

Goldbach's Conjecture — A Route to the Inconsistency of Arithmetic

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Abstract. This paper proves an inconsistency in Peano arithmetic (PA). We express two propositions — a strengthened form of the strong Goldbach conjecture and its negation — using a specific set that varies depending on these propositions. On the other hand, we show that this set remains unchanged under the assumptions of the two statements. This causes a contradiction.

Notations. Let \mathbb{N} denote the natural numbers starting from 1, let \mathbb{N}_n denote the natural numbers starting from $n > 1$ and let \mathbb{P}_3 denote the prime numbers starting from 3.

Strengthened strong Goldbach conjecture (SSGB): *Every even integer greater than 6 can be expressed as the sum of two different primes.*

Theorem. *PA is contradictory, i.e. the statement FALSE can be derived.*

Proof. We define the set $S_g := \{ (pk, mk, qk) \mid k, m \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q; m = (p + q) / 2 \}$.

SSGB is equivalent to saying that every integer $n \geq 4$ is the arithmetic mean of two different odd primes and so it is equivalent to saying that all integers $n \geq 4$ appear as m in a middle component mk of S_g . So, by the definition of S_g we have

$$\text{SSGB} \iff \forall n \in \mathbb{N}_4 \exists (pk, mk, qk) \in S_g \quad n = m$$

$$\neg\text{SSGB} \iff \exists n \in \mathbb{N}_4 \forall (pk, mk, qk) \in S_g \quad n \neq m.$$

The set S_g has the following two properties.

First, the whole range of \mathbb{N}_3 can be expressed by the triple components of S_g ("covering"), because every integer $x \geq 3$ can be written as some pk with $k = 1$ when x is prime, as some pk with $k \neq 1$ when x is composite and not a power of 2, or as $(3 + 5)k / 2$ when x is a power of 2; $p \in \mathbb{P}_3, k \in \mathbb{N}$. So we have

$$(C) \quad \forall x \in \mathbb{N}_3 \exists (pk, mk, qk) \in S_g \quad x = pk \quad \vee \quad x = mk.$$

A few examples of the covering:

$x = 19$: (**19·1**, 21·1, 23·1), (**19·1**, 60·1, 101·1)

$x = 27$: (**3·9**, 7·9, 11·9)

$x = 38$: (**19·2**, 21·2, 23·2)

$x = 42$: (**3·14**, 5·14, 7·14), (**7·6**, 9·6, 11·6)

$x = 4096$: (3·1024, **4·1024**, 5·1024)

$x = 10000$: (**5·2000**, 6·2000, 7·2000).

Second, according to the statement SSGB, all pairs (p, q) of distinct odd primes are used in the definition of the set S_g ("maximality"). So we have

(M) $\forall p, q \in \mathbb{P}_3, p < q \quad \forall k \in \mathbb{N} \quad (pk, mk, qk) \in S_g$, where $m = (p + q) / 2$.

The property (C) excludes the possibility that there is an $n \geq 4$ different from all m , i.e. \neg SSGB, for the reason that n is different from all S_g triple components pk and mk .

The property (M) excludes the possibility that there is an $n \geq 4$ different from all m , i.e. \neg SSGB, for the reason that n is the arithmetic mean of a pair of primes not used in S_g . Thus (M) excludes the possibility that \neg SSGB applies due to a missing prime number pair. The proof would no longer be possible if we left out any prime number pair in the formulation of SSGB and S_g .

Therefore, in the case of \neg SSGB, neither \neg (C) nor \neg (M) applies.

The basic idea is now the following.

There are two possibilities for S_g , exactly one of which must occur: Either there is an $n \in \mathbb{N}_4$ in addition to all the numbers m defined in S_g or there is not. The latter is equivalent to SSGB and the former is equivalent to \neg SSGB.

