Goldbach's Conjecture — A Route to the Inconsistency of Arithmetic

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Abstract. This paper proves an inconsistency in Peano arithmetic (PA). We express two propositions — a strengthened form of the strong Goldbach conjecture and its negation — using a specific set that varies according to which of the propositions holds. On the other hand, we show that this set remains unchanged under the assumptions of the two statements. This causes a contradiction.

Notations. Let $\mathbb{N}$ denote the natural numbers starting from 1, let $\mathbb{N}_n$ denote the natural numbers starting from n > 1 and let $\mathbb{P}_3$ denote the prime numbers starting from 3.

Strengthened strong Goldbach conjecture (SSGB): *Every even integer greater than 6 can be expressed as the sum of two different primes.*

Theorem. *PA is contradictory, i.e. the statement FALSE can be derived.*

Proof. We define the set S_g := { (pk, mk, qk) | k, m $\in \mathbb{N}$; p, q $\in \mathbb{P}_3$, p < q; m = (p + q) / 2 }.

SSGB is equivalent to saying that every integer n ≥ 4 is the arithmetic mean of two different odd primes and so it is equivalent to saying that all integers n ≥ 4 appear as m in a middle component mk of S_g. So, by the definition of S_g we have

$$\text{SSGB} \iff \forall n \in \mathbb{N}_4 \;\; \exists (pk, mk, qk) \in S_g \quad n = m$$

$$\neg\text{SSGB} \iff \exists n \in \mathbb{N}_4 \;\; \forall (pk, mk, qk) \in S_g \quad n \neq m.$$

The set S_g has the following two properties.

First, the whole range of $\mathbb{N}_3$ can be expressed by the triple components of S_g (”*covering*”), because every integer x ≥ 3 can be written as some pk with k = 1 when x is prime, as some pk with k ≠ 1 when x is composite and not a power of 2, or as (3 + 5)k / 2 when x is a power of 2; p $\in \mathbb{P}_3$, k $\in \mathbb{N}$. So we have

(C) $\forall x \in \mathbb{N}_3 \;\; \exists (pk, mk, qk) \in S_g \quad x = pk \;\vee\; x = mk.$

A few examples of the covering:

x = 19: (**19·1**, 21·1, 23·1), (**19·1**, 60·1, 101·1)

x = 27: (**3·9**, 7·9, 11·9)

x = 38: (**19·2**, 21·2, 23·2)

x = 42: (**3·14**, 5·14, 7·14), (**7·6**, 9·6, 11·6)

x = 4096: (3·1024, **4·1024**, 5·1024)

x = 10000: (**5·2000**, 6·2000, 7·2000).

Second, according to the statement SSGB, all pairs (p, q) of distinct odd primes are used in the definition of the set S_g ("*maximality*"). So we have

(M) $\forall$ p, q $\in \mathbb{P}_3$, p < q $\quad \forall$ k $\in \mathbb{N}$ $\quad$ (pk, mk, qk) $\in S_g$, where m = (p + q) / 2.

The property (C) excludes that there is an n ≥ 4 different from all m, i.e. ¬SSGB, for the reason that n is different from all S_g triple components pk and mk.

The property (M) excludes that there is an n ≥ 4 different from all m, i.e. ¬SSGB, for the reason that n is the arithmetic mean of a pair of primes not used in S_g. Thus (M) excludes the possibility that ¬SSGB applies due to a missing prime number pair. The proof would no longer be possible if we left out any prime number pair in the formulation of SSGB and S_g.

The basic idea is now the following.

There are two possibilities for S_g, exactly one of which must occur: Either there is an $n \in \mathbb{N}_4$ in addition to all the numbers m defined in S_g or there is not. The latter is equivalent to SSGB and the former is equivalent to ¬SSGB.

Since, due to (M), an n ≥ 4 different from all m cannot be the arithmetic mean of a pair of primes not used in S_g and since, due to (C), this n equals a component of some S_g triple that exists by definition, the covering of $\mathbb{N}_3$ by the S_g triples in the case n exists (¬SSGB) is equal to that in the case n does not exist (SSGB). This causes a contradiction because in the case SSGB the numbers m defined in S_g take all integer values x ≥ 4 whereas in the case ¬SSGB they don't.

We illustrate this by representing the covering of $\mathbb{N}_3$ by an infinite matrix where the k-th row, k ≥ 1, is determined by all triple components pk, mk, qk :

3 4 5	3 5 7	...	5 6 7	5 8 11	...	∘ ∘ ∘	∘ ∘ ∘	∘ ∘ ∘	● ∘ ∘	...	∘ ∘ ∘	∘ ∘ ∘	∘ ∘ ∘	...
6 8 10	∘ ∘ ∘	...	∘ ∘ ∘	∘ ∘ ∘	...	∘ ∘ ∘	∘ ∘ ∘	∘ ∘ ∘	∘ ∘ ∘		∘ ∘ ∘	∘ ∘ ∘	∘ ∘ ∘	...
...	...		...	...		...	...	...	...		...	...	...	...
...	...		...	...		...	...	...	...		...	...	...	...
∘ ∘ ∘	∘ ∘ ∘		∘ ∘ ∘	∘ ∘ ∘		∘ ∘ ∘	● ∘ ∘	...	∘ ∘ ∘		∘ ∘ ∘	∘ ∘ ∘	∘ ∘ ∘	...
...	...		...	...		...	...	...	...		...	...	...	...
...	...		...	...		...	...	...	...		...	...	...	...
∘ ∘ ∘	∘ ∘ ∘		∘ ∘ ∘	∘ ∘ ∘		∘ ∘ ∘	∘ ∘ ∘	∘ ∘ ∘	∘ ∘ ∘		∘ ∘ ∘	∘ ∘ ∘	∘ ∘ ∘	...
∘ ● ∘	∘ ∘ ∘		∘ ∘ ∘	∘ ∘ ∘		∘ ∘ ∘	∘ ∘ ∘	∘ ∘ ∘	∘ ∘ ∘		∘ ∘ ∘	∘ ∘ ∘	∘ ∘ ∘	...
∘ ∘ ∘	∘ ∘ ∘		∘ ∘ ∘	∘ ∘ ∘		∘ ∘ ∘	∘ ∘ ∘	∘ ∘ ∘	∘ ∘ ∘		∘ ∘ ∘	∘ ∘ ∘	∘ ∘ ∘	...
...	...		...	...		...	...	...	...		...	...	...	...
...	...		...	...		...	...	...	...		...	...	...	...

