \\ -arcsin_\alpha\left(sin_\alpha(p^\alpha)-cos_\alpha(p^\alpha)\right)-Ln_\alpha\left(\left|sin_\alpha(p^\alpha)+cos_\alpha(p^\alpha)+[sin_\alpha(2p^\alpha)]^{\otimes_\alpha\left(\frac{1}{2}\right)}\right|\right)\end{array}\right\}.$$

Q.e.d.

IV. CONCLUSION

In this paper, based on Jumarie's modified R-L fractional calculus and a new multiplication of fractional analytic functions, two fractional integrals involving fractional tangent function are studied. Using some techniques, we can find the exact solutions of these two fractional integrals. In addition, our results are generalizations of classical calculus results. In the future, we will continue to use Jumarie type of R-L fractional calculus and the new multiplication of fractional analytic functions to solve the problems in fractional differential equations and engineering mathematics.

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