Since, due to (M), an $n \geq 4$ different from all m cannot be the arithmetic mean of a pair of primes not used in S_g and since, due to (C), this n equals a component of some S_g triple that exists by definition, the covering of \mathbb{N}_3 by the S_g triples in the case n exists (\neg SSGB) is equal to that in the case n does not exist (SSGB). This causes a contradiction because in the case SSGB the numbers m defined in S_g take all integer values $x \geq 4$ whereas in the case \neg SSGB they don't.

The following steps are independent of the choice of n if, in the case of $\neg\text{SSGB}$, there is more than one that is different from all m . For example, the minimal such n works.

We split S_g into two complementary subsets in the following way. For any $y \in \mathbb{N}_3$, we write

$S_g = S_{g+(y)} \cup S_{g-(y)}$, with

$S_{g+(y)} := \{ (pk, mk, qk) \in S_g \mid \exists k' \in \mathbb{N} \quad pk = yk' \vee mk = yk' \vee qk = yk' \}$

$S_{g-(y)} := \{ (pk, mk, qk) \in S_g \mid \forall k' \in \mathbb{N} \quad pk \neq yk' \wedge mk \neq yk' \wedge qk \neq yk' \}$.

We define

$S_1 := \{ (pk, mk, qk) \in S_g \mid \text{SSGB} \}$

$S_2 := \{ (pk, mk, qk) \in S_g \mid \neg\text{SSGB} \wedge ((C) \wedge (M)) \}$.

Under the assumption SSGB there is no $n \in \mathbb{N}_4$ in addition to all m defined in S_g . Then, under this assumption, we can choose any $y \in \mathbb{N}_3$ such that $S_g = S_{g+(y)} \cup S_{g-(y)}$. So,

(1.1) $\forall y \in \mathbb{N}_3 \quad \text{SSGB} \Rightarrow S_1 = S_{g+(y)} \cup S_{g-(y)}$.

Under the assumption $\neg\text{SSGB}$ there is an $n \in \mathbb{N}_4$ as described above. Then, since the possibilities $\neg(C)$ and $\neg(M)$ for the existence of n are ruled out,

(1.2) $\neg\text{SSGB} \Rightarrow S_2 = S_{g+(n)} \cup S_{g-(n)}$.

$S_{g+(n)} \cup S_{g-(n)}$ is independent of n . Hence, from (1.2) we get

(1.2') $\forall y \in \mathbb{N}_3 \quad \neg\text{SSGB} \Rightarrow S_2 = S_{g+(y)} \cup S_{g-(y)}$.

Now, we will make use of the following principle.

If two sets of (possibly infinitely many) x -tuples are equal, then the sets of their corresponding i -th components are equal; $1 \leq i \leq x$.

To this end, for each $k \in \mathbb{N}$ we define

$$M_1(k) := \{ mk \mid (pk, mk, qk) \in S_1 \}$$

$$M_2(k) := \{ mk \mid (pk, mk, qk) \in S_2 \}.$$

Then, applying the principle above to the middle component of the triples (pk, mk, qk) , $((1.1) \wedge (1.2'))$ implies

$$(2.1) \quad \forall k \in \mathbb{N} \quad \forall y \in \mathbb{N}_3 \quad \text{SSGB} \Rightarrow M_1(k) = \{ mk \mid (pk, mk, qk) \in S_{g^+(y)} \cup S_{g^-(y)} \}$$

\wedge

$$(2.2) \quad \forall k \in \mathbb{N} \quad \forall y \in \mathbb{N}_3 \quad \neg\text{SSGB} \Rightarrow M_2(k) = \{ mk \mid (pk, mk, qk) \in S_{g^+(y)} \cup S_{g^-(y)} \}.$$

Setting $M_1 := M_1(1)$ and $M_2 := M_2(1)$, we get

$$(2.1') \quad \forall y \in \mathbb{N}_3 \quad \text{SSGB} \Rightarrow M_1 = \{ m \mid (p, m, q) \in S_{g^+(y)} \cup S_{g^-(y)} \}$$