The marked matrix elements ● represent an n ≥ 4 given in the case ¬SSGB, depending on which of the three types the number n is: prime, composite and not a power of 2, power of 2. There are infinitely many other matrix elements (unmarked) that are also equal to n or a multiple of n. In the case SSGB where there is no such n, the matrix is the same. On the other hand, there is a column mk = nk, k ≥ 1, in the case SSGB that does not exist in the case ¬SSGB. So, the matrix is different in the two cases. This is a contradiction.

After having excluded the possibility that ¬SSGB applies due to ¬(C) or ¬(M), we will now show that ((C) ∧ (M)) leads to a contradiction.

We split S_g into two complementary subsets in the following way. For any y ∈ $\mathbb{N}_3$, we write

$S_g = S_g+(y) \cup S_g-(y)$, with

$S_g+(y) := \{ (pk, mk, qk) \in S_g \mid \exists\, k' \in \mathbb{N} \quad pk = yk' \vee mk = yk' \vee qk = yk' \}$

$S_g-(y) := \{ (pk, mk, qk) \in S_g \mid \forall\, k' \in \mathbb{N} \quad pk \neq yk' \wedge mk \neq yk' \wedge qk \neq yk' \}$.

Then, after defining

$S_1 := \{ (pk, mk, qk) \in S_g \mid \ \ SSGB \ \wedge \ ((C) \wedge (M)) \}$

$S_2 := \{ (pk, mk, qk) \in S_g \mid \neg SSGB \ \wedge \ ((C) \wedge (M)) \}$,

we have

(1.1) there is a proof of $(\forall y \in \mathbb{N}_3 \quad SSGB \ \Rightarrow \ S_1 = S_g+(y) \cup S_g-(y))$

$\wedge$

(1.2) there is a proof of $(\forall y \in \mathbb{N}_3 \quad \neg SSGB \ \Rightarrow \ S_2 = S_g+(y) \cup S_g-(y))$.

Now, we will make use of the following principle.

If two sets of (possibly infinitely many) x-tuples are equal, then the sets of their corresponding i-th components are equal; $1 \leq i \leq x$.

To this end, for each $k \in \mathbb{N}$ we define

$M_1(k) := \{ mk \mid (pk, mk, qk) \in S_1 \}$

$M_2(k) := \{ mk \mid (pk, mk, qk) \in S_2 \}$.

Then, applying the principle above to the middle component of the triples (pk, mk, qk), $((1.1) \wedge (1.2))$ implies

(2.1) there is a proof of

$(\forall k \in \mathbb{N} \ \ \forall y \in \mathbb{N}_3 \quad SSGB \ \Rightarrow \ M_1(k) = \{ mk \mid (pk, mk, qk) \in \ S_g+(y) \cup S_g-(y) \})$

$\wedge$

(2.2) there is a proof of

$(\forall k \in \mathbb{N} \ \ \forall y \in \mathbb{N}_3 \quad \neg SSGB \ \Rightarrow \ M_2(k) = \{ mk \mid (pk, mk, qk) \in \ S_g+(y) \cup S_g-(y) \})$.

Setting $M_1 := M_1(1)$ and $M_2 := M_2(1)$, we get

(2.1') there is a proof of ($\forall y \in \mathbb{N}_3$ SSGB => $M_1 = \{ m \mid (p, m, q) \in S_g+(y) \cup S_g-(y) \}$)

$\wedge$

(2.2') there is a proof of ($\forall y \in \mathbb{N}_3$ $\neg$SSGB => $M_2 = \{ m \mid (p, m, q) \in S_g+(y) \cup S_g-(y) \}$).

Since by definition $S_g+(y) \cup S_g-(y)$ equals S_g for every $y \in \mathbb{N}_3$, there is a proof that for every $y \in \mathbb{N}_3$ $\{ m \mid (p, m, q) \in S_g+(y) \cup S_g-(y) \}$ equals a set $M := \{ m \mid (p, m, q) \in S_g \}$. So, from ((2.1') $\wedge$ (2.2')) we obtain

(3) there is a set X such that

(**(3.1)** there is a proof of (SSGB => $M_1 = X$)

$\wedge$

(3.2) there is a proof of ($\neg$SSGB => $M_2 = X$)).

Under the assumption SSGB the numbers m defined in S_g take all integer values $x \geq 4$ whereas under $\neg$SSGB they don't. Therefore,

(4.1) there is a proof of (SSGB => $M_1 = \mathbb{N}_4$)

$\wedge$

(4.2) there is a proof of ($\neg$SSGB => $M_2 = Y \neq \mathbb{N}_4$),

for some non-empty proper subset Y of $\mathbb{N}_4$.

Then, due to ((4.1) $\wedge$ (4.2)), we obtain that the statement (3) is false.

□