Variance Swap Introduction

A variance swap is a forward contract on annualized variance, the square of the realized volatility. The holder of a variance swap at expiration receives a notional amount of dollar for every point by which the stock's realized variance has exceeded the variance delivery price.

Valuation of the swap involves decomposition of the contract into two periods, one that has become historical, and the other with stock prices still unknown. The unknown variance from the value date to the swap maturity can be replicated by a portfolio of call and put options. Volatility skew can be incorporated into pricing of these options through bi-linear interpolation.

We developed pricing models for equity log forward contracts and variance swaps. The model for variance swaps can be used to calculate P&L and risk numbers.

Suppose there are n consecutive trading days $\{t_i, i = 0,1,\cdots,n\}$. At the maturity of a variance swap, the realized variance is defined as

$$\sigma_R^2 = 252\sum_{i=1}^{n}\frac{[\ln(S_i / S_{i-1})]^2}{n} \tag{1}$$

where S_i is the underlying stock price at time t_i. The payoff of the variance swap at maturity T is given by

$$V(T) = N \times \left(\sigma_R^2 - K^2\right) \tag{2}$$

where N is the notional amount and K^2 is the delivery price for variance.

Let t be the value date, $t < T$. If $t > t_m$, the stock prices at $\{t_i, i = 0,1,\cdots,m\}$ have become historical. We then have

$$\sigma_R^2 = \frac{m}{n} Var_{hist} + \frac{n-m}{n} Var_{fut} \tag{3}$$

where the first term at the right hand of the equation is known and can be calculated as

$$Var_{hist} = 252 \sum_{i=1}^{m} \frac{\left[\ln(S_i / S_{i-1})\right]^2}{m} \tag{4}$$

and the second term Var_{fut} is the future realized variance. This future realized variance can be replicated by a portfolio of call and put options with appropriate weights. In brief, the expectation of Var_{fut} can be written as

$$E(Var_{fut}) = \frac{2}{T-t} \left\{ E\left[\int_t^T \mu d\tau\right] - \frac{F(t,T) - S_*}{S_*} - \ln\left(\frac{S_*}{S_t}\right) \right\} + \frac{1}{df(t,T)} \Pi_{CP} \tag{5}$$

where μ is the drift of the stock price process, $F(t,T)$ is the forward price of stock at time t with expiration at T, S_* is the a reference strike price, $df(t,T)$ is the discount factor, and Π_{CP} is the present value of the portfolio, given by

$$T_{CP} = \sum_i w_p(K_{ip}) P(S, K_{ip}) + \sum_i w_c(K_{ic}) C(S, K_{ic}) . \tag{6}$$

Here, P and C are put and call option prices. We suppose that we can trade all options with strikes K_{ic} such that $K_0 = S_* < K_{1c} < K_{2c} < K_{3c} < \cdots$ and put options with strikes K_{ip} such that $K_0 = S_* > K_{1p} > K_{2p} > K_{3p} > \cdots$.

Weights of the portfolio can be determined by many approaches. One such approach is called the slope method, described as follows. Define a function

$$f(x) = \frac{2}{T-t}\left[\frac{x-S_*}{S_*}\right] - \ln\frac{x}{S_*}, \tag{7}$$

weights are then given by

$$w_c(K_0) = \frac{f(K_{1c}) - f(K_0)}{K_{1c} - K_0} \tag{8}$$

$$w_c(K_{ic}) = \frac{f(K_{n+1,c}) - f(K_{n,c})}{K_{n+1,c} - K_{n,c}} - \sum_{i=0}^{n-1} w_c(K_{i,c})\,. \tag{9}$$

$$w_p(K_0) = \frac{f(K_{1p}) - f(K_0)}{K_0 - K_{1p}} \tag{10}$$

$$w_p(K_{ip}) = \frac{f(K_{n+1,c}) - f(K_{n,c})}{K_{n,p} - K_{n+1,c}} - \sum_{i=0}^{n-1} w_p(K_{i,p})\,. \tag{11}$$

We have also proposed another approach, named the K-squared method.

Equation (5) can be rewritten as

$$E(Var_{fut}) = \frac{2}{T-t}\left\{\ln F(t,T) - \frac{F(t,T)-S_*}{S_*} - \ln S_*\right\} + \frac{1}{df(t,T)}\Pi_{CP}\,. \tag{12}$$

The term $\ln F(t,T)$ is where the calculation for equity log forward takes effective.

Reference:

https://finpricing.com/FinPricing-ProductBrochure.pdf