\wedge

$$(2.2') \quad \forall y \in \mathbb{N}_3 \quad \neg\text{SSGB} \Rightarrow M_2 = \{ m \mid (p, m, q) \in S_{g^+(y)} \cup S_{g^-(y)} \}.$$

Since for every $y \in \mathbb{N}_3$ $S_{g^+(y)} \cup S_{g^-(y)}$ equals S_g , there is a set X such that for every $y \in \mathbb{N}_3$ $\{ m \mid (p, m, q) \in S_{g^+(y)} \cup S_{g^-(y)} \}$ equals X . So, from $((2.1') \wedge (2.2'))$ we obtain

$$(3) (SSGB \Rightarrow M_1 = X) \wedge (\neg SSGB \Rightarrow M_2 = X).$$

The set X is a free variable in (3) that is either equal to \mathbb{N}_4 or to some non-empty proper subset Y of \mathbb{N}_4 .

Now, we make use of the following rule.

Let $P = P(A)$ be a proposition that depends on a set A . Then, for any set B ,

$$(\text{we have a proof of } P(A) \wedge \text{we have a proof of } A = B) \Rightarrow \text{we have a proof of } P(B).$$

In the special case that A is a free variable that is replaced by the value B , the above conjunct (we have a proof of $A = B$) is trivially true.

We apply this rule to the proven statement (3). Then, replacing the variable X , (3) splits as follows.

$$(3.1) \text{ we have a proof of } (SSGB \Rightarrow M_1 = \mathbb{N}_4 \wedge \neg SSGB \Rightarrow M_2 = \mathbb{N}_4)$$

∨

$$(3.2) \text{ we have a proof of } (SSGB \Rightarrow M_1 = Y \neq \mathbb{N}_4 \wedge \neg SSGB \Rightarrow M_2 = Y \neq \mathbb{N}_4).$$

This implies

$$(3.1') \left(\text{we have a proof of } (SSGB \Rightarrow M_1 = \mathbb{N}_4) \wedge \text{we have a proof of } (\neg SSGB \Rightarrow M_2 = \mathbb{N}_4) \right)$$

∨

$$(3.2') \left(\text{we have a proof of } (SSGB \Rightarrow M_1 = Y \neq \mathbb{N}_4) \wedge \text{we have a proof of } (\neg SSGB \Rightarrow M_2 = Y \neq \mathbb{N}_4) \right).$$

Now, we will establish a contradiction to $((3.1') \vee (3.2'))$.

Under the assumption SSGB the numbers m defined in S_g take all integer values $x \geq 4$ whereas under \neg SSGB they don't. Therefore,

(4.1) we have a proof of $(SSGB \Rightarrow M_1 = \mathbb{N}_4)$

\wedge

(4.2) we have a proof of $(\neg$ SSGB $\Rightarrow M_2 \neq \mathbb{N}_4)$.

Then, $((3.1') \vee (3.2'))$ together with $((4.1) \wedge (4.2))$ implies

(5.1) we have a proof of $(\neg$ SSGB $\Rightarrow M_2 = \mathbb{N}_4)$

\vee

(5.2) we have a proof of $(SSGB \Rightarrow M_1 = Y \neq \mathbb{N}_4)$.

Because of $((4.1) \wedge (4.2))$ and because

$SSGB \Rightarrow M_2 = \{ \} \neq \mathbb{N}_4$

and

\neg SSGB $\Rightarrow M_1 = \{ \} \neq Y$,

we have a proof that $(M_2 = \mathbb{N}_4)$ is false and we have a proof that $(M_1 = Y \neq \mathbb{N}_4)$ is false. So, $((5.1) \vee (5.2))$ yields

(6.1) we have a proof of SSGB

\vee

(6.2) we have a proof of \neg SSGB.

Since we have neither a proof of SSGB nor of \neg SSGB, both (6.1) and (6.2) are false.

Therefore, we obtain $(\text{FALSE} \vee \text{FALSE})$ and thus FALSE.

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